Fractals and geometric measure theory 2013

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Multifractal analysis

We write $\mathcal{M}$ for the set of measures μ satisfying:

- μ is a Radon measure,
- $\mathrm{spt}(\mu)$ is compact,
- $0 < \mu(\mathbb{R}^d) < \infty$.

Let

$$\mathcal{M}_1 := \{\mu \in \mathcal{M} : \mu \text{ is a probability measure } \}. \quad (1)$$

Let $A \subset \mathbb{R}^d$ be a Borel set. Further, we define

$$\mathcal{M}(A) := \{\mu \in \mathcal{M} : \mathrm{spt}(\mu) \subset A\},$$

$$\mathcal{M}_1(A) := \left\{\mu \in \mathcal{M}(A) : \mu(\mathbb{R}^d) = 1.\right\}.$$

Hausdorff dimension of a measure

Let $\mu \in \mathcal{M}$. Recall: we have introduced the definition:

Definition

$$\dim_{\mathrm{H}}(\mu) := \inf \left\{ \dim_{\mathrm{H}}(A) : \mu(\mathbb{R}^d \setminus A) = 0 \right\}.$$

Recall: we have proved the following theorem

Theorem

$$\dim_{\mathrm{H}}(\mu) = \operatorname{ess\,sup}_x \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}. \qquad (2)$$

Roughly speaking, $\dim_{\mathrm{H}}(\mu) = \delta$ if for a μ-typical x we have $\mu(B(x,r)) \approx r^{\delta}$ for small $r > 0$.

Local dimension I

Let $\mu \in \mathcal{M}_1$. From now we denote the local dimension by $d\mu(x)$ instead of $\dim_{\mathrm{loc}}(\mu, x)$. That is

$$\underline{d}_\mu(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\overline{d}_\mu(x) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

The lower local dimension, upper local dimension, local dimension is defined by:

$$\underline{d}_\mu(x) \quad , \overline{d}_\mu(x) \quad , d_\mu(x) \text{ respectively.}$$

What we have just proved it is a theorem due to Lai Sang Young:

Figure : Lai Sang Young

Theorem (L.S. Young)

Let $\Lambda \subset \mathbb{R}^d$ *be measurable and* $\mu(\Lambda) > 0$. *Suppose that for every* $x \in \Lambda$,

$$a \leq \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq b. \tag{3}$$

Then

$$a \leq \dim_{\mathrm{H}}(\Lambda) \leq b. \tag{4}$$

Clearly, all limits remains unchanged if instead of $r \to 0$ we change to a sequence $r_n \downarrow 0$ satisfying $\lim\limits_{n \to \infty} r_n / r_{n+1} = 1$.

self-similar measures I

For a probability vector

$$\mathbf{p} = (p_1, \dots, p_m).$$

we define the infinite product measure:

$$\mathbf{p}^{\mathbb{N}} := (p_1, \dots, p_m)^{\mathbb{N}}.$$

We are also given a self-similar IFS

$$\mathcal{S} = \{S_1, \dots, S_m\}$$

on $\mathbb{R}^d$ with contraction ratios

$$\boxed{0 < r_i < 1}.$$

self-similar measures II

Recall that the similarity dimension s was defined as the solution of the equation

$$\boxed{r_1^s \cdots + r_m^s = 1.} \tag{5}$$

Using the natural projection (coding) Π,

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} S_{i_1\dots i_n}(\mathbf{0}),$$

we consider the push down measure of $\mathbf{p}^{\mathbb{N}}$:

$$\nu := \Pi_* \mathbf{p}^{\mathbb{N}}, \tag{6}$$

self-similar measures III

that is, for Borel $A \subset \mathbb{R}^d$:

$$\nu(A) := \mathbf{p}^{\mathbb{N}}(\Pi^{-1}(A)).$$

Homework Prove that for Borel $A \subset \mathbb{R}^d$

$$\nu(A) = \sum_{i=1}^{m} p_i \nu(S_i^{-1}(A)). \tag{7}$$

Theorem

There is a unique measure $\mu \in \mathcal{M}_1$ satisfying (7).

Idea of the proof

By (7): $\mathrm{spt}(\nu) \subset \Lambda$. We introduce the metric $L(\mu, \eta)$ for $\mu, \eta \in \mathcal{M}_1(\Lambda)$:

$$L(\mu, \eta) := \sup \left\{ \mu(\phi) - \eta(\phi) \middle| \phi : \Lambda \to \mathbb{R},\ \mathrm{Lip}(\phi) \leq 1 \right\}.$$

Further, consider the operator $\mathcal{F} : \mathcal{M}_1(\Lambda) \to \mathcal{M}_1(\Lambda)$:

$$(\mathcal{F}\nu)(\phi) := \sum_{k=1}^{m} p_i \int \phi \circ S_i d\nu.$$

Then

(a) The metric space $(\mathcal{M}_1(\Lambda), L)$ is complete.
(b) $\mathcal{F}$ is a contraction on $(\mathcal{M}_1(\Lambda), L)$.

So, by Banach fixed point theorem we obtain that there is a unique fixed point of $\mathcal{F}$. $\square$

Natural measure (the definition)

Let s be the similarity dimension of the IFS $\mathcal{S}$. The important special case is:

$$\nu := \Pi_*(\mathbf{p}^{\mathbb{N}}) \text{ for } \mathbf{p} = (r_1^s, \dots, r_m^s). \tag{8}$$

The measure ν is called the natural measure on Λ.

The natural measure I

Fact

Assume that the IFS $\mathcal{S} = \{S_i(x) = r_i x + t_i\}_{i=1}^{m}$ *satisfies the OSC. Then*

$$\boxed{d_\nu(x) \equiv s} \text{ holds } \forall x \in \Lambda.$$

We remark that Young's Theorem and this Fact implies that

$$\dim_{\mathrm{H}} \nu = s. \tag{9}$$

Proof We give the proof in the special case when the Strong Separation Property holds that is we assume that the sets $\Lambda_i := S_i(\Lambda)$, $i = 1, \dots, m$ are pairwise disjoint.

The natural measure II

Without loss of generality we may assume that $|\Lambda| = 1$. Let

$$d := \min \operatorname{dist}(\Lambda_i, \Lambda_j), \ \ i \neq j.$$

Set $r_{\max} := \max\{r_1, \dots, r_m\}$. Fix an ℓ such that

$$r_{\max}^{\ell} < d.$$

Fix an arbitrary $x = \Pi(\mathbf{i})$ and $r > 0$. We define n such that

$$r_{i_1 \dots i_{n+\ell}} \leq r_{i_1 \dots i_n} d \leq r < r_{i_1 \dots i_{n-1}} d. \tag{10}$$

Then

$$\Lambda_{i_1 \dots i_{n+\ell}} \subset B(x, r) \cap \Lambda \subset \Lambda_{i_1 \dots i_n}. \tag{11}$$

The natural measure III

Hence

$$r^s_{i_1\dots i_{n+\ell}} \leq \nu(B(x,r)) \leq r^s_{i_1\dots i_n}, \tag{12}$$

Putting together (10) and (14) we obtain

$$\frac{\log r^s_{i_1\dots i_n}}{\log r_{i_1\dots i_{n+\ell}}} < \frac{\log \nu(B(x,r))}{\log r} \leq \frac{\log r^s_{i_1\dots i_{n+\ell}}}{\log r_{i_1\dots i_{n-1}} d} \tag{13}$$

Now let $r \to 0$ to get the assertion of the Fact. $\square$

Entropy, Lyapunov exponent

Recall that we are given a self-similar IFS $\mathcal{S} = \{S_1, \dots, S_m\}$ with contractions $\mathbf{r} := (r_1, \dots, r_m)$ respectively. As always Λ is the attractor and $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$ is the symbolic space. Further we write (as always) $\Pi : \Sigma \to \Lambda$,

$$\Pi(\mathbf{i}) := \lim_{n \to \infty} S_{i_1, \dots i_n}(0), \ \mathbf{i} = (i_1, i_2, \dots).$$

Entropy, Lyapunov exponent

For a given probability vector $\mathbf{p} = (p_1, \dots, p_m)$ we consider the self-similar measure:

$$\nu := \Pi_* \mathbf{p}^{\mathbb{N}}.$$

Set

$$h_{\mathbf{p}} := -\sum_{j=1}^{m} p_j \log p_j \text{ and } \kappa_{\mathbf{p},\mathbf{r}} := -\sum_{j=1}^{m} p_j \log r_j. \tag{14}$$

We call

- $h_{\mathbf{p}}$ the entropy.
- $\kappa_{\mathbf{p},\mathbf{r}}$ the Lyapunov exponent.

Local dimension of self-similar measures assuming OSC I

In what follows we always assume that the OSC holds.

Theorem

$$d_{\nu}(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \frac{h_{\mathbf{p}}}{\kappa_{\mathbf{p},\mathbf{r}}} \quad \textit{for } \nu\textit{-a.e } x \in \Lambda. \tag{15}$$

Local dimension of self-similar measures assuming OSC II

Proof: We give the proof for the case when the SSP holds. That is for

$$d := \min \operatorname{dist}(\Lambda_i, \Lambda_j), \ i \neq j,$$

$d > 0$. Like above, we set $r_{\max} := \max \{r_1, \dots, r_m\}$ and we fix an ℓ such that

$$r_{\max}^{\ell} < d.$$

We obtained on slide 16 that

$$\boxed{r^s_{i_1 \dots i_{n+\ell}} \leq \nu(B(x,r)) \leq r^s_{i_1 \dots i_n},} \tag{16}$$

Local dimension of self-similar measures assuming OSC III

A similar argument as on slide 16 yields that for μ-a.e. $\mathbf{i} \in \Sigma$

$$\lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \frac{-\lim\limits_{n\to\infty} \frac{1}{n}(\log p_{i_1} + \cdots + \log p_{i_n})}{-\lim\limits_{n\to\infty} \frac{1}{n}(\log r_{i_1} + \cdots + \log r_{i_n})} = \frac{h_{\mathbf{p}}}{\kappa_{\mathbf{p},\mathbf{r}}}, \tag{17}$$

where in the last step we used the LLN both in the nominator and denominator.

Local dimension of self-similar measures assuming OSC IV

Remark

It follows from (11) that

$$d_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \lim_{n \to \infty} \frac{\log p_{i_1 \dots i_n}}{\log r_{i_1 \dots i_n}}. \tag{18}$$

That is whenever the limit $\lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$ *exists then the limit on the right hand side in (18) also exists and the two limits are the same. This is true even in the (ν-atypical) case when this limit is not equal to* $\frac{h_{\mathbf{p}}}{\kappa_{\mathbf{p},\mathbf{r}}}$.

In this section we consider a self-similar measure ν and study the size (Hausdorff dimension) of the set K_α where the local dimension $d_\nu(x) = \lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}$ of the measure μ is equal to a given number α. That is

$$K_\alpha := \{x \in \Lambda : d_\nu(x) = \alpha\}. \tag{19}$$

The object of our study is, the function

$$\mathcal{D} : \alpha \mapsto \dim_{\mathrm{H}}(K_\alpha) \tag{20}$$

Clearly $K_\alpha = \emptyset$ if $\alpha \notin [\alpha_1, \alpha_2]$, where

$$\alpha_{\min} := \min_{1\le i\le m} \frac{\log p_i}{\log r_i}, \text{ and } \alpha_{\max} := \max_{1\le i\le m} \frac{\log p_i}{\log r_i}.$$

Principal assumptions and def. of $T(q)$

Principal assumptions:

(A1) $\mathcal{S}$ satisfies SSP. That is

$$d := \min_{i \neq j} \operatorname{dist}\left\{S_i(\Lambda), S_j(\Lambda)\right\} > 0. \tag{21}$$

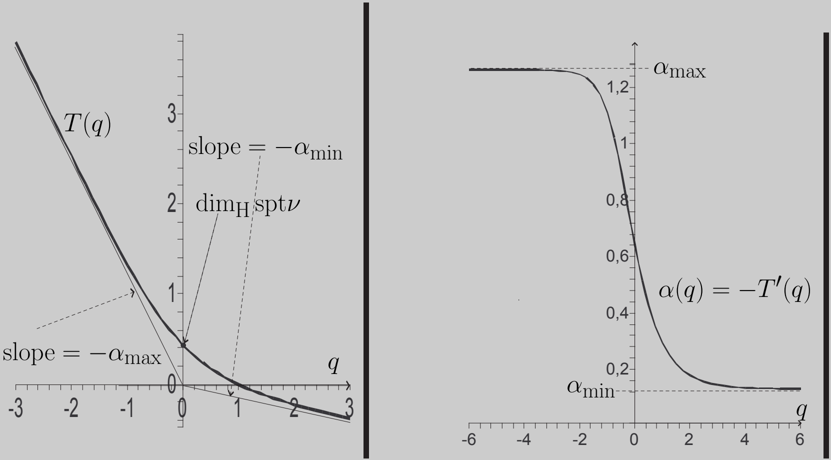
(A2) $\mathbf{p} \neq (r_1^s, \dots, r_m^s)$,

Definition

For a $q \in \mathbb{R}$, let $T(q)$ be the unique solution of the equation

$$\boxed{\sum_{i=1}^{m} p_i^q r_i^{T(q)} = 1} \tag{22}$$

Homework Prove that $T''(q) > 0$ for all $q \in \mathbb{R}$.



(a) The function $T(q)$

(b) The function $\alpha(q) := -T'(q)$

Figure : $m = 2$, $\mathbf{p} = \left(\frac{3}{4}, \frac{1}{4}\right)$, $\mathbf{r} = \left(\frac{1}{9}, \frac{1}{3}\right)$.

The main Theorem

We assume that assumptions **(A1)** and **(A2)** hold for the IFS $\mathcal{S}$. Then the multifractal spectrum of the self-similar measure $\nu = \Pi_* \left(\mathbf{p}^{\mathbb{N}}\right)$ is

$$\begin{aligned} \mathcal{D}(\alpha) &= \dim_{\mathrm{H}} \{x : d_\nu(x) = \alpha\} \\ &= \begin{cases} T^*(\alpha), & \text{if } \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

where T^* is the Legendre transform of the convex function T. That is

$$T^*(\alpha) := \inf_q \left(T(q) + \alpha \cdot q \right). \tag{23}$$

Example

Let us assume that $m = 2$ and $\mathbf{p} = \left(\frac{3}{4}, \frac{1}{4}\right)$, $\mathbf{r} = \left(\frac{1}{9}, \frac{1}{3}\right)$. That is we consider the IFS

$$\mathcal{S} = \left\{\frac{1}{9} \cdot x, \frac{1}{3} \cdot x + \frac{2}{3}\right\},$$

and we write ν for the self similar measure with probabilities $\mathbf{p}$. In this case we can find formulaes for

See Figure **??**.

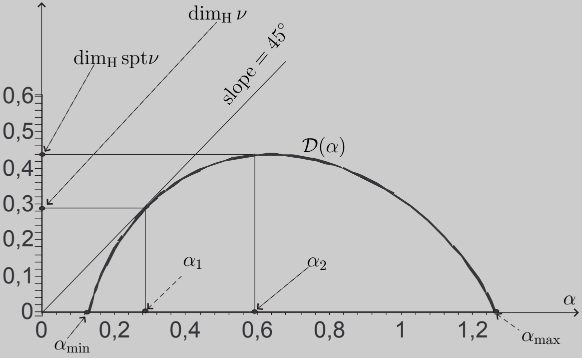


Figure : Dimension spectrum $\mathcal{D}(\alpha)$ in the case when $m = 2$ and $\mathbf{p} = \left(\frac{3}{4}, \frac{1}{4}\right)$, $\mathbf{r} = \left(\frac{1}{9}, \frac{1}{3}\right)$. $\alpha_1 := (\sum_{i=1}^{m} p_i \log p_i)/(\sum_{i=1}^{m} p_i \log r_i) = \dim_{\mathrm{H}} \nu$, $\alpha_2 := (\sum_{i=1}^{m} r_i^s \log p_i)/(\sum_{i=1}^{m} r_i^s \log r_i)$

Definition

1. $$\boxed{\mu_q} := \left\{ p_1^q \cdot r_1^{T(q)}, \dots, p_m^q \cdot r_m^{T(q)} \right\}^{\mathbb{N}}, \quad \boxed{\nu_q} := \Pi_*(\mu_q).$$

2. *Definition of $\alpha(q)$:*

$$\boxed{\alpha(q) := -T'(q) = \frac{\sum\limits_{i=1}^{m} p_i^q r_i^{T(q)} \log p_i}{\sum\limits_{i=1}^{m} p_i^q r_i^{T(q)} \log r_i}}$$

3. *Definition of $q(\alpha)$*. For $\alpha \in (\alpha_{\min}, \alpha_{\max})$ we define the function $q(\alpha)$ as the inverse function of $\alpha(q)$. ($T''(q) > 0$ so this makes sence.)

Lemma

For ν_q a.e. $x = \Pi(\mathbf{i})$ the following two assertions hold

$$d_\nu(x) = \lim_{n\to\infty} \frac{\log p_{i_1\dots i_n}}{\log r_{i_1\dots i_n}} = \alpha(q). \tag{24}$$

and

$$d_{\nu_q}(x) = T(q) + q\cdot\alpha(q) \Leftrightarrow d_\nu(x) = \alpha(q). \tag{25}$$

The proof of the Lemma is a simple application of LLN and left as an exercise. Let

$$f(\alpha) := T(q(\alpha)) + \alpha\cdot q(\alpha),\ E_\alpha := \left\{x \in \Lambda : d_{\nu_{q(\alpha)}}(x) = f(\alpha)\right\}.$$

It follows from (25) that

$$K_\alpha = \{x : d_\nu(x) = \alpha\} = E_\alpha.$$

Using L.S. Young's Theorem for the measure $\nu_{q(\alpha)}$ we obtain that

$$f(\alpha) = \dim_{\mathrm{H}}(K_\alpha).$$

Now we prove that

$$f(\alpha) = T^*(\alpha). \tag{26}$$

First observe that by differentiation a function $q \to T(q) + \alpha \cdot q$ attains its minimum at the q, where $\alpha = -T'(q)$, which is equal to $\alpha(q)$ by definition.

J.E. HUTCHINSON
Fractals and self-similarity
Indiana Univ. Math. J. **30** (1981), 713-747.

P. Mattila.
Geometry of sets and measures in Euclidean Spaces
Cambridge Univ. Press 1995.

K. Falconer
The geometry of fractal sets.
Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge, 1986.

K. Falconer
Fractal geometry. Mathematical foundations and applications. Second edition.
Wiley, 2003

K. Falconer
Techniques in fractal geometry.
Wiley, 1997

Y. Peres, C.J. Bishop
Book in preparation I do not know the title
Wiley, 1997