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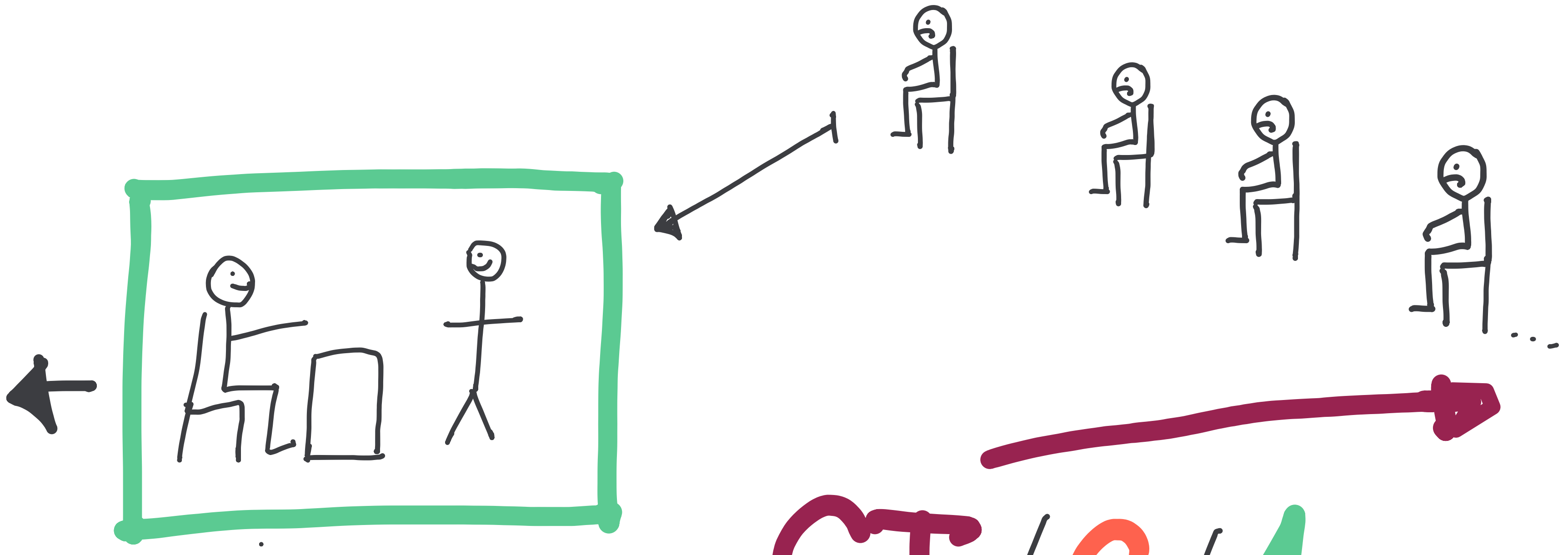
QUEUEING THEORY CONT.



ARRIVAL

$t_i \sim F$, iid.

$\mathbb{E}(F) = 1/\lambda$

(rate of arrival is λ)

GI/G/1

SERVICE TIME

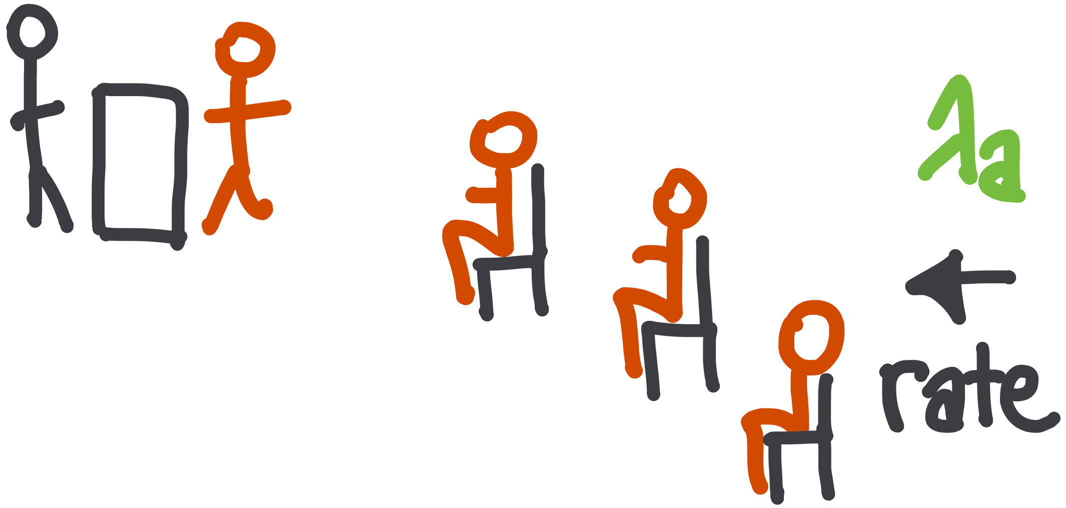
$s_i \sim G$, iid. $\mathbb{E}(G) = 1/\mu$

(rate of service μ)

GI/G/1 Q

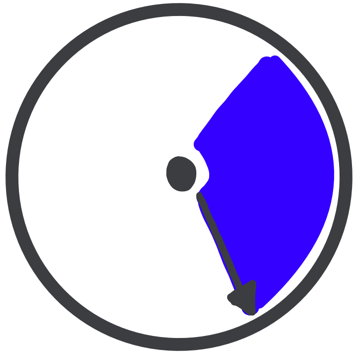
X_s = #customers in the system at time s (people in queue + people being served)

L = the avg number of —"— $= \lim_{t\to\infty} \frac{1}{t}\int_0^t X_s\,ds$



W_m = time the m-th cust. spends in the system

W = avg. time spent in the system $= \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} W_n$



$N_a(t)$ = #people who arrive before t & stand in the line

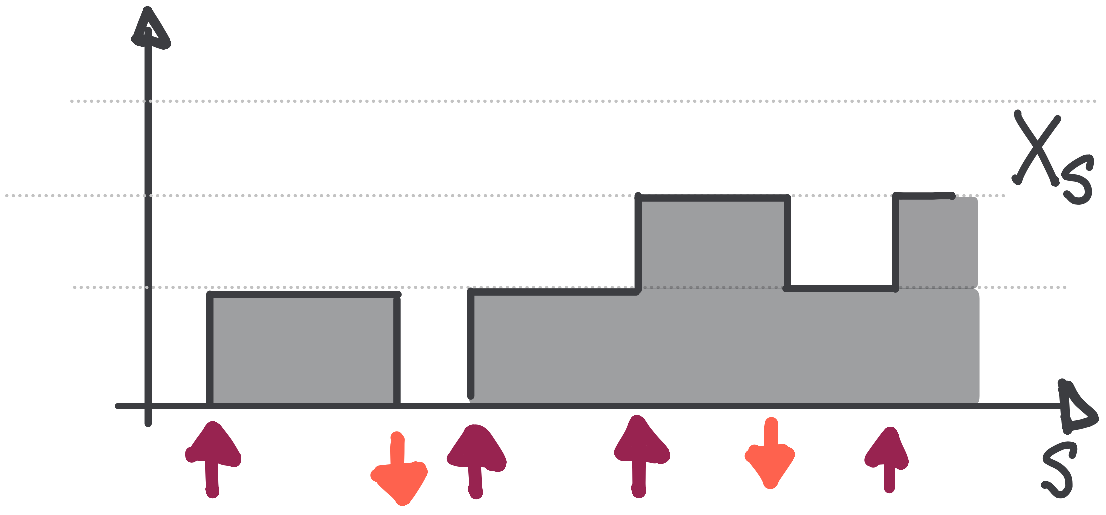
$$\lambda_a = \lim_{t\to\infty} \frac{N_a(t)}{t}$$

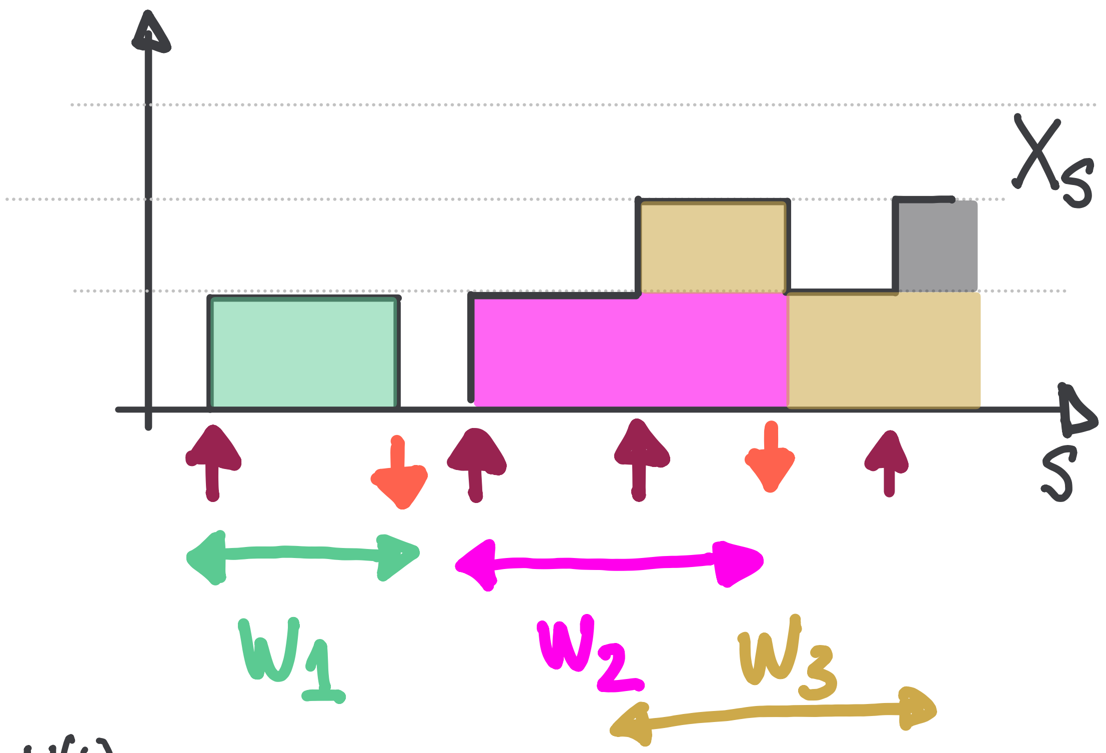
Little's law: $\boxed{L = \lambda_a W}$

Little's law: $\boxed{L = \lambda_a W}$

$L = \lim_{t\to\infty} \frac{1}{t} \cdot \int_0^t X_s ds,$

$\lambda_a = \lim_{t\to\infty} \frac{N_a(t)}{t},\quad W = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} W_k$



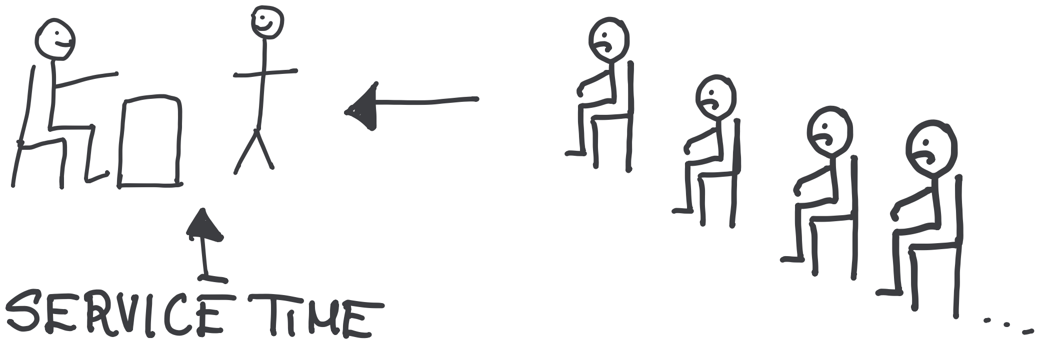


$\int_0^t X_s ds = \quad \approx \sum_{m=1}^{N(t)} W_m \approx N(t)\cdot W$

$\frac{1}{t}\int_0^t X_s ds \quad \approx \quad \frac{1}{t} N(t)\cdot W \approx \lambda_a \cdot W$

MARKOVIAN Q-S – M/M/1 QUEUES

1 server ∞– waiting room



ARRIVAL
← according to
Poisson(λ) ($\Rightarrow$ interarrival $\sim EXP(\lambda)$)
the rate:
$\left(\lambda_d = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E(t_i)} = \lambda\right)$

SERVICE TIME
$s_i \sim EXP(\mu)$, iid. $E(s_i) = 1/\mu$
(rate of service μ)

Markovian

X_t = number of customers in the system. $\in \{0,1,2,\dots\}$ is a continuous time **Markov process**
with infinitesimal generator

$$q(n,n+1) = \lambda, \quad q(n,n-1) = \mu$$

Brief heuristic

X_t – # customers in the system

Markov prop. for $0 \le s_0 < s_1 < \dots < s_n < s$ times, & $i_0, \dots, i_n, i, j$ states

$$\mathbb{P}(X_{t+s}=j \mid X_s = i, \underbrace{X_{s_n}=i_n, \dots, X_{s_0}=i_0}_{\text{does not matter because of memoryless prop. of EXP.}}) = \mathbb{P}(X_t=j \mid X_0=i)$$

Brief heuristic

X_t - # customers in the system

Infinitesimal generator

$$q(i,j) = \lim_{h \to 0} \frac{\mathbb{P}(X_h = j \mid X_0 = i)}{h} \Rightarrow$$

$$\boxed{\begin{aligned} q(n, n+1) &= \lambda \\ q(n, n-1) &= \mu \end{aligned}}$$

A/ more than 1 departure in time h

$\Updownarrow$

Sum of two or more exp r.v. $< h$

probability $o(h)$

B/ More than 1 arrival in time h

prob $o(h)$

C/ one arrival, no dep: $(\lambda h e^{-\lambda h} + o(h)) e^{-\mu h}$

D/ no arrival, one dep. $(\mu h e^{-\mu h} + o(h)) e^{-\lambda h}$

M/M/1 Q-S ARE B&D PROCESSES

$$q(n, n+1) = \lambda$$
$$q(n, n-1) = \mu$$

BIRTH & DEATH PROCESS. (with $\lambda_n = \lambda, \mu_n = \mu$)

Assume $\boxed{\lambda < \mu}$ (the arrival rate < the departure rate)

Recall: (Thm. 3.1.)

Let X_s be a birth & death process with $q(n, n+1) = \lambda$, $q(n, n-1) = \mu$, with infinite state space.

Then $\pi(n) = \lambda^n/\mu^n \cdot \pi(0)$ satisfies the detailed balanced condition,

Hence if $\sum \pi(i) = 1$ (attainable if $\sum_{n=1}^{\infty} (\lambda/\mu)^n < \infty$) $\Rightarrow$ π is a stationary distr.

M/M/1 Q-S, STATIONARY DISTR.

If $\pi(n) = \left(\lambda/\mu\right)^n \pi(0)$ &

$\sum \pi(i) = 1$ (attainable if $\sum_{n=1}^{\infty} \left(\lambda/\mu\right)^n < \infty$) $\Rightarrow$ π is a stationary distr.

$\sum \pi(i) = \pi(0) \cdot \frac{1}{1-\lambda/\mu} \Rightarrow$ $\pi(0) = 1 - \lambda/\mu$, $\pi(n) = \left(1-\lambda/\mu\right)\left(\lambda/\mu\right)^n$

M/M/1 Q-S, OTHER PROPERTIES

We are in the stationary state

T_Q = time spent in the queue (r.v.)

$$^{*}\boxed{T_Q \mid \{T_Q > 0\} \sim EXP(\mu-\lambda)}$$

1. $\mathbb{P}(T_Q = 0) = \mathbb{P}(Q=0) = 1 - \lambda/\mu$

2. on $(0,\infty)$ it has a density:
if there is n people already in the system one has to wait $\underbrace{G_1 + \dots + G_n}_{n\text{-service}}$, $G_i \sim EXP(\mu)$, hence

$$f_{T_Q}(x) = \sum_{n=1}^{\infty} \underbrace{f(x \mid n \text{ people in the system})}_{\Gamma(n,\mu)} \cdot \underbrace{\mathbb{P}(n \text{ people in the system})}_{\pi(n)} = \sum_{n=1}^{\infty} \frac{\mu^n \cdot x^{n-1} \cdot e^{-\mu x}}{(n-1)!} \cdot \left(\frac{\lambda}{\mu}\right)^n \cdot \left(1 - \lambda/\mu\right) =$$

$$= e^{-\mu x} \lambda \left(1 - \lambda/\mu\right) \sum_{n=1}^{\infty} \frac{(x\lambda)^{n-1}}{(n-1)!} = e^{-\mu x} \lambda \left(1-\lambda/\mu\right) e^{\lambda x} = e^{-(\mu-\lambda)x} \cdot (\mu-\lambda) \cdot \lambda/\mu \; ^{*}$$

M/M/1 Q-S, OTHER PROPERTIES

- $W_Q = \mathbb{E}(T_Q) = \mathbb{E}(T_Q \cdot \mathbb{1}\{T_Q > 0\}) = \lambda/\mu \cdot \frac{1}{\mu-\lambda}$

  ↑ average waiting time inside the queue

- $\mathbb{E}(W) = W_Q + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$

  ↑ average overall waiting time (inside queue & in service)

- $L = \lim_{t\to\infty} \frac{1}{t} \int_0^t X_t \, dt$, **Little Law:** $L = W \cdot \lambda = \frac{\lambda}{\mu-\lambda}$

  ↑ the (time) average of the number of customers

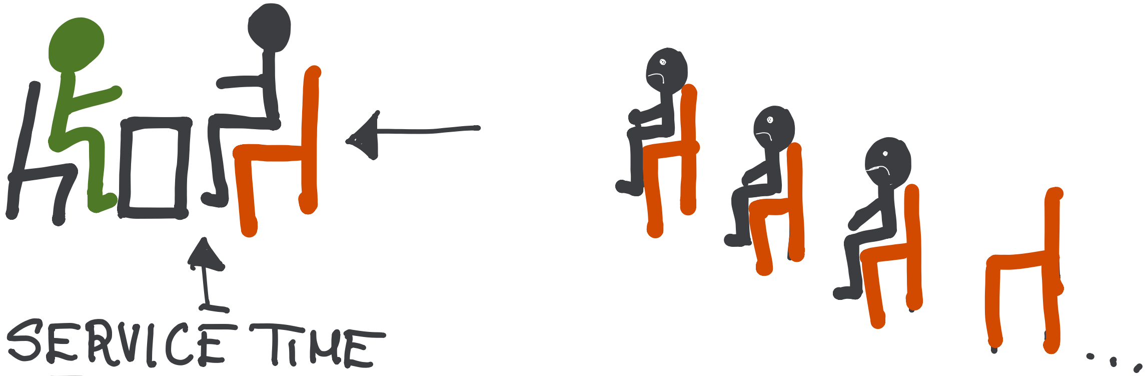
  or: L "(= time avg) = space avg" $= \mathbb{E}(X_t) = \sum_{n=0}^{\infty} n \cdot \pi(n) = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$

  $\sim \text{geom}(1-\lambda/\mu)$

M/M/1 Q-S, FINITE WAITING ROOM

1 server

there is only N space in the system.



SERVICE TIME

$s_i \sim EXP(\mu)$ iid. $E(s_i) = 1/\mu$

(rate of service μ)

ARRIVAL according to

Poisson(λ)

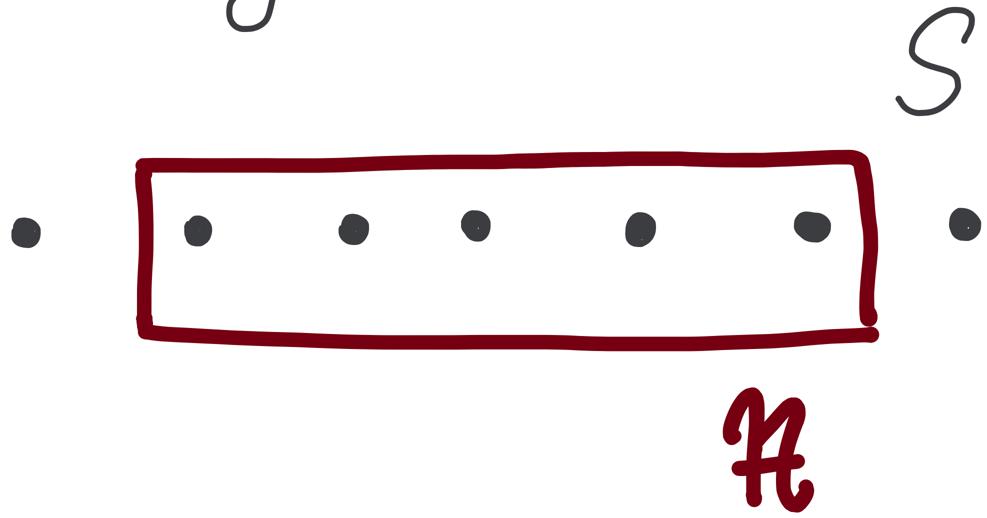
the rate:

$\left(\lambda = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E(t_i)} = \lambda\right)$

E.g. barbershop: 2 chairs to wait.

M/M/1 Q-S, FWR, STATIONARY DISTR.

How to get the stationary dist. from the ∞-waiting room case.



Lemma 4.2.

Given X_t, a M.C. on the state space S, with stationary distr. π satisfying the detailed balance condition.

Let $\{q(i,j)\}$ be the infinitesimal generator of X_t.

Let $A \subseteq S$, Y_t is the restriction of X_t to A.

$$\tilde{q}(x,y) = \begin{cases} q(x,y), & \text{if } x,y \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\boxed{\nu = \frac{1}{\sum_{x \in A} \pi(x)} \cdot \pi}$ is a stationary distr. for Y_t.

Proof: D.B.C. $x,y \in A$: $\nu(x)\,\tilde{q}(x,y) = \frac{\pi(x)}{\sum_{z \in A} \pi(z)} \cdot q(x,y) = \frac{\pi(y)}{\sum \pi(z)} \cdot q(y,x) = \nu(y) \cdot \tilde{q}(y,x)$

& $\sum \nu(x) = \sum_{x \in A} \pi(x) \Big/ \sum_{x \in A} \pi(x) = 1$

M/M/1 Q-S, FWR, STATIONARY DISTR.

From prev. lemma + stat. distr. for the ∞-waiting room.

$S = \{0, 1, \dots\} \Rightarrow \mathcal{A} = \{0, \dots, N\}$

$$\underset{i \in S}{\pi(i) = \left(1 - \lambda/\mu\right)\left(\lambda/\mu\right)^i} \quad \Rightarrow \quad \nu(i) = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} \left(\frac{\lambda}{\mu}\right)^i, \quad i \in \{0, \dots, N\}$$

$$\sum_{k=0}^{N} \left(1 - \lambda/\mu\right)\left(\lambda/\mu\right)^k = \left(1 - \lambda/\mu\right) \frac{1 - (\lambda/\mu)^{N+1}}{1 - \lambda/\mu}$$

$$\underline{\nu} = \frac{1}{\sum_{x \in \mathcal{A}} \pi(x)} \cdot \underline{\pi}$$

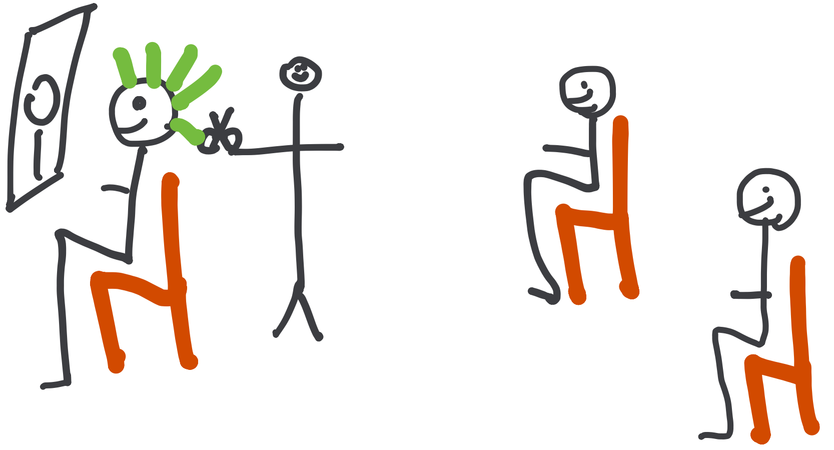
$$\pi(i) = \left(1 - \lambda/\mu\right)\left(\lambda/\mu\right)^i$$

Remark. On this finite state space, this is a stat. distr. even if $\lambda > \mu$.

For $\lambda = \mu$ we have: $\pi(i) \cdot \lambda = \pi(i+1) \cdot \lambda \Rightarrow \pi(i) = \pi(j) \ \forall i, j \in \mathcal{A}$

Hence $\pi(i) = \dfrac{1}{N+1} \ \forall i \in \mathcal{A}$.

BACK TO THE BARBER



waiting room: 3 chairs $(N=3)$
service: EXP(3) $(\mu=3)$
arrival: Poisson(2) $(\lambda=2)$

Little $\lambda_a W$:

$$L = \lambda_a \cdot W$$

$W = \frac{66}{114}$ (I would say its about half an hour)

Stat. distr.: $\pi(i) = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \left(\frac{\lambda}{\mu}\right)^i = \frac{1-2/3}{1-(2/3)^4} \cdot \left(\frac{2}{3}\right)^i \Rightarrow \left(\frac{27}{65}, \frac{18}{65}, \frac{12}{65}, \frac{8}{65}\right) = \underline{\pi}$

$$L = \mathbb{E}_\pi(X_t) = \frac{18}{65} + 2 \cdot \frac{12}{65} + 3 \cdot \frac{8}{65} = \frac{66}{65}$$

λ_a = rate of arrival $= \lim_{t\to\infty} \frac{N_a(t)}{t}$

ratio of time when the waiting room is full

$\Rightarrow$ heuristics for finding λ_a: $\lambda_a = \pi(3) \cdot 0 + (1-\pi(3)) \cdot \lambda = \frac{114}{65}$

ratio of time, when its not!

BACK TO THE BARBER

average waiting time again:

wait for 2 harcut to finish $\frac{1}{3}=\frac{1}{\mu}$ amount of time expectadly

no waiting

$$W=\frac{1}{1-\pi(3)}\left[\pi(0)\cdot\frac{1}{3}+\pi(1)\cdot\frac{2}{3}+\pi(2)\cdot\frac{3}{3}\right]=\frac{66}{114}$$

one can step inside the line

wait for 1 harcut to finish $\frac{1}{3}=\frac{1}{\mu}$ amount of time expectadly

M/M/S Q-S, STATIONARY DISTR.

For $\lambda < s\cdot\mu$ $\exists \pi$ stationary distribution, satisfying the detailed balance condition.

Keep in mind

$$q(n,n+1) = \lambda$$

$$q(n,n-1) = \begin{cases} n\cdot\mu, & 1 \le n \le s \\ s\cdot\mu & n \ge s \end{cases}$$

Proof.

d.b.c.

$$\pi(n)\, q(n,n+1) = \pi(n+1)\, q(n+1,n)$$

A/ $n+1 \le s$

$$\pi(n)\cdot\lambda = \pi(n+1)\cdot(n+1)\cdot\mu \Longrightarrow \pi(n+1) = \frac{\pi(n)}{n+1}\cdot\frac{\lambda}{\mu}$$

B/ $n+1 > s$

$$\pi(n)\cdot\lambda = \pi(n+1)\cdot s\cdot\mu \Longrightarrow \pi(n+1) = \frac{\pi(n)}{s}\cdot\frac{\lambda}{\mu}$$

$$\sum_{k=0}^{\infty}\pi(i) = \pi(0)\sum_{k=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^k\frac{1}{k!} + \pi(0)\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{s+k}\frac{1}{s!}\,\frac{1}{s^k} = \pi(0)\cdot\tilde{C}$$

$\tilde{C} < \infty \iff \lambda < \mu\cdot s \iff$ we have a stationary distr.

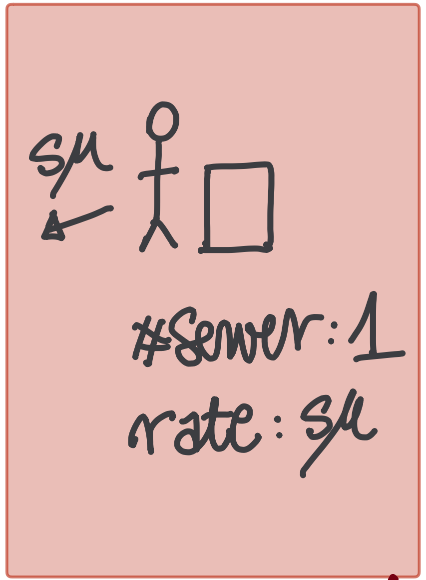
$$\pi(i) = \begin{cases} \left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{k!}\,\frac{1}{\tilde{C}}, & k \le s \\ \left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{s!\,s^{k-s}}\cdot\frac{1}{\tilde{C}}, & k > s \end{cases}$$

M/M/S Q-S, TRANSIENCE & REC.

if $\lambda < s\cdot\mu \Rightarrow$ the chain is recurrent

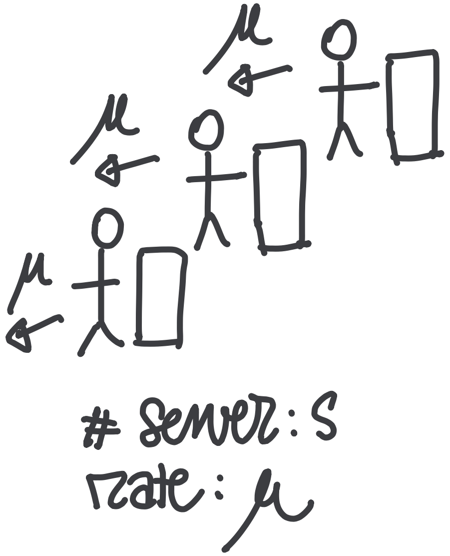
if $\lambda > s\cdot\mu \Rightarrow$ the chain is transient

Why? Compare with the M/M/1 queue with 1 server with serving rate $s\cdot\mu$. This system is more efficient whenever not all servers are serving.



more efficient

vs.



(e.g. 1 person is served at $\frac{1}{s\cdot\mu}$ or $\frac{1}{\mu}$ time)

If $\lambda > s\cdot\mu \Rightarrow$ the M/M/1 queue is already <u>TRANSIENT</u>, and it is more efficient than the M/M/S one.

EXERCISES

1. Find the stationary distr. for the M/M/S queue for $\mu=1, \lambda=2, S=3$
2. — " — for the M/M/1 queue for $\mu=3, \lambda=2$.
3. Find the stationary distr. for the M/M/1 queue for $\mu=3, \lambda=2$, when only 5 people can be in the system.

Summary of stat. distr.

M/M/1 QUEUE (∞-waiting room)

$\lambda<\mu$: $\pi(i)=\left(1-\lambda/\mu\right)\left(\lambda/\mu\right)^i$

$\lambda\geq\mu$: $\nexists$

M/M/1 QUEUE, at most N ppl. in the system:

$\lambda\neq\mu$: $\pi(i)=\dfrac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}\left(\dfrac{\lambda}{\mu}\right)^i$

$i\in\{0,\dots,N\}$

$\lambda=\mu$: $\pi(i)=\dfrac{1}{N+1}$

$i\in\{0,\dots,N\}$

M/M/S Queue (∞ waiting room)

$\lambda<s\cdot\mu$:

$$\pi(i)=\begin{cases}\left(\lambda/\mu\right)^k\cdot\dfrac{1}{k!}\cdot\dfrac{1}{\tilde{c}}, & k\leq s\\[2ex] \left(\dfrac{\lambda}{\mu}\right)^k\cdot\dfrac{1}{s!\,s^{k-s}}\cdot\dfrac{1}{\tilde{c}}, & k>s\end{cases}$$

$$\tilde{c}=\sum_{k=0}^{s-1}\left(\lambda/\mu\right)^k\frac{1}{k!}+\left(\frac{\lambda}{\mu}\right)^s\frac{1}{s!}\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu\cdot s}\right)^k$$

1. Find the stationary distr. for the M/M/S queue for $\mu=1, \lambda=2, S=3$

Recall.

$$\pi(k)=\begin{cases}\left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{k!}\frac{1}{\tilde{c}}, & k\le s\\ \left(\frac{\lambda}{\mu}\right)^k\cdot\frac{1}{s!\,s^{k-s}}\cdot\frac{1}{\tilde{c}}, & k>s\end{cases}$$

where $\tilde{c}=\sum_{k=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^k\frac{1}{k!}+\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{s+k}\frac{1}{s!}\frac{1}{s^k}$

why? $\pi(0)\cdot\tilde{c}=\sum_{k=0}^{\infty}\pi(k)=1$,

$\frac{1}{\tilde{c}}=\pi(0)$

Find $\tilde{c}$: $\boxed{1+\frac{2}{1}+\frac{2^2}{2!}}+\frac{2^3}{3!}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k=5+\frac{2^3}{3!}\cdot 3=9$

$\pi(0)=\frac{1}{\tilde{c}}=\frac{1}{9}$, $\pi(1)=\frac{2}{9}$, $\pi(2)=\left(\frac{2}{3}\right)^2\cdot\frac{1}{2!}$, for $k>2$: $\pi(k)=\left(\frac{2}{3}\right)^k\cdot\frac{1}{3!}$

2. Find the stationary distr. for the M/M/1 queue, for $\mu=3, \lambda=2$.

$$\pi(k) = \left(1-\frac{2}{3}\right)\left(\frac{2}{3}\right)^k = \frac{1}{3}\left(\frac{2}{3}\right)^k$$

$$\pi(0) = \frac{1}{3},\ \pi(1) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9},\ \pi(2) = \frac{4}{27},\ \ldots$$

3. Find the stationary distr. for the M/M/1 queue for $\mu=3, \lambda=2$, when only 5 people can be in the system.

$$\pi(i) = \frac{1-\frac{2}{3}}{1-\left(\frac{2}{3}\right)^6}\left(\frac{2}{3}\right)^i = \frac{\frac{1}{3}}{1-\frac{64}{729}}\left(\frac{2}{3}\right)^i = \frac{3^5}{665}\left(\frac{2}{3}\right)^i \qquad i \in \{0, \ldots, 5\}$$