Fractals and geometric measure theory 2012

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1st week

Introduction

Outline

1. Self-similar fractals
2. Self-affine sets
3. Julia and Mandelbrot sets
4. Dragons
5. Fractal percolation
6. Brownian motion
7. Blaschke selection theorem
8. References

An application of fractals in Numb3rs

Click here to see a way how to apply fractals.

Application of fractals

We use fractals to describe objects or phenomena in which some sort of scale invariance exists.

Fractals appear physics, astronomy, biology, chemistry, market fluctuation analysis, and so on.

At the conference

Practical Applications of Fractals
17 - 19 November 2004
Miramare, Trieste, Italy the following main applications were discussed:

Fractals in industry and man-made fractals:

- Fractal antennae,
- Fractal sound barriers,
- Use of fractal polymeric surfaces,
- Fractal reactor design,
- Fractal studies of heterogeneous catalysis,
- Petroleum research.

Natural fractal objects:

- Fractal bronchial trees in mammals,
- Growth of fractal trees in nature,
- Optimal fractal distribution,
- Absolute limitations of tree distributive structures,
- River Networks,
- Fractals and allometry (relative growth of a part in relation to an entire organism or to a standard; also: the measure and study of such growth).

Applications of fractal concepts to the study of complex systems:

- Image analysis and compression
- Multifractal signal analysis
- Scaling topology of the internet and the www
- Fractal aviation communication network

How long is the coast of Britain?

Figure: Britain coastline, 200km: 2400km, 100km, 50 km:3400km

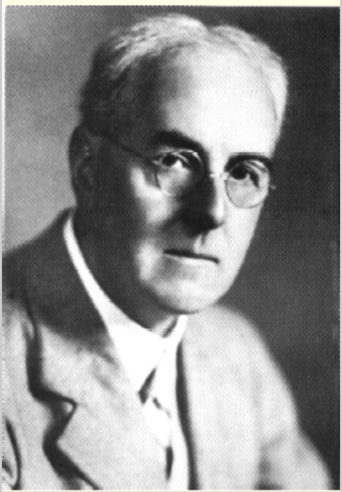


Figure: Lewis R. Richardson 1881-1953

Richardson conjectured: The measured length $L(G)$ of a geographic boarder is

$$L(G) \approx M \cdot G^{1-D},$$

M is a constant and D is the dimension. Namely:

$$L(G) = N(G) \cdot G$$

$$\frac{\log N(G)}{\log G^{-1}} \approx D \Longrightarrow L(G) \approx G^{1-D}.$$

Britain: $D = 1.25$, Germany: $D = 1.14$, South Africa $D = 1.02$.

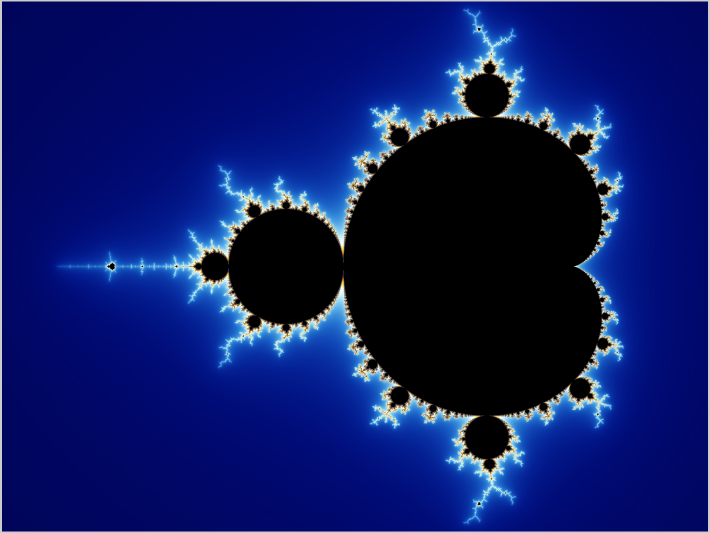


Figure: The famous Mandelbrot set (we do not learn much about it on this course).

Beniot Mandelbrot

Figure: The father of fractal geometry

- In École Polytechnique, student of Julia, Lévy.
- Later post. doc. working with J. Neumann at Princeton.
- Worked for IBM for 35 years. Then moved to Yale. Books:
- Fractals: Form, Chance and Dimension 1975.
- The Fractal Geometry of Nature, 1982.

Figure: Waclaw Sierpinski

- Born in Warsaw 1882.
- Ph.D. in 1908 at Univ. of Krakkow (Poland).
- 1919-1969 worked at the Univ of Warsaw, died: 1969
- A GREAT mathematician. Very important results in: set theory, real analysis and topology.

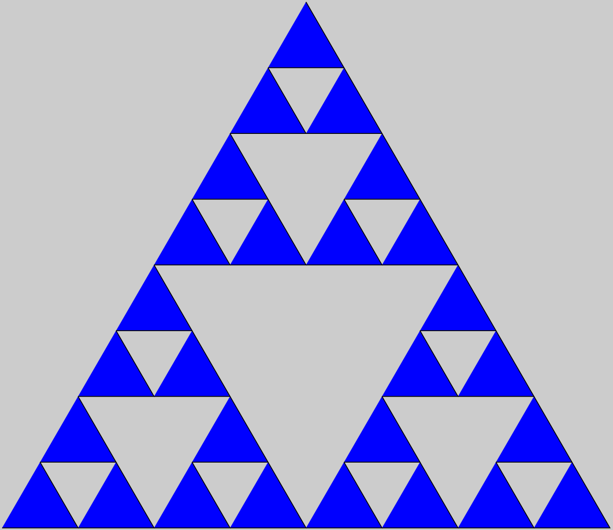


Figure: The third approximation of the Sierpinski gasket

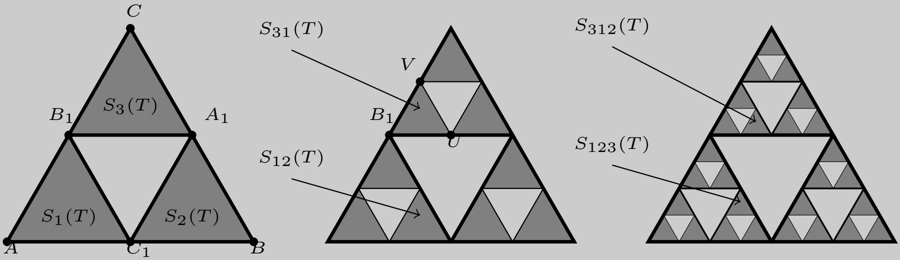


Figure: $S_{312}(x) := S_3 \circ S_1 \circ S_2(x) = S_3(S_1(S_2(x)))$

S_i are translations of the appropriate homothety-transformatons of the form:

$$\boxed{S_i(x) = \frac{1}{2}x + t_i}.$$

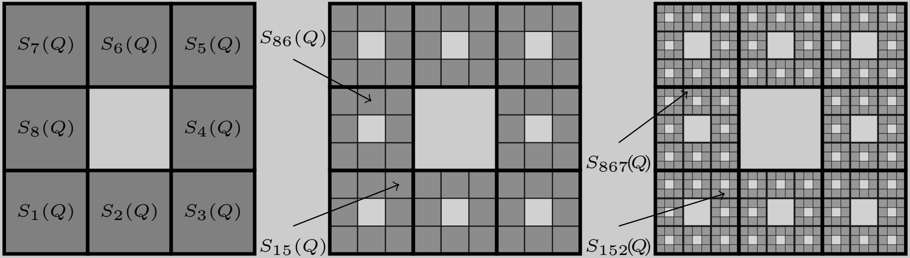


Figure: The third approximation of the Sierpinski carpet

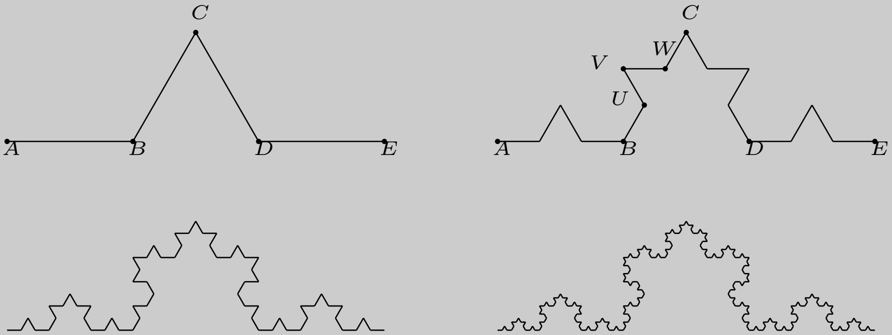
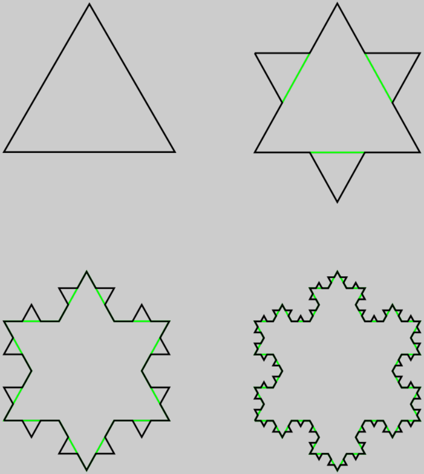


Figure: von Koch snowflake (from Wikipedia)



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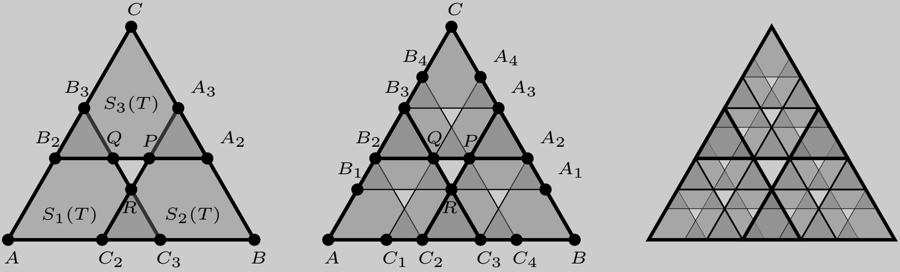


Figure: The third approximation of the golden gasket

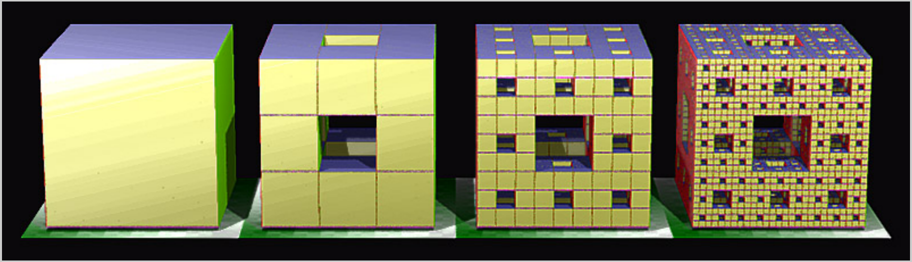


Figure: Menger Sponge (from Wikipedia)

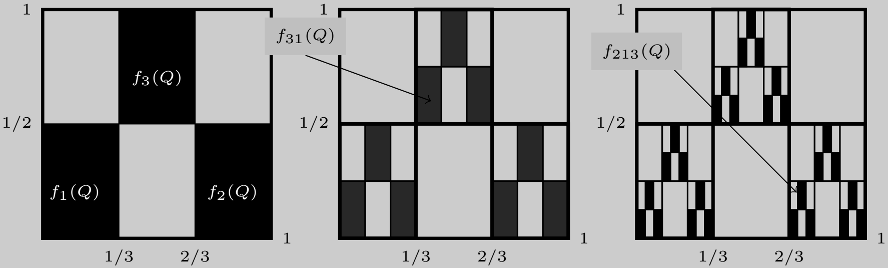


Figure: The Hironika curve

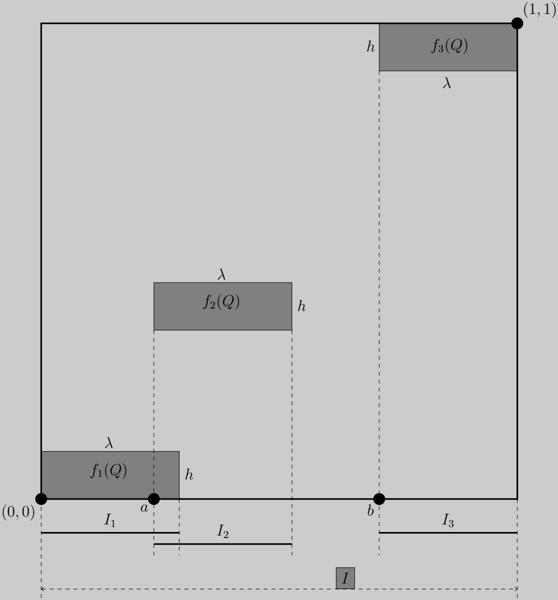


Figure: The generator of a self-affine set

$$A_1 := \begin{bmatrix} 0.3464101616 & -0.1250000000 \\ 0.2 & 0.2165063510 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 0.2 & 0.2165063510 \\ -0.3464101616 & 0.1250000000 \end{bmatrix}$$

$$t_1 := [0.5196152, 0.3], t_2 := [-0.4688749, 0.5721152]$$

Let $f_1(x) := A_1x + t_1$ and $f_2(x) := A_2x + t_2$ and

$$D_{i_1 \dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}(D),$$

where D is the unit disk.

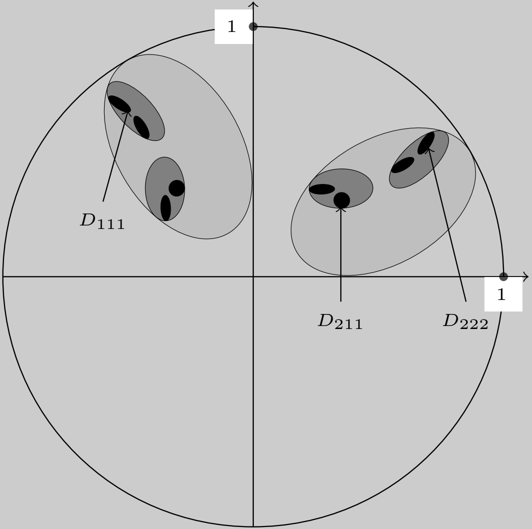


Figure: The third approximation of the attractor of the self affine IFS.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ .16 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .2 & -.26 \\ .22 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .23 \\ 1.6 \end{pmatrix};$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -.15 & .28 \\ .24 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .26 \\ .44 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .75 & -.04 \\ .85 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -.04 \\ 1.6 \end{pmatrix}.$$

Figure: The Barnsly's fern

Julia sets: the definition

Let $f(z) = p(z)/q(z)$, where $p(z), q(z)$ complex polynomials. A point z_0 is a periodic point of f with period $n \geq 1$ if $f^n(z_0) = z_0$ but $f^k(z_0) \neq z_0$ for $0 < k < n$. We say that such a z_0 is repelling if $|f'(z_0)| > 1$.

Definition (Julia set of f)

The Julia set of f denoted by $J(f)$, is the closure of the repelling periodic points of f.

Julia sets: properties

(a) $f^{-1}(J(f)) = f(J(f)) = J(f)$

(b) $\forall z, w \in J(f)$, we have $|f(z) - f(w)| > |z - w|$ if w is close enough to z.

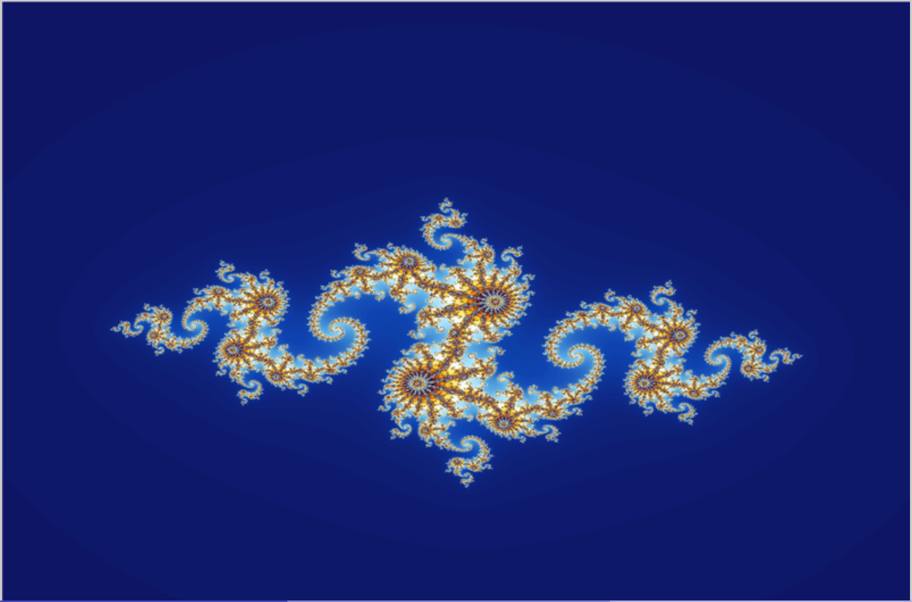
(c) For all but at most two z, $J(f)$ is the set of limit points of $\cup_n f^{-n} z$.

If f is a polynomial then

1. $J(f)$ is the boundary of the set of points whose orbit (the sequence $\{f^n\}_{n=1}^{\infty}$) tend to infinity.
2. $J(f)$ is the boundary of the set of points whose orbit is bounded. (We call this set filled Julia set.)

Next three pictures are from the Wikipedia.

$J(f_c)$ for $f_c = z^2 + c$, $c = -0.80.156i$



$J(f_c)$ for $f_c = z^2 + c$, $c = -0.70176 - 0.3842i$

Mandelbrot set

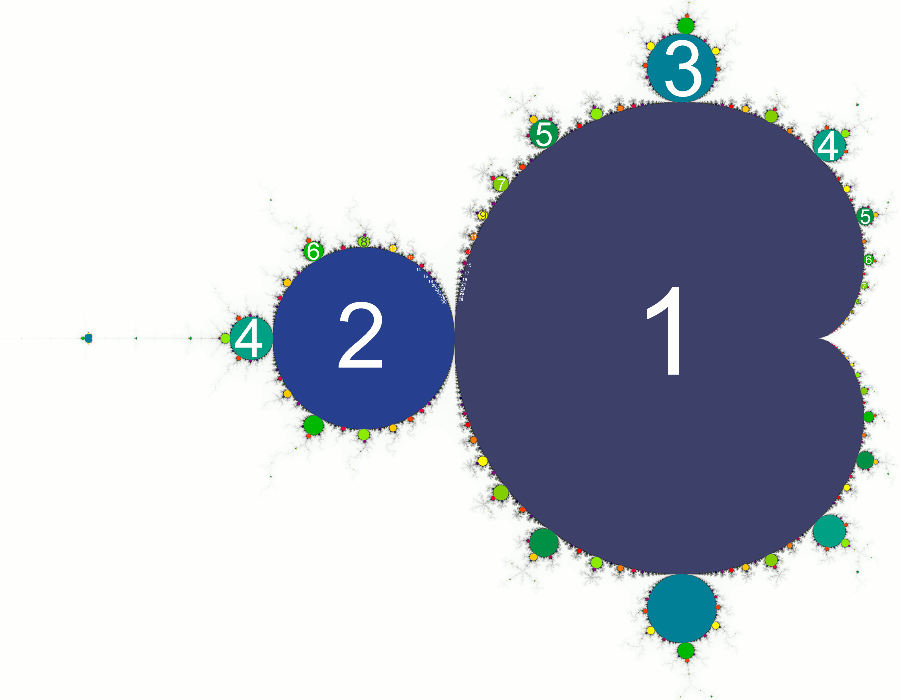
$$\mathcal{M} := \left\{ c \in \mathbb{Z} \middle| \left\{ f_c^n(0) \right\}_{n=0}^{\infty} \text{ is bounded.} \right\},$$

where $f_c(z) = z^2 + c$. E.g. $1 \notin \mathcal{M}$ but $i \in \mathcal{M}$.
Equivalently,

$$\mathcal{M} := \left\{ c \in \mathbb{Z} : \left| \left\{ f_c^n(0) \right\}_{n=0}^{\infty} \right| \leq 2, \quad \forall n \right\}.$$

Equivalently:

$$\mathcal{M} := \left\{ c : J(f_c) \text{ is a connected set. } \right\}$$


3
5
4
7
9
5
6
1
6
8
2
4

The main cardioid consists of $c \in \mathcal{M}$ for which f_c has an attracting fixed point.

For $(p, q) = 1$, there is a bulb tangent to the main cardioid at

$$c_{\frac{p}{q}} = \frac{e^{2\pi i \frac{p}{q}}}{2}\left(1 - \frac{e^{2\pi i \frac{p}{q}}}{2}\right)$$

which bulb consists of those $c \in \mathcal{M}$ for which f_c has an attracting periodic orbit of period q.

click here for further information about the Mandelbrot set.

To see the corresponding Julia set for a $c \in \mathcal{M}$ click here

click here

Heighway Dragon I

Click here to see a vidio on youtube how the Heighway dragon fractal builds up.

Heighway Dragon II

$$S_1(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \mathbf{x}$$

$$S_2(\mathbf{x}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathcal{S}_H := \{S_1(\mathbf{x}), S_2(\mathbf{x})\}. \tag{1}$$

Heighway Dragon III

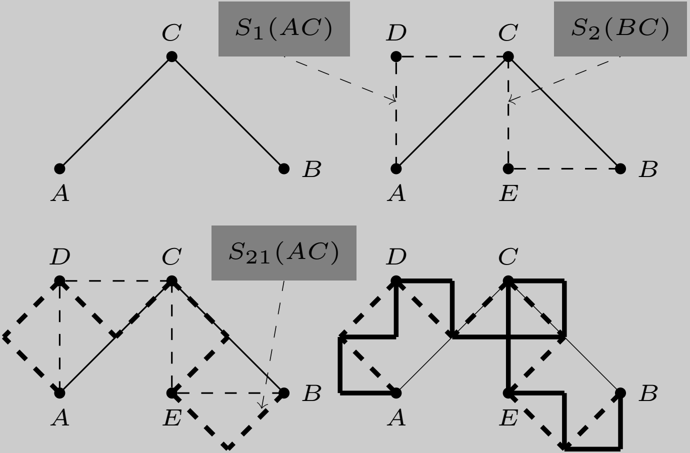


Figure: The first four approximations of the Heighway dragon.

Heighway Dragon IV

Let P_n the broken line that we obtain after n steps. Then $\{P_n\}_{n=1}^{\infty}$ is a Cauchy sequence of compact sets in the Hausdorff metric (defined later). It converges to a set Λ (the attractor) which is called Heighway dragon .

- The interior of Λ is non-empty
- The plane can be tiled with congruent copies of Λ.
- The Hausdorff dimension (to be defined later) of the boundary is $2\log\lambda/\log 2 = 1.5236270862\ldots$, where λ is the largest real zero of $\lambda^3 - \lambda^2 - 2$.

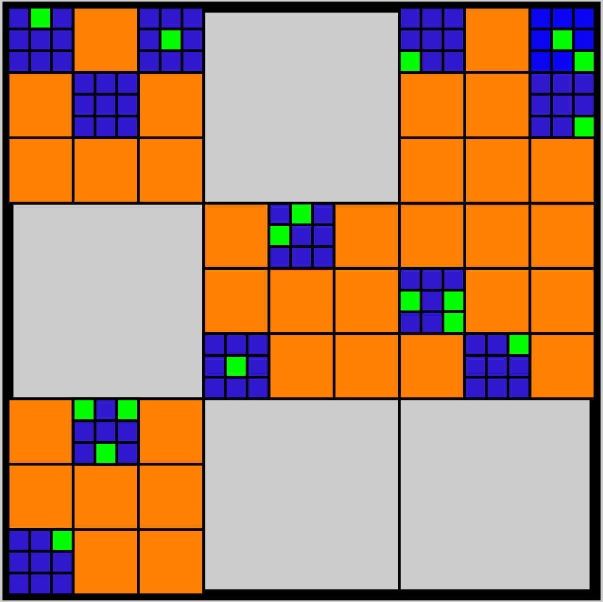


Figure: Fractal percolation

Figure: Brownian motion

Hausdorff metric

Let $E \subset \mathbb{R}^d$ be a bounded set. The δ-parallel body of E is

$$\boxed{[E]_\delta} := \left\{ x \in \mathbb{R}^d : \inf_{y \in E} |x-y| \right\}$$

The Hausdorff metric dist_H is defined for the non-empty compact sets $E, F \subset \mathbb{R}^d$ as follows:

$$\boxed{\mathrm{dist}_H(E,F)} := \inf\{\delta : F \subset [E]_\delta \text{ and } E \subset [F]_\delta\}$$

Let $Q \subset \mathbb{R}^d$ be a compact set and let $\boxed{\mathfrak{C}_Q}$ be the space of compact sets of $\mathbb{R}^d$ which are contained in Q. Then dist_H is a metric on $\mathfrak{C}_Q$.

Blaschke selection theorem

Given a $Q \subset \mathbb{R}^d$ non-empty compact set.

(a) Let $\{E_i\}_{i=1}^{\infty}$ be a Cauchy sequence in the metric space $(\mathfrak{C}_Q, \mathrm{dist}_H)$. Set

$$E := \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} E_i}, \tag{2}$$

where bar denotes the closure. Then $E_i \to E$ in Hausdorff metric. (In particular, $(\mathfrak{C}_Q, \mathrm{dist}_H)$ is a complete metric space.) Moreover,

(b) $(\mathfrak{C}_Q, \mathrm{dist}_H)$ is a compact metric space.

Proof of part (a) first slide

Fix an arbitrary $\varepsilon > 0$ and choose N such that

$$\forall n > N, \quad \mathrm{dist}_H(E_i, E_j) < \varepsilon/3. \tag{3}$$

It is enough to prove that

$$\forall k > N, \quad E \subset [E_k]_\varepsilon \tag{4}$$

and

$$\forall k > N, \quad E_k \subset [E]_\varepsilon\,. \tag{5}$$

Proof of (4)

Fix an arbitrary $x \in E$. By the definition of E, we can find an $i > N$ such that

$$B(x, \varepsilon/3) \cap E_i \neq \emptyset.$$

Choose $e_i \in B(x, \varepsilon/3) \cap E_i$. Then for every $k > N$ there is an e_k such that $|e_i - e_k| < \varepsilon/3$ (since $E_i \subset [E_k]_{\varepsilon/3}$). Thus

$$|x - e_k| < 2\varepsilon/3.$$

This completes the proof of (4).

Proof of (5)

Fix an $y \in E_k$ It follows from (3) that

$$\forall j > N, \ \exists e_j \in E_j \text{ such that } |y - e_j| < \varepsilon/3.$$

Clearly, $e_j \in Q$ for all j. So, there exists an $e \in Q$ and $\{j_k\}_{k=1}^{\infty}$ such that $e_{j_k} \to e$.

It is immediate that $e \in E$ and $|y - e| \leq \varepsilon/3$. This completes the proof of (5).

Proof of (b) first slide

It is well known that a metric space is compact if every infinite subset has an accumulation point contained in the metric space. (See [1, p. 112].)

Let $\mathcal{D} \subset \mathfrak{C}_Q$ be an infinite subset and let $\{E_{1,i}\}_{i=1}^{\infty}$ be an arbitrary sequence of distinct elements of $\mathcal{D}$. For each $k > 1$ we define a subsequence of distinct elements $\{E_{k,i}\}_{i=1}^{\infty}$ of $\{E_{k-1,i}\}_{i=1}^{\infty}$ as follows:

Cover Q with finitely many balls of diameter $1/k$ in any particular way. Let $\mathcal{B}$ be the finite collection of these balls. Let $\{E_{k,i}\}_{i=1}^{\infty}$ be an infinite subsequence of $\{E_{k-1,i}\}_{i=1}^{\infty}$ such that for every i, j:

$$\{B \in \mathcal{B} : B \cap E_{k,i} \neq \emptyset\} = \{B \in \mathcal{B} : B \cap E_{k,j} \neq \emptyset\} .$$

Proof of (b) second slide

Let

$$F := \bigcup_{B \in \mathcal{B}} B.$$

Then

$$\forall i, \quad E_{k,i} \subset F \subset [E_{k,i}]_{1/k} .$$

Hence

$$\forall i,\ \mathrm{dist}_H(F, E_{k,i}) < 1/k \Longrightarrow \forall i,j,\ \mathrm{dist}_H(E_{k,i}, E_{k,j}) < 1/k.$$

This implies that the sequence $E_i := E_{i,i}$ is a Cauchy sequence in $(\mathfrak{C}_Q, \mathrm{dist}_H)$, which tends to a compact set $E \subset Q$ according to part (a).

A.N. Kolmogorov, Sz.V. Fomin
A függvényelmélet és a funkcionálanalízis elemei
Mûszaki Könykiadó, 1981.