Queueing

Recall the sum of n independent $Exp(\lambda)$ is a Gamma (n,μ) distribution which has PDF $f_T(t)=\mu e^{-\mu t}\frac{(\mu t)^{n-1}}{(n-1)!}$ for $t\geq 0$.

M/M/1 queue $\begin{cases} q(n,n+1)=\lambda & n\geq 0 \\ q(n,n-1)=\mu & n\geq 1 \end{cases}$

Birth & death process $\pi(n)=\frac{\lambda_{n-1}\cdots\lambda_0}{\mu_n\cdots\mu_1}\pi(0)$

$\lambda<\mu \Rightarrow$ (*) $\boxed{\pi(n)=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n}$. In particular

$\boxed{\pi(0)=1-\frac{\lambda}{\mu}}$ (**)

Let V be the event that there is at least one customer in the system.

$\mathbb{P}(V)=1-\pi(0)=1-\left(1-\frac{\lambda}{\mu}\right)=\frac{\lambda}{\mu}$.

Let T_Q be the time spent in the queue.

Let $f(x)$ be the conditional density function of T_Q condition on V.

$G_n = \{n \text{ customer in the system}\}$

$$\mathbb{P}(T_Q \in (x-\varepsilon, x+\varepsilon) \mid V) = \sum_{n=1}^{\infty} \mathbb{P}(T_Q \in (x-\varepsilon, x+\varepsilon) \cap G_n \mid V)$$

$$= \frac{\sum_{n=1}^{\infty} \mathbb{P}(T_Q \in (x-\varepsilon, x+\varepsilon) \cap G_n)}{\mathbb{P}(V)} =$$

$$\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\mu}{\lambda} \cdot \sum_{n=1}^{\infty} \mathbb{P}(T_Q \in (x-\varepsilon, x+\varepsilon) \mid G_n) \cdot \mathbb{P}(G_n)$$

the sum n $\mathrm{Exp}(\mu)$ indep r.v.

$= \Gamma(n, \mu)$ whose PDF is: $\boxed{e^{-\mu x} \frac{\mu^n x^{n-1}}{(n-1)!}}$

$$= \frac{\mu}{\lambda} \sum_{n=1}^{\infty} \left(e^{-\mu x} \frac{\mu^n x^{n-1}}{(n-1)!} \cdot 2\varepsilon \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \right)$$

So, the conditional of T_Q conditioned V is:

$$f(x) = \frac{\mu}{\lambda} \sum_{n=1}^{\infty} \left(e^{-\mu x} \frac{\mu^n x^{n-1}}{(n-1)!} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \right)$$

$$= \cancel{\lambda} \cdot \frac{\cancel{\mu}}{\cancel{\lambda}} e^{-\mu x} \cdot \frac{\mu-\lambda}{\cancel{\mu}} \underbrace{\sum_{n=1}^{\infty} \frac{\cancel{\mu^n} \lambda^{n-1}}{\cancel{\mu^n}} \frac{x^{n-1}}{(n-1)!}}_{e^{\lambda x}} = (\mu-\lambda) e^{(\lambda-\mu)x}.$$

Lemma 4.1 ① is trivial, $\int_V T_Q \, d\mathbb{P}$ $\int_{V^c} T_Q \, d\mathbb{P}$

② $\mathbb{E}[T_Q] = \mathbb{E}[T_Q; V] + \mathbb{E}[T_Q; V^c]$

$= \mathbb{E}[T_Q | V] \cdot \underbrace{\mathbb{P}(V)}_{\frac{\lambda}{\mu}} + \underbrace{\mathbb{E}[T_Q | V^c]}_{0} = \circledast$

$$\mathbb{E}[T_Q | V] = \int (\mu-\lambda) \cdot x \cdot e^{-(\mu-\lambda)x} dx = \frac{1}{\mu-\lambda}$$

$$\circledast = \frac{1}{\mu-\lambda} \cdot \frac{\lambda}{\mu} + 0 = \frac{\lambda}{\mu} \cdot \frac{1}{\mu-\lambda}$$

3.) $\mathbb{E}[W] = \mathbb{E}[W_Q] + \frac{1}{\mu}$.

4.) Let $p = 1 - \frac{\lambda}{\mu}$ then "success" happens with probability p and "failure" happens with $1-p = \frac{\lambda}{\mu}$ The shifted geometric distribution gives the number Y of failures before the first success. $\mathbb{P}(Y=n) = (1-p)^n \cdot p$ $n=0,1,2,...$

in our case $\pi(n) = \Big(\underbrace{\frac{\lambda}{\mu}}_{1-p}\Big)^n \cdot \Big(\underbrace{1-\frac{\lambda}{\mu}}_{p}\Big)$

We know that the expectation of the shifted geometric distribution is $\frac{1-p}{p} = \frac{\lambda/\mu}{1-\frac{\lambda}{\mu}} = \frac{\frac{\lambda}{\mu}}{\frac{\mu-\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$.