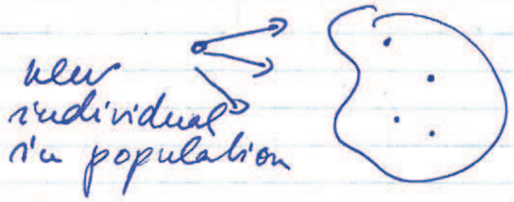
Preferential attachment model

ER Random graph
Generalized random graph
Config. model

} Static models, size of graph fixed

Dynamic models → lead to power law degree sequences (scale-free nature)

- Vertices added sequentially, ~~connect them~~ with a number of edges connected proportionally to the degree of the receiving vertex (favoring v with deg v large)



ER: new edge with equal prob.

bias towards more social individuals (hubs, high degree nodes in network)

Barabási & Albert
(Albert László) Réka

$t_0=0$: m_0 vertices

$\forall t$, 1 new vertex with m stubs linking to m different vertices

$\mathbb{P}(v_{new}$ connected to $v_{old}) \sim$

informal $\sim \deg(v_{old})$

rigorous

Bollobás, Riordan, Spencer, Tusnadi

lack of formal def.

1, how to start ($\deg v = 0$ initially)

2, $\sum_i \mathbb{P}(v \sim v_{old}) = 1$ expected degree = 1 not m.

3, giving marginals distribution you can get anything.

(Yule ~ group model with PA
evolution of species

Simon ~ occurence of words in large text

ⓘ prob. (k+1)th word already appeared i times ~ # words occured i times

ⓘⓘ constant prob. (k+1)th word is new

frequency distr. ~ power law

Model

graph sequence $\{PA_t(m,\delta)\}_{t=1}^{\infty}$ PA_t : graph t vertices mt edges

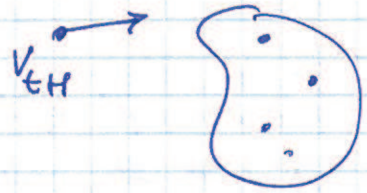
$\{PA_{(m,\delta)}(t)\}$

1.) $m=1$

$t=1$  1 vertex, selfloop

vertices $\{v_1, v_2, \dots v_t\}$ $\deg v_i \to D_i(t)$ self-loop $=2$

$PA_{t+1}(1,\delta) \mid PA_t(1,\delta)$ is given by



$$P(v_{t+1} \Leftrightarrow v_{t+1}) = \frac{1+\delta}{t\cdot(2+\delta)+(1+\delta)}$$

$$P(v_{t+1} \Leftrightarrow v_i) = \frac{D_i(t)+\delta}{t(2+\delta)+(1+\delta)} = p_i$$

$\boxed{\delta > -1 \text{ parameter}}$

Ex. 1. $D_i(t) \geq 1 \quad \forall t \geq i;$

$\Rightarrow D_i(t)+\delta \geq 0 \quad \forall \delta \geq -1$

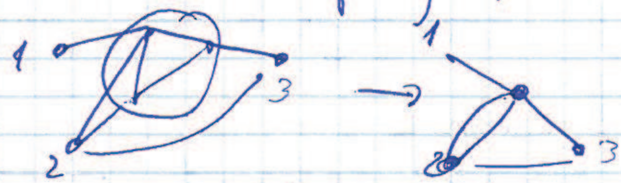
Ex. 2. $\sum_{i=1}^{t+1} p_i = 1$ (prove) $\frac{(1+\delta)+\sum(D_i(t)+\delta)}{t(2+\delta)+(1+\delta)} =$

$$= \frac{(1+\delta)+t\delta+2t}{t(2+\delta)+(1+\delta)} = 1 \checkmark$$

2.) $m>1$ Consider $PA_{mt}(1,\frac{\delta}{m})$

$$\underbrace{v_1, v_2, \dots v_m}_{V_1^{(m)}}, \underbrace{v_{m+1}, \dots v_{2m}}_{V_2^{(m)}}, \dots, \underbrace{v_{m(t-1)+1}, \dots v_{mt}}_{V_t^{(m)}}$$

identify vertices (collapsing of vertices)



multigraph with t vertices, mt edges
$\sum D_i(t) = 2mt$

$PA_{mt}(1,\frac{\delta}{m})$ $P(v \sim v_1) \sim \deg v_1 + \frac{\delta}{m}$

$P(v \sim \underbrace{v_1, v_2, \dots v_m}_{V_1^{(m)}}) \sim \sum_{i=1}^{m} \deg v_i + \delta \checkmark$

$\Rightarrow$ In this model

Degrees updated after each stub is connected

$v_{m(t-1)+1}, v_{m(t-1)+2}, \dots v_{mt}$ $\boxed{\text{Intermediate updating}}$

② Barabási Albert model $\delta = 0$

$PA_t(m, \delta)$ ~ constructed via $PA_{mt}\left(1, \frac{\delta}{m}\right)$

immediate construction.

$PA_1(m, \delta)$ $t=1$  m loops

$PA_{t+1}(m, \delta)$ add  m stubs } added sequentially, intermediate update of degrees

j: th stub:

$\mathbb{P}(v_{t+1}, j\text{'th stub} \sim v_i) \sim D_i(j-1, t) + \delta$

$D_i(j, t)$ = degree of v_i after j th stub added to the graph.

Ex. 3. Prove, $\boxed{\delta = -1}$ $PA_t(1, \delta) =$ 

$\delta = -1$, $m > 1$?

③ Variation NO SELF LOOP = MODEL Ⓑ

$PA_t^{(B)}(m, \delta)$

$t = 2$ 

growth rule $\mathbb{P}(v_{t+1} \rightarrow v_i \mid PA_t^{(B)}(1, \delta)) = \frac{D_i(t) + \delta}{t(2+\delta)}$

Advantage: · no self loop
· connected graph

$m \geq 2$, $\delta > -m$ $PA_{mt}^{(b)}\left(1, \frac{\delta}{m}\right)$

$\boxed{\mathbb{P}_t(\text{loop}) \text{ very small if } t \text{ large}}$

④ Remark $\delta = 0 \longrightarrow \delta = \infty$
interpolation between BA & ER

$\boxed{m=1}$

$\alpha \in [0, 1]$ draw r.v. i.i.d. $X_{t+1} = \begin{cases} 0 & \alpha \\ 1 & 1-\alpha \end{cases}$

$X_{t+1} = 0 \rightarrow$ apply uniform attachment
$\mathbb{P}(v_{t+1} \rightarrow v_i) = \frac{1}{t}$

$= 1 \rightarrow$ pref. attachment $\frac{D_i(t)}{2t}$

$PA_t^{B'}(1, \alpha)$ $\alpha > 0$

Prove that $\alpha = \frac{\delta}{2+\delta}$; $\left\{PA_t^{b'}(1, \alpha)\right\}_{t=2}^{\infty} \sim \left\{PA_t^{B}(1, \delta)\right\}_{t=1}^{\infty}$

⑤ $\boxed{m=1}$ Prove $D_i(t) \rightarrow \infty$ a.s.

$I_t := BER\left(\frac{1+\delta}{t(2+\delta)+1+\delta}\right)$ $\underbrace{\sum_{s=i}^{t} I_s}_{\rightarrow \infty \text{ a.s.}} \leq D_i(t)$

8.2. Degrees of fixed vertices

$$\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx; \quad \Gamma(t+1)=t\Gamma(t) \qquad \frac{\Gamma(t+a)}{\Gamma(t)}=t^a\left(1+O\left(\frac{1}{t}\right)\right)$$

(Stirling formula)

$$e^{-t}\frac{t^{t+\frac{1}{2}}}{\sqrt{2\pi}} \leq \Gamma(t+1) \leq \dots e^{\frac{1}{12t}}$$

Thm. 8.1. $m=1$; $\delta>-1$.

$$\frac{D_i(t)}{t^{\frac{1}{2+\delta}}} \longrightarrow \xi_i \quad \text{a.s. as } t\to\infty$$

$$\mathbb{E}(D_i(t)+\delta) = (1+\delta)\cdot\frac{\Gamma(t+1)\,\Gamma\left(i-\frac{1}{2+\delta}\right)}{\Gamma\left(t+\frac{1+\delta}{2+\delta}\right)\cdot\Gamma(i)}$$

Proof. Via martingales.

$$\mathbb{E}(D_i(t+1)+\delta\,|\,D_i(t)) = D_i(t)+\delta+\mathbb{E}(\underbrace{D_i(t+1)-D_i(t)}_{=0\text{ v. }1}\,|\,D_i(t)) =$$

$$= D_i(t)+\delta+\frac{D_i(t)+\delta}{(2+\delta)t+(1+\delta)} =$$

$$=(D_i(t)+\delta)\cdot\left[\underbrace{\frac{\overset{(t+1)(2+\delta)}{(2+\delta)t+2+\delta}}{(2+\delta)t+(1+\delta)}}_{c_t}\right]$$

initializing

$$\mathbb{E}(D_i(i)+\delta) = (1+\delta)+\frac{1+\delta}{(2+\delta)(i-1)+1+\delta} = \frac{(1+\delta)(2+\delta)\,i}{(2+\delta)(i-1)+(1+\delta)} =$$

$$=(1+\delta)c_{i-1}$$

c_i

$\Rightarrow$ we can get a martingale in $\textcircled{t}$

$$M_i(t)=\frac{D_i(t)+\delta}{1+\delta}\cdot\prod_{s=i-1}^{t-1}\frac{1}{c_s} \qquad \mathbb{E}M_i(t)=1$$

$$\mathbb{E}(M_i(t)\,|\,M_i(t)) = \mathbb{E}(D_i(t+1)+\delta)\cdot\frac{1}{1+\delta}\cdot\prod_{s=i-1}^{t}\frac{1}{c_s} =$$

M_i

$$= c_t\cdot(D_i(t)+\delta)\cdot\frac{1}{1+\delta}\prod_{s=i-1}^{t}\frac{1}{c_s} = (D_i(t)+\delta)\cdot\prod_{s=i-1}^{t-1}\frac{1}{c_s}\cdot\frac{1}{1+\delta} = M_i(t) \checkmark$$

$$\mathbb{E}M_i(i)=\frac{\mathbb{E}(D_i(i)+\delta)}{1+\delta}\cdot\frac{1}{c_{i-1}} = 1 \checkmark$$

martingale convergence theorem

$\boxed{2.21.}$ M_n be a martingale wrt X_n

$$\mathbb{E}|M_n| \leq B \quad \forall n\geq 0 \Rightarrow M_n \xrightarrow{a.s.} M_\infty$$

for some limiting random variable M_∞ which

$\mathbb{P}(M_\infty<\infty)=1$ (If $\mathbb{E}M_n^2 < M \ \forall n \Rightarrow \lim \mathbb{E}M_n^2 = \mathbb{E}M_\infty^2$

§. $$\prod_{s=i-1}^{t-1}\frac{1}{c_s}=\prod_{s=i-1}^{t-1}\frac{\overset{s}{\overbrace{(2+\delta)s+(1+\delta)}}}{(2+\delta)(s+1)}=\prod_{s=i-1}^{t-1}\frac{s+\frac{1+\delta}{2+\delta}}{s+1}=$$

$$=\frac{\Gamma\left(t+\frac{1+\delta}{2+\delta}\right)}{\Gamma\left(i-\frac{1}{2+\delta}\right)}\cdot\frac{1}{\frac{\Gamma(t+1)}{\Gamma(i)}}=\frac{\Gamma(i)}{\Gamma\left(i-\frac{1}{2+\delta}\right)}\cdot\frac{\Gamma\left(t+\frac{1+\delta}{2+\delta}\right)}{\Gamma(t+1)}\approx$$

$$\approx i^{-\frac{1}{2+\delta}}\cdot t^{\left(\frac{1+\delta}{2+\delta}-1\right)}\longrightarrow t^{\cancel{\ \ }-\frac{1}{2+\delta}}$$

$$\frac{D_i(t)+\delta}{1+\delta}\cdot\frac{\Gamma(i)}{\Gamma\left(i-\frac{1}{2+\delta}\right)}\,t^{-\frac{1}{2+\delta}}\left(1+o\left(\frac{1}{t}\right)\right)\longrightarrow M_{i;\infty}$$

$$\mathbb{E}M_{i,\infty}=1$$

$$\Rightarrow\frac{D_i(t)}{t^{\frac{1}{2+\delta}}}\longrightarrow(1+\delta)\frac{\Gamma\left(i-\frac{1}{2+\delta}\right)}{\Gamma(i)}M_{i,\infty}$$

$$\mathbb{E}\left(\frac{D_i(t)+\delta}{t^{\frac{1}{2+\delta}}}\right)^{2}\Rightarrow\ ?\quad\mathbb{P}(\xi_i=0)=0\ ?$$

or not ?

Result for $m>1$ $\quad PA_t(m,\delta)\Longleftrightarrow PA_{mt}\left(1,\frac{\delta}{m}\right)$

$$\mathbb{E}_{m,\delta}\left(D_i(t)\right)=\sum_{s=1}^{m}\mathbb{E}_{1,\frac{\delta}{m}}\left(D_{m(i-1)+s}(mt)\right)$$

$$\frac{D_i(t)}{(mt)^{\frac{-1}{2+\frac{\delta}{m}}}}\xrightarrow{a.s}\xi_i'\qquad \xi_i'=\sum_{j=(i-1)m+1}^{mi}\xi_j$$

↗ ξ_j as above in $PA\left(1,\frac{\delta}{m}\right)$

$\boxed{\text{Power law heuristic}}$

$$\mathbb{E}D_i(t)\sim a_m\cdot\left(\frac{t}{i}\right)^{\frac{1}{2+\frac{\delta}{m}}}$$

$$N_{\geq k}(t)=\#\{i:\ D_i(t)\geq k\}$$

heuristic:

$$N_{\geq k}(t)\approx\#\{i:\ \mathbb{E}D_i(t)\geq k\}\overset{?}{\cancel{\approx}}\#\left\{i:\ a_m\left(\frac{t}{i}\right)^{\frac{1}{2+\frac{\delta}{m}}}\geq k\right\}$$

$$=\sum_{i=1}^{t}\mathbb{1}\left(i<t\,a_m^{2+\frac{\delta}{m}}\cdot k^{-\left(2+\frac{\delta}{m}\right)}\right)=\underline{\underline{t\cdot a_m^{2+\frac{\delta}{m}}\cdot k^{-\left(2+\frac{\delta}{m}\right)}}}$$

$$\frac{N_{\geq k}(t)}{t}\approx k^{-\left(2+\frac{\delta}{m}\right)}\leadsto\tau-1=2+\frac{\delta}{m}\qquad\tau=3+\frac{\delta}{m}$$

Degree sequence?

$$P_k(t) = \frac{1}{t}\sum_{i=1}^{t} \mathbb{1}(D_i(t) = k)$$ proportion of vertices with degree k at time t

$(m \geq 1,\ \delta > -m,\ p_k = 0$ for $k \leq m-1$

Let's define

$$\circledast \quad p_k = \left(2+\frac{\delta}{m}\right)\frac{\Gamma(k+\delta)\,\Gamma\left(m+2+\delta+\frac{\delta}{m}\right)}{\Gamma(m+\delta)\,\Gamma\left(k+3+\delta+\frac{\delta}{m}\right)}$$
$k = m, m+1, \dots$

$m = 1,$ ~~p_k~~ $=$

$\delta = 0;\ k \geq m \qquad p_k = 2\,\frac{\Gamma(k)}{\Gamma(m)}\cdot\frac{\Gamma(m+2)}{\Gamma(k+3)} = \frac{2\cdot m(m+1)}{k(k+1)(k+2)}$

Aim is to prove this is the limiting distribution

$\boxed{1.}$ $\{p_k\}_{k=1}^{\infty}$ prob. distribution.

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{b-a-1}\left(\frac{\Gamma(k+a)}{\Gamma(k-1+b)} - \frac{\Gamma(k+1+a)}{\Gamma(k+b)}\right) \qquad \boxed{\Gamma(t+1) = t\,\Gamma(t)}$$

Proof elem.

$$\frac{1}{b-a-1}\ \frac{\Gamma(k+a)(k+\overset{-1}{b}) - \Gamma(k+a)(k+a)}{\Gamma(k+b)} = \frac{\Gamma(k+a)(b-a-1)}{\Gamma(k+b)(b-a+1)} = ✓$$

$a = \delta \qquad b = 3+\delta+\frac{\delta}{m}$

$2+\frac{\delta}{m}$ missing

$$p_k = \frac{\Gamma\left(m+2+\delta+\frac{\delta}{m}\right)}{\Gamma(m+\delta)}\cdot\left(\frac{\Gamma(k+\delta)}{\Gamma\left(k+2+\delta+\frac{\delta}{m}\right)} - \frac{\Gamma(k+1+\delta)}{\Gamma\left(k+3+\delta+\frac{\delta}{m}\right)}\right)$$

$\Rightarrow$ telescopic sum first term

$$\sum_{k\geq 1} p_k = \sum_{k\geq m} p_k = \frac{\Gamma\left(m+2+\delta+\frac{\delta}{m}\right)}{\Gamma(m+\delta)}\cdot\frac{\Gamma(m+\delta)}{\Gamma\left(m+2+\delta+\frac{\delta}{m}\right)} = 1$$

$+\ \boxed{p_k \geq 0}$

Theorem

Degree sequence $\quad \delta > -m,\ m \geq 1$ fix; $\exists\, C = C(m,\delta) > 0$

$$\mathbb{P}\left(\max_k |P_k(t) - p_k| \geq C\cdot\sqrt{\frac{\log t}{t}}\right) = o(1)$$ as $t \to \infty$

$$p_k = c_{m,\delta}\cdot k^{-\tau}\left(1 + O\left(\frac{1}{k}\right)\right)$$

$\boxed{\tau = 3 + \frac{\delta}{m}} > 2 \qquad c_{m,\delta} = \left(2+\frac{\delta}{m}\right)\frac{\Gamma\left(m+2+\delta+\frac{\delta}{m}\right)}{\Gamma(m+\delta)}$

Proof of Thm 8.2. In $PA_t(m,\delta)$ $\forall C > m\sqrt{8}$, as $t \to \infty$

$$N_k(t) = \sum_{i=1}^{t} \mathbb{1}(D_i(t)=k) = t P_k(t)$$

$$\mathbb{P}\left(\max_k |P_k(t) - p_k| \geq C \cdot \sqrt{\frac{\log t}{t}}\right) = o(1)$$

① $\mathbb{P}\left(\max_k |N_k(t) - \mathbb{E}N_k(t)| \geq C \cdot \sqrt{t \log t}\right) = o(1)$

② $|\mathbb{E}N_k(t) - t p_k| \leq C$

Proof of ①: $N_k(t) = 0$ $k > m(t+1)$

$$\mathbb{P}\left(\max_k |N_k(t) - \mathbb{E}N_k(t)| \geq C\sqrt{t\log t}\right) = \mathbb{P}\left(\max_{k \leq m(t+1)} |N_k(t) - \mathbb{E}(\,)| \geq C\sqrt{t \log t}\right)$$

max ≤ Σ

$$\leq \sum_{k=1}^{m(t+1)} \underbrace{\mathbb{P}\left(|N_k(t) - \mathbb{E}N_k(t)| \geq C\sqrt{t\log t}\right)}_{\text{prove, each term } o(\frac{1}{t})}$$

conditional ~~do~~ up to time n: $n \leq t$

$$M_n = \mathbb{E}(N_k(t) \mid \mathcal{F}_n) \leftarrow PA_n(m,\delta)$$

Doob mart.

1., $N_k(t)$ bounded ⇒ $\mathbb{E}N_k(t) \leq t < \infty$ (total number of vertices)

$$\mathbb{E}(M_{n+1} \mid PA_n) = \mathbb{E}\big(\mathbb{E}(N_k(t) \mid PA(n+1)) \mid PA_n\big) = \mathbb{E}(N_k \mid PA_n) = M_n$$

$M_0 = \mathbb{E}N_k(t)$ $M_t = N_k(t)$

$$\Rightarrow |M_t - M_0| \leq \sum_{i=1}^{t} |M_i - M_{i-1}|$$

Azuma–Hoeffding →

> Martingales with bounded differences $|X_k - X_{k-1}| < c_k$
> $\Rightarrow \forall N, \forall t > 0$
> $$\mathbb{P}(|X_N - X_0| \geq t) \leq e^{-\frac{t^2}{2\sum c_k^2}}$$

$|M_n - M_{n-1}| \leq 2m$ $\forall n \in [t]$ a.s.

$$M_n - M_{n-1} = \sum_{i=1}^{t} \mathbb{P}(D_i(t)=k \mid PA_n) - \mathbb{P}(D_i(t)=k \mid PA_{n-1}) =$$

= extra information gained in step n

PA_s: info to which the first ms edges are attached

extra infó

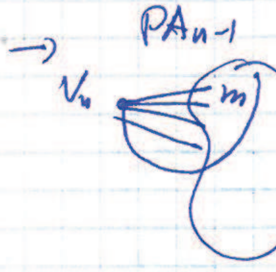
PA_n: $G(v_1 \ldots v_{n-1})$ ismert + v_n hová kapcsolódik

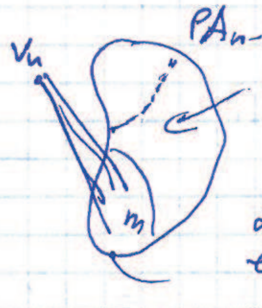
ez ≤ m db élek fokát változtatja

$\mathbb{P}(\ldots \mid PA_n)$ ← expectation w.r.t. all possible ways of attaching the remaining vertices $v_{n+1}, \ldots v_t$

← ez 1 db megvalósulás

$= (\mathbb{1}(D_i(t) = k \mid \ldots)$ egy menetben melyik ≤ m csúcshoz kötődbe V_n-t. ⇝ legfeljebb 2m csúcs foka különbözik





ezeknek ugyanannyi a foka

2m csúcsnak eltér az eloszlása]

fokuk ugyanaz

<u>A többiek nem változnak</u>
(a bekötési valószínűségek
IGEN
de csak
ebben a 2m
csúcs közt
van máshogy
szétbontva)
a többi változatlan

Hoeffding:

$$\mathbb{P}\Big(\underbrace{\sum_{n=1}^{t}|M_n - M_{n-1}|}_{M_T - M_0} \geq a\Big) \leq 2\cdot e^{-\frac{a^2}{2\cdot(2m)^2 t}} = 2e^{-\frac{a^2}{8m^2 t}} =$$

$$a := C\sqrt{t\log t}$$

$$= 2e^{-\frac{C^2 t\log t}{8m^2\cdot t}} = 2\left(\frac{1}{t}\right)^{\frac{C^2}{8m^2}} \qquad C^2 > 8m^2 \text{ ez } o\left(\frac{1}{t}\right)$$

Remark 1. $N_{\geq k}(t) = \sum_{\ell=k}^{\infty} N_\ell(t)$ concentrates as well

Try to prove it by using $M_n' = \mathbb{E}(N_{\geq k}(t) \,|\, PA_n)$

$$\mathbb{P}\left(|N_{\geq k}(t) - \mathbb{E}N_{\geq k}(t)| \geq C\sqrt{t\log t}\right) = o\left(\frac{1}{t}\right)$$

<u>Second part</u> $\quad \overline{N_k}(t) = \mathbb{E}(N_k(t)) = t P_k(t)$

Claim: $\delta > -n$, $m \geq 1$; $\exists C(m,\delta)$; $\forall t \geq 1$, $\forall k \in \mathbb{N}$

$$|\overline{N_k}(t) - p_k\cdot t| \leq C$$

Exercise: Total degree of High Degree vertices

$\ell = \ell(t) \to \infty$; $t \to \infty$, $t\cdot \ell^{2-\tau} \geq K\sqrt{t\log t}$ $\quad$ K large enough

$\exists B > 0$ s.t. $\quad \mathbb{P}\Big(\forall$ ilyen ℓ-re $\sum_{i:D_i(t)\geq \ell} D_i(t) \geq Bt\ell^{2-\tau}$

és $\quad N_{\geq \ell}(t) \gg \sqrt{t}\Big) = 1 - o\left(\frac{1}{t}\right)$

<u>Expected degree sequence</u> $\quad m=1$

$$\mathbb{E}(N_k(t+1)\,|\,PA_t) = N_k(t) + \mathbb{E}(\underbrace{N_k(t+1) - N_k(t)}_{\neq 0}\,|\,PA_t)$$

4 ~~[illegible]~~ ways $\neq 0$

$v_{t+1} \bullet \longrightarrow \bullet\, k-1 \nearrow k \qquad \frac{k-1+\delta}{t(2+\delta)+(1+\delta)} \qquad ⊕1$

$v_{t+1} \longrightarrow \bullet\, k \nearrow k+1 \qquad \frac{k+\delta}{t(2+\delta)+(1+\delta)} \qquad ⊖1$

$v_{t+1} \curvearrowright N_1(t) \nearrow ⊕1 \qquad \frac{}{t(2+\delta)+(1+\delta)} \longrightarrow 1 - \frac{(1+\delta)}{t(2+\delta)+(1+\delta)}$

v_{t+1} ↺ $N_2(t) \nearrow +1 \qquad \frac{1+\delta}{t(2+\delta)+1+\delta}$

$$\mathbb{E}\left(N_k(t+1)-N_k(t)\,/\,PA_t\right) = \frac{k-1+\delta}{t(2+\delta)+(1+\delta)}\cdot N_{k-1}(t) - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\cdot N_k(t)$$

$k \geq 1,\ N_0(t) := 0$

$$+\,\mathbb{I}(k=1)\left(1-\frac{1+\delta}{t(2+\delta)+(1+\delta)}\right) + \mathbb{I}(k=2)\,\frac{1+\delta}{t(2+\delta)+1+\delta}$$

$\Longrightarrow$ taking expectation we get

I)

$$\boxed{\begin{aligned}\overline{N_k}(t+1) = \overline{N_k}(t) + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)}\,\overline{N_{k-1}}(t) - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\,\overline{N_k}(t)\\ +\,\mathbb{I}(k=1)\left(1-\frac{1+\delta}{t(2+\delta)+1+\delta}\right) + \underbrace{\mathbb{I}(k=2)\,\frac{1+\delta}{t(2+\delta)+1+\delta}}_{\approx 0}\end{aligned}}$$

let $t \to \infty$ $\left.\begin{aligned}\overline{N_k}(t+1) &\sim (t+1)\,p_k\\ \overline{N_k}(t) &\sim t\cdot p_k\end{aligned}\right\} \Longrightarrow$ in the limit we get

II.

$$\boxed{p_k = \frac{k-1+\delta}{2+\delta}\cdot p_{k-1} - \frac{k+\delta}{(2+\delta)}\,p_k + \mathbb{I}(k=1)} \quad (p_0=0)$$

megoldjuk:

$$p_k\left(\frac{2+\delta+k+\delta}{2+\delta}\right) = \frac{k-1+\delta}{2+\delta}\,p_{k-1} + \mathbb{I}(k=1)$$

$$p_k = \frac{k-1+\delta}{2+k+2\delta}\,p_{k-1} + \mathbb{I}(k=1)\,\frac{2+\delta}{k+2+2\delta} \qquad (p_0=0)$$

$$p_1 = \frac{2+\delta}{1+2+2\delta} = \frac{2+\delta}{3+2\delta}\ ; \quad k>1 \quad p_k = \frac{k-1+\delta}{2+k+2\delta}\,p_{k-1}$$

$$\Longrightarrow p_k = \prod_{j=2}^{k}\frac{j-1+\delta}{2+j+2\delta}\cdot\frac{2+\delta}{3+2\delta} = \frac{\Gamma(k+\delta)}{\Gamma(1+\delta)}\cdot\frac{\Gamma(4+2\delta)}{\Gamma(k+3+2\delta)}\cdot\frac{2+\delta}{3+2\delta} =$$

$p_2 = \frac{\delta}{4+2\delta}\,p_1$

$$= \frac{\Gamma(k+\delta)}{\Gamma(k+3+2\delta)}\cdot\frac{\Gamma(3+2\delta)}{\Gamma(1+\delta)}\cdot(2+\delta) \quad ✓$$

Most össze akarjuk vetni a kettőt. $\varepsilon_k(t) := \overline{N_k}(t) - t\,p_k.$

kell $\max_k |\varepsilon_k(t)| \leq C$

$$(t+1)\,p_k = t\,p_k + \frac{k-1+\delta}{(2+\delta)}\,p_{k-1} - \frac{k+\delta}{2+\delta}\,p_k + \mathbb{I}(k=1)$$

[addig gyötörjük míg olyan nem lesz, mint I. és a „különbség" lesz.]

$$= t\,p_k + \frac{k-1+\delta}{t(2+\delta)+1+\delta}\,p_{k-1}\cdot t - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\cdot t\,p_k + \mathbb{I}(k=1)$$

$$\left.\begin{aligned}&+\left(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+1+\delta}\right)(k-1+\delta)\cdot p_{k-1}\\ &-\left(\frac{1}{2+\delta} - \frac{t}{t(2+\delta)+1+\delta}\right)(k+\delta)\,p_k\end{aligned}\right\} =: K_k(t)$$

$$\gamma_k(t) = \frac{1+\delta}{t(2+\delta)+1+\delta} \cdot \left(\mathbb{I}(k=2) - \mathbb{I}(k=1) \right)$$

$\Rightarrow$ $\circledast$ − I yields

$$\varepsilon_k(t+1) = \left(1 - \frac{k+\delta}{t(2+\delta)+(1+\delta)}\right) \varepsilon_k(t) + \frac{k-1+\delta}{t(2+\delta)+(1+\delta)} \varepsilon_{k-1}(t) + \kappa_k(t) + \gamma_k(t).$$

Indukció t-re. $t=1$: $\bar{N}_k(1) = \mathbb{I}(k=2)$ $p_k \le 1$,

$$|\varepsilon_k(1)| = |\bar{N}_k(1) - p_k| \le \max\{\bar{N}_k(1), p_k\} \le 1$$

$k=1$

$|\kappa_k(t)| \le C_\kappa \frac{1}{t+1}$ $\gamma_k(t) \le C_\gamma \cdot \frac{1}{t+1}$ $C_\gamma = 1$, $C_\kappa = \frac{(1+\delta)(2+\delta)}{3+2\delta}$ jó len.

$\varepsilon_0(t) = N_0(t) - p_0 = 0$

$$|\varepsilon_1(t+1)| = \underset{\ge 0}{\left(1 - \frac{1+\delta}{t(2+\delta)+1+\delta}\right)} \underset{\substack{\le C \\ \text{by ind.}}}{|\varepsilon_1(t)|} + \underset{\le C_\kappa \cdot \frac{1}{t+1}}{|\kappa_1(t)|} + \underset{\le C_\gamma \frac{1}{t+1}}{|\gamma_1(t)|}$$

$$\frac{1}{t(2+\delta)+1+\delta} \le \frac{1}{(t+1)(2+}$$

$$\varepsilon_1(t+1) \le C - \frac{1}{t+1}\left(C \cdot \frac{1+\delta}{2+\delta} - (C_\kappa + C_\gamma)\right) \le C \quad \text{ha } C > \frac{2+\delta}{1+\delta}(C_\kappa + C_\gamma$$

$k=1$ ✓ kész.

$k \ge 2$-re

$$|\varepsilon_k(t+1)| \le \underbrace{\left(1 - \frac{k+\delta}{t(2+\delta)+1+\delta}\right)}_{0 \le \text{ ha } k \le t(2+\delta)+1} \underset{\le C}{|\varepsilon_k(t)|} + \frac{k-1+\delta}{t(2+\delta)+1+\delta} \underset{\le C}{|\varepsilon_{k-1}(t)|} + \underset{\le (C_\gamma + C_\kappa)\frac{1}{t+1}}{|\kappa_k(t)| + |\gamma_k(t)|}$$

$$\le C\left(1 - \frac{1}{t(2+\delta)+1+\delta}\right) + (C_\kappa + C_\gamma)\frac{1}{t+1} \le$$

$$\le C - \frac{1}{t+1}\left(\frac{C}{2+\delta} - (C_\kappa + C_\gamma)\right) \le C \quad C > (2+\delta)(C_\kappa + C_\gamma)$$

$k > t(2+\delta)+1$

max degree in $PA_t(1,\delta)$ is $(t+2)$ $\Rightarrow \bar{N}_k(t+1) = 0$ $k \ge t(2+\delta)+$

$$\Rightarrow |\varepsilon_k(t+1)| = (t+1)p_k$$

$$p_k \le C_p \cdot \left(\frac{1}{t+1}\right)^{-(3+\delta)} \leftarrow C_p(\delta)$$

$$\Rightarrow |\varepsilon_k(t+1)| = (t+1)p_k \le C_p \cdot \left(\frac{1}{t+1}\right)^{-(2+\delta)} \le C_p$$

$$C := \max\left\{(2+\delta)(C_\kappa + C_\gamma), \frac{(2+\delta)(C_\kappa + C_\gamma)}{1+\delta}, C_p\right\}$$