Fractals and geometric measure theory

Károly Simon

Department of Stochastics
Institute of Mathematics
Technical University of Budapest
www.math.bme.hu/~simonk

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IFS

Let A_i be $d \times d$ non-singular matrices with $\|A_i\| < 1$ and $t_i \in \mathbb{R}^d$ for $i = 1, \dots, m$. Let

$$\mathcal{F} := \{f_i\}_{i=1}^m = \{A_i \cdot x + t_i\}_{i=1}^m , \tag{1}$$

where we always assume that

$$\|A_i\| < 1.$$

We study the attractor Λ of the IFS $\mathcal{F}$.

The attractor Λ (definition I)

Let $B = B(0, r)$ be any closed ball centered at the origin with radius r such that

$$r > \max_{1 \le i \le m} \frac{\|t_i\|}{1 - \max\limits_{1 \le i \le m} \|A_i\|}$$

then

$$\forall i = 1, \dots, m : \qquad f_i(B) \subset B. \tag{2}$$

Thus

$$\begin{aligned} \bigcup_{i_1 \dots i_{n+1}} f_{i_1 \dots i_{n+1}}(B) &= \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n} \left(\bigcup_{i_{n+1}=1}^{m} f_{i_{n+1}}(B) \right) \\ &\subset \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B) \end{aligned} \tag{3}$$

The attractor Λ (definition II)

So we can define the non-empty compact set

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B). \tag{4}$$

The definition is independent of B. Then Λ is the only non-empty compact set satisfying

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda). \tag{5}$$

Coding the points of Λ

To code the elements of Λ we use the symbolic space

$$\Sigma := \{1, \dots, m\}^{\mathbb{N}} .$$

To code the elements of Λ with the infinite sequences from Σ we choose a sufficiently big closed ball B centered at the origin. We have seen that $f_i(B) \subset B$ for all $i = 1, \dots, m$. This follows that for all infinite sequence $\mathbf{i} := (i_1, i_2, \dots) \in \Sigma$ the sequence of sets

$$\{f_{i_1 \dots i_n}(B)\}_{n=1}^{\infty}$$

converge to a single point as $n \to \infty$. We call this point $\Pi(\mathbf{i})$.

Coding of the points of Λ (cont.)

For an $\mathbf{i} = (i_1, i_2, \dots) \in \Lambda$ we have

$$\begin{tikzcd} \mathbf{i} \arrow[r, "\sigma"] \arrow[d, "\Pi"'] & \sigma\mathbf{i} \arrow[d, "\Pi"] \\ \Pi(\mathbf{i}) & \Pi(\sigma\mathbf{i}) \arrow[l, "f_{i_1}"] \end{tikzcd} \tag{6}$$

Separation conditions

Strong separation Property

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \text{ for all } i \neq j \tag{7}$$

Open Set Condition (OSC)

There exists a non-empty open set V such that

1. $f_i(V) \subset V$ holds for all $i = 1, \dots, m$
2. $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$.

Box dimension

Let $E \subset \mathbb{R}^d$, $E \neq \emptyset$, bounded. $N_\delta(E)$ be the smallest number of sets of diameter δ which can cover E. Then the lower and upper box dimensions of E:

$$\underline{\dim}_{\mathrm{B}}(E) := \liminf_{r \to 0} \frac{\log N_\delta(E)}{-\log \delta}, \tag{8}$$

$$\overline{\dim}_{\mathrm{B}}(E) := \limsup_{r \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{9}$$

If the limit exists then we call it the box dimension of E.

Equivalent definitions I.

The definition of the box dimension does not change if we define $N_\delta(E)$ in any of the following ways:

1. the smallest number of closed balls of radius δ that cover E,
2. the smallest number of cubes of side δ that cover E,
3. the number of δ-mesh cubes that intersect E
4. the smallest number of sets of diameter at most δ that cover E,
5. the largest number of disjoint balls of radius δ with centers in E.

Equivalent definitions II.

$$\underline{\dim}_{\mathrm{B}}(E) := d - \limsup_{r \to 0} \frac{\log \mathrm{vol}^n([E]_\delta)}{-\log \delta}, \tag{10}$$

$$\overline{\dim}_{\mathrm{B}}(E) := d - \liminf_{r \to 0} \frac{\log \mathrm{vol}^n([E]_\delta)}{-\log \delta}, \tag{11}$$

where $[E]_\delta$ is the δ parallel body of E.

Hausdorff measure on $\mathbb{R}^d$

Let $\Lambda \subset \mathbb{R}^d$ and let $t \geq 0$. We define

$$\mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \underbrace{\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \Lambda \subset \bigcup_{i=1}^{\infty} A_i ; |A_i| < \delta \right\}}_{\mathcal{H}^t_\delta(\Lambda)} \right\} \tag{12}$$

Then $\mathcal{H}^t$ is a metric outer measure. The t-dimensional Hausdorff measure is the restriction of $\mathcal{H}^t$ to the σ-field of $\mathcal{H}^t$-measurable sets which include the Borel sets.

Hausdorff dimension I.

Let $\Lambda \subset \mathbb{R}^d$ and $0 \leq \alpha < \beta$. Then

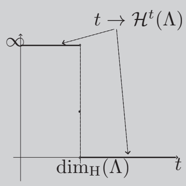
$$\mathcal{H}^\beta_\delta(\Lambda) \leq \delta^{\beta-\alpha} \mathcal{H}^\alpha_\delta(\Lambda).$$

Using that $\mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(\Lambda)$

$$\mathcal{H}^\alpha(\Lambda) < \infty \Rightarrow \mathcal{H}^\beta(\Lambda) = 0 \text{ for all } \alpha < \beta.$$

$$0 < \mathcal{H}^\beta(\Lambda) \Rightarrow \mathcal{H}^\alpha(\Lambda) = \infty \text{ for all } \alpha < \beta.$$

Hausdorff dimension II.



The Hausdorff dimension of Λ

$$\dim_{\mathrm{H}}(\Lambda) = \inf\{t : \mathcal{H}^t(\Lambda) = 0\} = \sup\{t : \mathcal{H}^t(\Lambda) = \infty\}.$$

Mass Distribution Principle

We say that a Borel measure μ on the set X is a mass distribution if $0 < \mu(X) < \infty$.

Lemma 1 (Mass Distribution Principle)

If $A \subset X$ supports a mass distribution μ such that for a constant C and for every Borel set D we have

$$\mu(D) \leq \mathrm{const} \cdot |D|^t$$

Then $\dim_{\mathrm{H}}(A) \geq t$.

Proof For all $\{A_j\}_{j=1}^{\infty}$

$$A \subset \bigcup_{j=1}^{\infty} A_j \Rightarrow \sum_j |A_j|^t \geq C^{-1} \sum_j \mu(A_j) \geq \frac{\mu(A)}{C}.$$

Frostman's Energy method

Let μ be a mass distribution on $\mathbb{R}^d$. The t-energy of μ is defined by

$$\mathcal{E}_t(\mu) := \iint |x-y|^{-t} d\mu(x) d\mu(y).$$

Lemma 2 (Frostman (1935))

For a Borel set $\Lambda \subset \mathbb{R}^d$ and for a mass distribution μ supported by Λ we have

$$\mathcal{E}_t(\mu) < \infty \Longrightarrow \dim_{\mathrm{H}}(\Lambda) \geq t.$$

In this case $\mathcal{H}^t(\Lambda) = \infty$.

Proof of Frostman Lemma I

This proof if due to Y. Peres. Let

$$\Phi_t(\mu, x) := \int \frac{d\mu(y)}{|x-y|^t}.$$

Then $\mathcal{E}_t(\mu) = \int \Phi_t(\mu, x) d\mu(x)$. Let

$$\Lambda_M := \{x \in \Lambda : \Phi_t(\mu, x) \leq M\}.$$

Since $\int \Phi_t(\mu, x) d\mu(x) = \mathcal{E}_t(\mu) < \infty$ we have M such that $\mu(\Lambda_M) > 0$. Fix such an M.

Proof of Frostman Lemma II

Let

$$\nu := \mu|_{\Lambda_M}$$

Then ν is a mass distribution supported by Λ. (That is ν satisfies one of the assumptions of the Mass Distribution Principle above.) Now we show that for every bounded set D:

$$\nu(D) < \mathrm{const} \cdot |D|^t. \tag{13}$$

If $D \cap \Lambda_M = \emptyset$ then (13) holds obviously. From now we assume that D is a bounded set such that $D \cup \Lambda_m \neq \emptyset$.

Proof of Frostman Lemma III

Pick an arbitrary $x \in D \cap \Lambda_M$. We define

$$m := \max\left\{k \in \mathbb{Z} : B(x, 2^{-k}) \supset D\right\}.$$

Then

$$|D| \geq 2^{-(m+1)} \text{ and } |D| < 2 \cdot 2^{-m}. \tag{14}$$

Proof of Frostman Lemma IV

Observe that from the right hand side of (14): $y \in D$ we have $|x-y|^{-t} \geq |D|^{-t} \geq 2^{-t} \cdot 2^{mt}$. So,

$$M \geq \int \frac{d\nu(y)}{|x-y|^t} \geq \int_D \frac{d\nu(y)}{|x-y|^t} \geq \nu(D) \cdot 2^{-t} \cdot 2^{m \cdot t}.$$

Using this and the left hand side of (14) we obtain

$$\nu(D) \leq M \cdot 2^t \cdot 2^t \cdot 2^{-(m+1)t} \leq M \cdot 2^{2t} \cdot |D|^t.$$

So, the mass distribution ν satisfies the assumptions of the Mass Distribution Principle which completes the proof of the Lemma.

Hausdorff dimension of a measure

Let μ be a mass distribution on $\mathbb{R}^d$.

Definition 1

$$\dim_{\mathrm{H}}(\mu) := \inf\left\{\dim_{\mathrm{H}}(A) : \mu(\mathbb{R}^d \setminus A) = 0\right\}.$$

Lemma 3

$$\dim_{\mathrm{H}}(\mu) = \mathrm{ess\ sup}_x \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Roughly speaking, $\dim_{\mathrm{H}}(\mu) = \delta$ if for a μ-typical x we have

$$\mu(B(x,r)) \approx r^\delta$$

for small $r > 0$.

Packing measure

δ-packing of $E \subset \mathbb{R}^d$ is a finite or countable collection of disjoint balls $\{B_i\}_i$ of radii at most δ and with centers in E. For $\delta > 0$

$$\mathcal{P}^s_\delta(E) := \sup\left\{\sum_{i=1}^{\infty} |B_i|^s : \{B_i\} \text{ is a } \delta\text{-packing of E}\right\}$$

Since $\mathcal{P}^s_0(E) := \lim_{\delta \to 0} \mathcal{P}^s_\delta(E)$ is NOT countably-sub additive therefore we need one more step:

$$\mathcal{P}^s(E) := \inf\left\{\sum_{i=1}^{\infty} \mathcal{P}^s_0(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\right\}$$

Packing dimension

$$\begin{aligned} \dim_{\mathrm{P}}(E) &:= \inf\{s : \mathcal{P}^s(E) = 0\} \\ &= \sup\{s : \mathcal{P}^s(E) = \infty\}. \end{aligned}$$

$$\dim_{\mathrm{P}}(E) = \inf\left\{\sup_i \overline{\dim}_B E_i : E \subset \bigcup_{i=1}^{\infty} E_i\right\},$$

where the sup is taken for all covers $\{E_i\}_{i=1}^{\infty}$ of E.

$$\dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{P}}(E) \leq \overline{\dim}_B E.$$

Self-similar sets with OSC

Assume that $\mathcal{F} := \{f_i\}_{i=1}^m$ is a self-similar IFS on $\mathbb{R}^d$. The similarity dimension $s = s(\mathcal{F})$ is defined as the only positive solution of the equation

$$r_1^s + \cdots + r_m^s = 1, \tag{15}$$

where r_i is the similarity ratio for f_i.

Hutchinson Theorem

Hutchinson (1981)

Theorem 1

Given a self similar IFS $\mathcal{F}$ which satisfies the OSC. Let $s = s(\mathcal{F})$ be the similarity dimension. Then

$$0 < \mathcal{H}^s(\Lambda) < \infty. \tag{16}$$

Further,

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} = s.$$

For the proof see: [1].

Hausdorff measure for self-similar attractors

We cannot easily estimate the appropriate dimensional Hausdorff measure of a self similar-set in the plane or higher dimension. If Λ is the Sierpinski triangle then the we know that $s = \dim_{\mathrm{H}} \Lambda = \frac{\log 3}{\log 2}$. The best estimate for s-dimensional Hausdorff measure:

$$0.77 \leq \mathcal{H}^s(\Lambda) \leq 0.81$$

The upper bound is old (proved in 1999) but the lower bound is new. It was given By Peter Móra (PhD student).

M. Keane's " $\{0, 1, 3\}$ " problem:

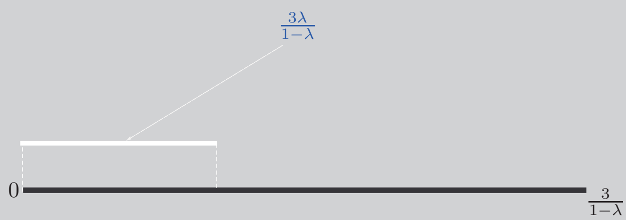
For every $\lambda \in (\frac{1}{4}, \frac{2}{5})$ consider the following self-similar set:

$$\Lambda_\lambda := \left\{\sum_{i=0}^{\infty} a_i \lambda^i : a_i \in \{0, 1, 3\}\right\}.$$

Then Λ_λ is the attractor of the one-parameter (λ) family IFS:

$$\left\{S_i^\lambda(x) := \lambda \cdot x + i\right\}_{i=0,1,3}$$

$\{0, 1, 3\}$ problem II.

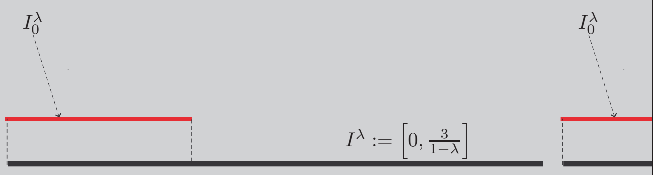


$\Pi_\lambda : \{0,1,3\}^{\mathbb{N}} \mapsto \Lambda_\lambda$

Let $k \in \mathbb{N}$ and $\mathbf{i} = (i_0, i_1, \dots) \in \underbrace{\{0,1,3\}^{\mathbb{N}}}_{\Sigma}$.

$I^\lambda_{i_0,\dots,i_k} := S^\lambda_{i_0} \circ \dots \circ S^\lambda_{i_k}(I^\lambda)$ and $\Pi_\lambda(\mathbf{i}) := \bigcap_{k=1}^{\infty} I^\lambda_{i_0,\dots,i_k}$.

Example: $\Pi_\lambda(0,3,1,0,\dots)$



Let $k \in \mathbb{N}$

$I^\lambda_{i_0,\dots,i_k} :=$

Example:

I_0^λ

Basic Measure Theory

Let $\mathcal{S}$ be a σ-field of a given set X. We say that a function $\mu : \mathcal{S} \to [0,\infty]$ is a measure if

- $\mu(\emptyset) = 0$
- $\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for every disjoint sequence of sets $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{S}$.

An outer measure ν on X is defined on all subsets of X takes values from $[0,\infty]$ such that

- $\nu(\emptyset) = 0$,
- $\nu(A) \le \nu(B)$ if $A \subset B$,
- $\nu\left(\cup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \nu(E_i)$ for all sequence of sets $\{E_i\}_{i=1}^{\infty}$.

Measurable subsets

A set E is measurable with respect to the outer measure ν if for every $A \subset X$ we have $\nu(A) = \nu(A \cap E) + \nu(A \setminus E)$.
Let $\mathcal{M}$ be the collection of all measurable set for an outer measure ν. Then $\mathcal{M}$ is a σ-field and the restriction of ν to $\mathcal{M}$ is a measure.
Further, assume that (X,d) is a metric space. We say that the outer measure ν is a metric outer measure if $\nu(A \cup B) = \nu(A) + \nu(B)$ holds for all $A, B \subset X$ with $\inf\{d(a,b) : a \in A, b \in B\} > 0$.
In this case the restriction of ν to the σ-field of the measurable sets $\mathcal{M}$ is a Borel measure.

Definitions

Let (X,d) be a separable metric space and let μ be a measure on X.

1. μ is locally finite if $\forall x \in X$, $\exists r > 0$, such that $\mu(B(x,r)) > 0$.
2. μ is a Borel measure if all Borel sets are μ measurable. (The family of Borel sets in X is the smallest σ-algebra containing all open sets.)
3. The measure μ is Borel regular if
   - (a) Borel measure and
   - (b) $\forall A \subset X$, $\exists A \subset B \subset X$ Borel set s.t. $\mu(A) = \mu(B)$.

Radon measure definition

μ is a Radon measure if

- (a) Borel measure,
- (b) $\forall K \subset X$ compact: $\mu(K) < \infty$,
- (c) $\forall V \subset X$ open: $\mu(V) = \sup\{\mu(K) : K \subset V$ is compact $\}$
- (d) $\forall A \subset X$: $\mu(A) = \inf\{\mu(V) : A \subset$ and V is open $\}$.

Theorem 2

A measure μ on $\mathbb{R}^d$ is a Radon measure if and only if it is locally finite and Borel regular

Proof: See Mattila's book [2, p. 11-12].

Radon measure examples

1. The Lebesgue measure $\mathcal{L}\mathrm{eb}_d$ on $\mathbb{R}^d$ is a Radon measure.
2. The Dirac measure $\delta_a(A) := 1$ if $a \in A$ and $\delta_a(A) = 0$ if $a \notin A$ is a Radon measure.
3. For every $s \ge 0$ the Hausdorff measure is a Borel regular measure but it need not be locally finite. So, in general the Hausdorff measure is not a Radon measure. However, for an $A \subset \mathbb{R}^d$, $\mathcal{H}^s(A) < \infty$ the restriction $\mathcal{H}^s|_A$ is a Radon measure. (See Mattila's book: [2, p. 57].)

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