17. **Homework 3.A.** ([redacted]) Let X, Y be two independent $\mathrm{Exp}(\lambda)$ random variables and $Z := X + Y$. Show that for any non-negative measurable h we have $\mathbb{E}[h(X)|Z] = \frac{1}{Z}\int_0^Z h(t)\,\mathrm{d}t$.

<u>Solutions</u> The density function of $\Gamma(\alpha,\lambda)$ is $\boxed{f(x)=\lambda e^{-\lambda x}\frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)};\ \text{if } x\ge 0 \text{ and } f(x)=0 \text{ if } x<0}$ where $\Gamma(\alpha)=\int_0^\infty e^{-y}y^{\alpha-1}\,dy$

So the density function of $\Gamma(2,\lambda)$ is

$X, Y \sim Exp(\lambda)$, indep.

$\boxed{f(x)=\lambda^2 x e^{-\lambda x} \text{ if } x\ge 0 \ \&\ f(x)=0 \text{ if } x<0}$ ← This is the density function of $Z=X+Y$.

<u>General theory</u>: $Y=g(X)$ & $g'(\underline{x})=\begin{bmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}\end{bmatrix}$. We assume that $\det(g'(x))\neq 0$. Then Y is absolute continuous with density $\boxed{f_Y(y)=f_X(g^{-1}(y))\cdot|\det(g'(g^{-1}(y)))|^{-1}.}$ (*) Here $g(x,y)=(x,x+y)$ $\forall (x,y)$

$g^{-1}(x,z)=(x,z-x)$; $g'(x,y)=\begin{bmatrix}1&0\\1&1\end{bmatrix}$

$(X,Z)=g(X,Y)$, $f_{X,Z}=f_{X,Y}(x,z-x)\cdot 1=f_X(x)\cdot f_Y(z-x)=\lambda e^{-\lambda x}\cdot\mathbb{1}_{\{x\ge 0\}}\cdot\lambda e^{-\lambda z}\cdot e^{\lambda x}\mathbb{1}_{\{z\ge x\ge x\}}=\lambda^2 e^{-\lambda z}\mathbb{1}_{\{z\ge x\ge 0\}}$

$f_{X|Z}(x|z)=\frac{f_{X,Z}(x,z)}{f_Z(z)}\cdot\mathbb{1}_{\{z:f_Z(z)>0\}}$,

the density of $\Gamma(2,\lambda)$ $=\lambda^2 z e^{-\lambda z}$ for $z\ge 0$

Hence, $\mathbb{E}[h(X)|Z]=\int_{\mathbb{R}} h(x)\cdot\frac{f_{X,Z}(x,z)}{f_Z(z)}\,dx\cdot\mathbb{1}_{\{f_Z(z)>0\}}=\int h(x)\frac{\lambda^2 e^{-\lambda z}}{\lambda^2 e^{-\lambda z}\cdot z}\cdot\mathbb{1}_{\{z\ge x\ge 0\}}\,dx$

$f_{X|Z}(x|z)$

$=\frac{1}{z}\int_0^z h(x)\,dx$.

18. **Homework 3.B.** [redacted] Let $\xi_1, \xi_2, \ldots$ be a sequence of iid. random variables with $\mathbf{P}(\xi_1 = 1) = \mathbf{P}(\xi_1 = -1) = 1/2$ and define the simple symmetric random walk $S_n = \xi_1 + \cdots + \xi_n$. For the integers k and l, define the hitting times $T_{-k} = \min\{n : S_n = -k\}$ and $T_l = \min\{n : S_n = l\}$ and the stopping time given by their minimum $T = \min(T_{-k}, T_l)$.

(a) Find $\mathbf{E}(S_T)$ by using the optional stopping theorem for the stopped martingale $S_{n\wedge T}$.

(b) What is $\mathbf{P}(T_{-k} < T_l)$? *Hint:* Note that the random variable S_T can only take two values.

(c) We have shown previously that $M_n = S_n^2 - n$ is a martingale. Apply the optional stopping theorem for M_n and T and compute $\mathbf{E}(T)$.

Solution (a+b) We can apply Doob's optional stopping time theorem since its condition (ii) holds. Namely, with probability 1 in finite time the s.s.RW leaves $[-k, l]$ so $T < \infty$ a.s. & $|S_{n\wedge T}| \leq n$. So (ii) holds. So, $\mathbb{E}[S_{n\wedge T}] = \mathbb{E}[S_1] = 0$.

$\lim_{n\to\infty} \mathbb{E}[S_{n\wedge T}] \underset{T<\infty \text{ a.s.}}{=} \mathbb{E}[T]$. Hence, $\mathbb{E}[T] = 0$.

$$S_T = \begin{cases} -k & \text{if } T_{-k} < T_l \\ l & \text{if } T_l < T_{-k} \end{cases}$$

$$0 = \mathbb{E}[S_T] = -k\,\mathbb{P}(T_{-k} < T_l) + l \cdot \mathbb{P}(T_l < T_{-k}) \Rightarrow \mathbb{P}(T_{-k} < T_l) = \frac{l}{l+k}.$$

(C) $M_n = S_n^2 - n$ is a martingale. We know that $\mathbb{P}(T<\infty)=1$.

$M_{n\wedge T} = S_{n\wedge T}^2 - n\wedge T$. $\quad S^2_{n\wedge T} \le \max\{k^2, l^2\}$ $\quad n\wedge T \le T$ which is finite a.s.

So, $M_{n\wedge T}$ is finite a.s. $\quad \mathbb{E}[M_{n\wedge T}]=0 \ \forall n$, $\quad \mathbb{E}[M_{n\wedge T}]=\mathbb{E}[S^2_{n\wedge T}]-\mathbb{E}[n\wedge T]$

dom. conv. thm. $\rightarrow$

$$\mathbb{E}[S_T^2]-\mathbb{E}[T]=\overset{0}{\overset{\parallel}{\mathbb{E}[M_T]}}.$$

$$0=\mathbb{E}[M_T]=\mathbb{E}[S_T^2]-\mathbb{E}[T]=k^2\,\mathbb{P}(T_{-k}<T_l)+l^2\,\mathbb{P}(T_l<T_{-k})-\mathbb{E}[T] = \underbrace{k^2\cdot\frac{l}{l+k}+\frac{l^2k}{l+k}}_{\frac{(k+l)\,lk}{l+k}=l\cdot k}-\mathbb{E}[T].$$

Hence,

$$\boxed{\mathbb{E}[T]=kl}.$$

$\blacksquare$

Let $X \sim \mathcal{N}(\mu, \Sigma)$, let B be an $\ell \times k$ full-rank matrix and $A \in \mathbb{R}^{\ell}$. We define the vector r. v. $\boxed{Y := A + BX}$. What kind of vector random variable is Y?

We know that the moment generating function of X is

$M_X(t) := \mathbb{E}[e^{t^T \cdot X}] = \exp(t^T \mu + \frac{1}{2} t^T \cdot \Sigma \cdot t)$. Then the moment generating function of Y is:

$$M_Y(t) = \mathbb{E}[e^{t^T \cdot A} \cdot e^{t^T \cdot BX}] = e^{t^T \cdot A} \cdot \mathbb{E}[e^{\underbrace{t^T \cdot B}_{(B^T \cdot t)^T} X}] = e^{t^T \cdot A} \cdot M_X(B^T t) = e^{t^T \cdot A} \cdot \exp\left((B^T t)^T \cdot \mu + \frac{1}{2} (B^T t)^T \Sigma B^T t\right)$$

$$= \exp(t^T A) \cdot \exp(t^T B \mu + \frac{1}{2} t^T B \Sigma B^T \cdot t) = \exp\left(t^T \cdot (A + B\mu) + \frac{1}{2} t^T B \Sigma B^T \cdot t\right)$$

Using that B is a full-rank matrix, we get that $B \Sigma B^T$ is positive definite. $(B \Sigma B^T)^T = B \Sigma^T B^T = B \Sigma B^T$ since $\Sigma = \Sigma^T$. Hence $B \Sigma B^T$ is a positive definite symmetric matrix. This and $M_Y(t) = \mathbb{E}[e^{t^T \cdot Y}] = \exp\left(t^T (A + B\mu) + \frac{1}{2} t^T B \Sigma B^T t\right)$ imply that $B = A + BX$ is a multivariate (ℓ variate, which can be univariate if $\ell = 1$) normal distribution with expectation $A + B\mu$ and Covariance matrix $B \Sigma B^T$. That is

$$\boxed{Y \sim \mathcal{N}(A + B\mu, B \Sigma B^T).} \quad *$$

Next we apply this in the special case: $k = 2$, $\ell = 1$.

Example Let $\mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$; $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma)$.

Find the distribution of $Y = X_1 + 2X_2$.

Solution Observe that $Y = B \cdot X$, where B is the 1×2 matrix: $B = [1 \ \ 2]$, $A = 0$. Then Y is a normal distribution with mean:

$$A + B\mu = 0 + [1 \ \ 2] \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 7,$$

and covariance matrix (which is a 1×1 matrix so it is the variance)

$$\underline{\mathrm{Var}(Y)} = \underbrace{[1 \ \ 2]}_{B} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{B^T} = [1 \ \ 2] \begin{bmatrix} 2 \cdot 1 + 1 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = [1 \ \ 2] \cdot \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 7 = \underline{\underline{18}}.$$

That is

$$\boxed{Y \sim \mathcal{N}(7, 18).}$$

$$[1 \ \ 2] \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 + 2X_2$$