2011. 05. 07. Configuration model -1-

Aim: investigate properties of graph with a **prescribed** degree sequence.

may NOT exist even if $\sum d_i = 2m$ even

let's consider random MULTIGRAPHS  and condition on being SIMPLE

Model $\underline{d} = \{d_i\}_{i=1}^n$ vertex i has degree d_i

$l_n = \sum_{j=1}^n d_j$ even half-edge = stub

1. Number the stubs arbitrarily $1 \dots l_n$
2. Connect randomly a stub to remaining, repeat (produces a uniform matching on stubs) (Stub = left or right end)
3. then forget the numbers

not all multigraphs have the same prob., ie.

$\boxed{d_1 = 2,\ d_2 = 2}$ 

$CM_n(\underline{d})$ weight $1/j!$ for each j-edge, weight $1/2$ for each loop

Other description

Second graph, vertices $1 \dots l_n$ → uniform matching on it $\mathrm{Conf}_n(d)$ collapse vertices to get $v_1, \dots v_n$.

$\to \underbrace{1 \dots d_1}_{v_1} \underbrace{d_1 \dots d_1+d_2}_{v_2} \dots \underbrace{\dots}_{v_n}$

Uniform → $\mathrm{Conf}_n(d)$ these multigraphs (with numbered stubs) are equally likely.

Ex. All degree $d_i = d$ → random d-regular graph.

Ex2. $D_i \sim$ iid. → Mind $L_n = \sum_{i=1}^n D_i$ $\mathbb{P}(L_n \text{ odd}) \approx 1/2$

D_i iid (not concentrated on even numbers)

$$\mathbb{P}(L_n \text{ odd}) = \frac{1}{2} \cdot \left[1 - \mathbb{E}(-1)^{L_n}\right] = \frac{1}{2}\left(1 - \phi_D(\pi)^n\right)$$

$\phi_D(t) = \mathbb{E}\left(e^{itD_1}\right)$ $|\phi_D(\pi)| < 1$ ⇒ $\mathbb{P}(L_n \text{ odd})$ exp close to $1/2$ in n.

2 main results

1. When we erase loops & combine multiple edges into 1 edge → degree sequence remains asymptotically the same

ERASED CM

2. $\mathbb{P}(CM_n(d) \text{ is SIMPLE})$

② $\mathbb{P}(\ CM_n(\underline{d})$ is simple $)$

(A4) $\nu = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{d_i(d_i-1)}{l_n} =$

$= \frac{1}{\mu} \cdot \sum_{j=1}^{\infty} j(j-1)p_j \quad < \infty$

$p_k^{(n)} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(d_i = k)$

Assume (A1) $\lim_{n\to\infty} p_k^{(n)} = p_k$

(assymptotic prob. of $\mathbb{P}(\deg X = k)$)

(A2) $\mu = \sum_{k=1}^{\infty} k\,p_k$;

(A3) $\lim_{n\to\infty} \frac{l_n}{n} = \mu < \infty$

$\boxed{\text{average degree finite}}$

satisfied ~~for~~ iid degrees if $\mathbb{E}D < \infty$

<u>Theorem</u> 7.3

If (A1) $\lim_{n\to\infty} p_k^{(n)} = p_k$ & (A4) $\nu < \infty$,

then $\mathbb{P}(CM_n(\underline{d})\text{ simple}) \xrightarrow[n\to\infty]{} e^{-\frac{\nu}{2} - \frac{\nu^2}{4}}$.

<u>Lemma</u> $S_n = \#$ loops in $CM_n(d)$

$M_n = \#$ multiple edges in $CM_n(d)$

$S_n = \sum_{i=1}^{n} s_i$; $\quad M_n = \frac{1}{2}\sum_{i=1}^{n} m_i$

If (A1) & (A4) holds, then (S_n, M_n) converges (jointly) in distribution to (S, M), independent $S \sim POI\left(\frac{\nu}{2}\right)$

$M \sim POI\left(\frac{\nu^2}{4}\right)$

(Thm: Consequence : $\mathbb{P}(S_\cdot = M_\cdot = 0) = e^{-\nu/2 - \nu^2/4}$.

Thm. ~~Convergence to indep. POI-s.~~

A vector of integer-valued r.vs. $(X_{1,n}, X_{2,n}, \dots X_{d,n})_{n=1}^{\infty}$ converges in distribution to $(Y_1, Y_2, \dots Y_d)$ $Y_i \sim POI(\lambda_i)$ independent; if for all $r_1, r_2, \dots r_d \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbb{E}\left((X_{1,n})_{r_1} (X_{2,n})_{r_2} \dots (X_{d,n})_{r_d} \right) = \lambda_1^{r_1} \dots \lambda_d^{r_d}$$

all factorial moments converge.

Thm. Factorial moments of sums of indicators.

$X_{(l,n)} = \sum_{i \in I_l} I_i^{(l,n)}$,

$\mathbb{E}\left((X_{1,n})_{r_1} \dots (X_{d,n})_{r_d}\right) =$

$= \sum^{*}_{i_1^{(1)} \dots i_{r_1}^{1} \in I_1} \dots \sum^{*}_{i_1^{d} \dots i_{r_d}^{d} \in I_d}$

$\mathbb{P}\left(I_{i_s}^{(l,n)} = 1 \ \forall l = 1 \dots d \ \ \forall s = 1 \dots r_l \right)$

$\boxed{\text{Proof by induction on } r}$

$\boxed{\text{method of moments}}$

$S = \sum \mathbb{1}_i$

$\mathbb{E}((S)_r) = \sum^{*}_{i_1 \dots i_r \in I} \mathbb{E}\left(\prod_{l=1}^{r} I_{i_l}\right) =$

$* = i_1, \dots i_r$ different indices

$= \sum^{*}_{i_1, \dots i_r \in I} \mathbb{P}(I_{i_1} = \dots = I_{i_r} = 1)$

Ex. Prove it for $r = 2$ $\mathbb{E}[X(X-1)]$

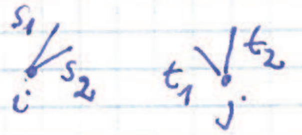
Ex Prove $BIN(n, \frac{\lambda}{n}) \to POI(\lambda)$

We want to use this thm, but M_n NOT a sum of indicators

$$S_n = \sum_{i=1}^{n} \sum_{1 \le s < t < d_i} \mathbb{1}_{st,i} \qquad \mathbb{1}_{st,i}: \text{ stub } s \text{ \& stub } t \text{ are paired in } V_i$$

$$\mathbb{E} S_n = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} \cdot \mathbb{E}\left(\mathbb{1}_{12,i}\right) = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} \cdot \frac{1}{l_n-1} \le \sum_i \frac{d_i^2}{l_n}$$

$$M_n < \tilde{M}_n = \frac{1}{2} \sum_{1 \le i \ne j \le n} \sum_{1 \le s_1 < s_2 < d_i} \sum_{1 \le t_1 \ne t_2 \le d_j} \mathbb{1}_{s_1 t_1, s_2 t_2, ij}$$



$$\mathbb{E} M_n \le \frac{1}{2} \sum_{1 \le i \ne j \le n} \sum_{1 \le s_1 \le s_2 < d_i} \sum_{1 \le t_1 \ne t_2 < d_j} \mathbb{E}\left(\mathbb{1}_{s_1 t_1, s_2 t_2, ij}\right) \doteq$$

$$= \frac{1}{4} \sum_{1 \le i \ne j \le n} d_i(d_i-1) \cdot d_j(d_j-1)\, \mathbb{E}\left(\mathbb{1}_{11, 22, ij}\right) \le$$

$$\frac{1}{(l_n-1)(l_n-3)}$$

$$\le \sum_{i,j} \frac{d_i(d_i-1) d_j(d_j-1)}{4(l_n-1)(l_n-3)} = \left(\sum_i d_i(d_i-1)\right)^2 \Big/ 4(l_n-1)(l_n-3) \le$$

$$\le 2\left(\frac{\sum d_i^2}{l_n}\right)^2$$

$M_n \ne \tilde{M}_n$ if $\exists$ triple edge

$$\mathbb{P}(\exists \text{ triple edge}) \le \sum_{i,j} \frac{d_i(d_i-1)(d_i-2) \cdot d_j(d_j-1)(d_j-2)}{(l_n-1)(l_n-3)(l_n-5)} = o(1)$$

(unió korlát)

$\boxed{d_i = o(\sqrt{n}); \ l_n > n}$

$$(S_n, M_n) \longrightarrow (S, M)$$

$$\Downarrow \ \mathbb{P}(M_n \ne \tilde{M}_n) = o(1) \longrightarrow \quad (S_n, \tilde{M}_n) \longrightarrow (S, M)$$

Kell $\lim_{n \to \infty} \mathbb{E}\left[(S_n)_s (\tilde{M}_n)_r\right] = \left(\frac{\nu}{2}\right)^s \cdot \left(\frac{\nu^2}{4}\right)^r$

$$\mathbb{E}\left((S_n)_s (M_n)_r\right) = \sum^{\infty}_{\substack{m_1^{(1)} \dots m_s^{(1)} \in I_1 \\ m_1^{(2)} \dots m_r^{(2)} \in I_2}} \mathbb{P}\left(I_{m_1^{(1)}} = 1 \cdots = I_{m_r^{(2)}} = 1\right) =$$

$$I_1 = \{(st, i) : i \in [n] \quad 1 \leq s < t \leq d_i\}$$

$$I_2 = \{(s_1 t_1, s_2 t_2, i, j) : 1 \leq i < j \leq n ;\ 1 \leq s_1 < s_2 \leq d_i ,\ 1 \leq t_1 \neq t_2 \leq d_j\}$$

Since all stubs are uniformly paired, this has the probability

$$= \sum^{\infty} \frac{1}{(l_n-1)\cdots(l_n-1-2s-4r)} =$$ if no conflict with indices.

one stub to two

$$\leq \frac{|I_1|\cdot|I_1-1|\cdots|I_1-s+1|\cdot|I_2|\cdots|I_2-r+1|}{(l_n-1)\cdots(l_n-1-2s-4r)} \leq \left(\lim_{n\to\infty}\frac{|I_1|}{l_n}\right)^s \cdot \left(\lim \frac{|I_2|}{l_n}\right)^r$$

$$\lim_{n\to\infty} \frac{|I_1|}{l_n} = \lim_{n\to\infty} \frac{1}{l_n} \cdot \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} = \frac{\nu}{2}$$

$$\lim \frac{|I_2|}{l_n^2} = \lim \frac{1}{l_n^2} \cdot \sum \frac{d_i(d_i-1)}{2} d_j(d_j-1) =$$

$$= \underbrace{\left(\lim_{n\to\infty} \frac{1}{l_n} \sum \frac{d_i(d_i-1)}{2}\right)^2}_{\frac{\nu^2}{4}} - \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \frac{d_i^2(d_i-1)^2}{2 l_n^2}}_{o(1)}$$

$\left(\frac{\sum d_i^2}{l_n} = o(\sqrt{n}) \quad \max d_i = o(\sqrt{n})\right)$

Lower bound: conflicting cases in the indices are:
$\exists\, i\ s$, matched to two different stubs. (rare)

Corollary Number of graphs with a given degree sequence.

$\underline{d} = (d_i)_{i=1}^n$ satisfys ... ; $\lim p_k^{(n)} \to p_k$

l_n even $\quad \nu < \infty$

Then the number of simple graphs with $\underline{d} = \{d_i\}_{i=1}^n$ is equal to

$$e^{-\frac{\nu}{2} - \frac{\nu^2}{4}} \frac{(l_n-1)!!}{\prod_{i=1}^n d_i!} (1 + o(1))$$

(There are $\prod_{i=1}^n d_i!$ ways of permuting the different stubs)