NORTH-HOLLAND

$\mathbf{P}_*$-Matrices Are Just Sufficient

Hannu Väliaho
Department of Mathematics
University of Helsinki
Helsinki, Finland

Submitted by Richard W. Cottle

ABSTRACT

The classes of sufficient matrices and of $\mathbf{P}_*$-matrices have recently arisen in connection with the linear complementarity problem. It is known that $\mathbf{P}_*$-matrices are sufficient. We show that, conversely, every sufficient matrix is a $\mathbf{P}_*$-matrix.

1. INTRODUCTION

The classes of sufficient matrices and of $\mathbf{P}_*$-matrices have recently arisen in connection with the linear complementarity problem (LCP). The class **SU** of sufficient matrices was defined by Cottle, Pang, and Venkateswaran [6], and the class $\mathbf{P}_*$ by Kojima, Megiddo, Noma, and Yoshise [8]. These classes are defined as follows.

A matrix $A \in \mathbb{R}^{n\times n}$ is *column sufficient* if for all $x \in \mathbb{R}^n$

$$x_i(Ax)_i \leqslant 0, \quad i = 1,\dots,n \Rightarrow x_i(Ax)_i = 0, \quad i = 1,\dots,n,$$

and *row sufficient* if A^T is column sufficient. A is *sufficient* if it is both row and column sufficient. Row sufficient matrices are linked to the existence of solutions to the LCP, and column sufficient matrices are associated with the convexity of the solution set.

For $\kappa \geqslant 0$, the class $\mathbf{P}(\kappa)$ consists of all matrices $A \in \mathbb{R}^{n\times n}$ satisfying

$$(1 + 4\kappa) \sum_{i\in I_+(x)} x_i y_i + \sum_{i\in I_-(x)} x_i y_i \geqslant 0 \qquad \text{for all} \quad x \in \mathbb{R}^n,$$

LINEAR ALGEBRA AND ITS APPLICATIONS 239:103–108 (1996)

0024-3795/96/$15.00
SSDI 0024-3795(94)00170-I

where $y = Ax$ and

$$I_+(x) = \{i \mid x_i y_i > 0\} \quad \text{and} \quad I_-(x) = \{i \mid x_i y_i < 0\}.$$

The class $\mathbf{P}_*$ is now defined by $\mathbf{P}_* = \bigcup_{\kappa \geqslant 0} \mathbf{P}_*(\kappa)$. In [8] it is shown that any feasible LCP with a $\mathbf{P}_*$-matrix can be solved by means of the unified interior point method.

It is known that $\mathbf{P}_* \subset \mathbf{SU}$ (see [8] for column sufficiency and [7] for row sufficiency). Guu and Cottle [7] have shown that, for 2×2 matrices, these classes are identical. On the basis of this result and some computational evidence with matrices of order greater than two they conjectured that $\mathbf{SU} = \mathbf{P}_*$. In this paper we show that this really is the case. As a consequence, all that has been proved about $\mathbf{P}_*$ in [8] holds for $\mathbf{SU}$ also, and all results on $\mathbf{SU}$ in [1–3, 6, 9] are valid for $\mathbf{P}_*$ too. For example, the existing finite tests for sufficient matrices can be used as criteria for membership in $\mathbf{P}_*$ (up to now, no finite criteria for membership in $\mathbf{P}_*$ have been known).

2. PRELIMINARIES

If $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ (A is a real $m \times n$ matrix), we write A^T for its transpose. If $R \subset \{1, \dots, m\}$ and $S \subset \{1, \dots, n\}$, we denote the submatrix of A induced by rows $i \in R$ and columns $j \in S$ by A_{RS}. A diagonal matrix $D \in \mathbb{R}^{n \times n}$ with the diagonal elements $d_1, \dots, d_n$ is denoted by $D = \operatorname{diag}(d_1, \dots, d_n)$. By a *principal permutation* of a square matrix we mean simultaneous permutation of the rows and the columns. Any vector $x \in \mathbb{R}^n$ is interpreted as an $n \times 1$ matrix and denoted by $x = [x_1, \dots, x_n]^T$ or, for simplicity, by $x = (x_1, \dots, x_n)$. We write x_R for the subvector of x consisting of components $i \in R$. Moreover, we define $N = \{1, \dots, n\}$, denote the empty set by $\varnothing$, and abbreviate $R - r = R \setminus \{r\}$. The symbol $:=$ will be used for definition.

If $A \in \mathbb{R}^{n \times n}$, $R \subset \{1, \dots, n\}$, and A_{RR} is nonsingular, the *principal pivotal operation* $\mathscr{P}_R$ transforms the equation $y = Ax$ into an equivalent equation in which the variables y_R and x_R have been exchanged; see e.g. [5, pp. 68–78]. If the matrix of the latter equation is $\hat{A}$, we write $\hat{A} = \mathscr{P}_R A$. Moreover, we define $\mathscr{P}_\varnothing A = A$. The single principal pivotal operation $\mathscr{P}_{\{r\}}$ will simply be denoted by $\mathscr{P}_r$. Any matrix obtained from A by means of a principal pivotal operation followed by a principal permutation is called a *principal transform* of A.

The classes **SU**, $\mathbf{P}_*(\kappa)$, and $\mathbf{P}_*$ have much in common (see [1, 6, 8]):

(1) Each of these classes is contained in $\mathbf{P}_0$, the class of matrices with nonnegative principal minors. (So all matrices in these classes have nonnegative diagonal elements.)

(2) Each of these classes contains **PSD**, the class of positive semidefinite matrices; **SU** and $\mathbf{P}_*$ contain **P**, the class of matrices with positive principal minors.

(3) If a matrix A is in one of these classes, then so is any principal permutation of A, any principal submatrix of A, and any principal transform of A.

It should be noted that $\mathbf{P}_*(0) = \mathbf{PSD}$. The following two theorems contain some additional facts about sufficient and $\mathbf{P}_*$-matrices.

THEOREM 2.1 [1]. *If $A \in \mathbb{R}^{n\times n}$ is sufficient and $a_{kk} = 0$, then $a_{ik} = a_{ki} = 0$ or $a_{ik}a_{ki} < 0$ for all $i \neq k$.*

THEOREM 2.2 (cf. [8]). *Let $A \in \mathbb{R}^{n\times n}$; let $P = \operatorname{diag}(p_1, \dots, p_n)$, $Q = \operatorname{diag}(q_1, \dots, q_n)$, where $p_i q_i > 0$ for all $i \in N$; and let $B = PAQ$. Then:*

(i) *If A is sufficient, then so is B.*

(ii) *If $A \in \mathbf{P}_*(\kappa)$ for some $\kappa \geqslant 0$, then $B \in \mathbf{P}_*(\kappa')$, where $\kappa' \geqslant \kappa$ is such that*

$$\frac{1 + 4\kappa'}{1 + 4\kappa} = \frac{\max_{i \in N}(p_i/q_i)}{\min_{i \in N}(p_i/q_i)}.$$

In what follows, the concept of the handicap of a sufficient matrix $A \in \mathbb{R}^{n\times n}$ [4] will be of crucial importance. If $x \in \mathbb{R}^n$ and $x^T Ax < 0$, then necessarily $I_+(x) \neq \emptyset$, and the ratio

$$F(x) := \frac{-x^T Ax}{\sum_{i \in I_+(x)} x_i (Ax)_i}$$

is well defined. The *handicap* $\hat{\kappa}(A)$ of A is now defined by

$$\hat{\kappa}(A) := \begin{cases} 0 & \text{if} \quad A \in \mathbf{PSD}, \\ \frac{1}{4}\sup\{F(x) \mid x^T Ax < 0\} & \text{otherwise}. \end{cases}$$

If $\hat{\kappa}(A) < \infty$, then $\hat{\kappa}$ is the smallest nonnegative number κ such that $A \in \mathbf{P}_*(\kappa)$. So $A \in \mathbf{P}_*$ if and only if $\hat{\kappa}(A) < \infty$. Finally we state two facts about handicaps:

(1) The handicaps of a sufficient matrix and all its principal transforms are the same.

(2) The handicap of a sufficient matrix is at least as large as the handicap of any of its proper principal submatrices.

3. RESULTS

Now we proceed to establish $\mathbf{SU} \subset \mathbf{P}_*$. The proof consists in showing that every sufficient matrix has a finite handicap.

THEOREM 3.1. $\mathbf{SU} \subset \mathbf{P}_*$.

Proof. By induction on the order n of the matrix. The case $n = 1$ is trivial. Then assume that the theorem holds for matrices of order $n - 1$, and consider a sufficient matrix $A \in \mathbb{R}^{n \times n}$ where $n \geqslant 2$. If $A \in \mathbf{P}$, then it is a $\mathbf{P}_*$-matrix. Otherwise there is a principal transform of A whose trailing diagonal element is zero. So, without loss of generality, we may assume that $a_{nn} = 0$. Note that $a_{in} = a_{ni} = 0$ or $a_{in}a_{ni} < 0$ for all $i \neq n$ by Theorem 2.1. In view of Theorem 2.2 it suffices to consider a matrix A of the form

$$A = \begin{bmatrix} \hat{A} & \delta \\ -\delta^T & 0 \end{bmatrix} \tag{3.1}$$

where $\delta = (\delta_1, \dots, \delta_{n-1})$ with $\delta_i = 0$ or 1 for any i. We shall show that any sufficient matrix of the form (3.1) has a finite handicap.

If $A \in \mathbf{PSD}$, then $\hat{\kappa}(A) = 0$. Otherwise we show that $F(x)$ is uniformly bounded by a finite number in the set $\{x \in \mathbb{R}^n | x^T A \psi x < 0\}$. Let $x^0 \in \mathbb{R}^n$ be such that $x^{0T}Ax^0 = \hat{x}^{0T}\hat{A}\hat{x}^0 < 0$, where $\hat{x}^0 = (x_1^0, \dots, x_{n-1}^0)$. It will turn out that the cases where x^0 or $y^0 = Ax^0$ has a zero component or A has a zero column are almost immediately covered by the induction hypothesis. If $x_i^0 y_i^0 \neq 0$ for all $i \in N$ and all columns of A are nonzero, then a more elaborate treatment is needed.

If $x_h^0 = 0$ for some $h \in N$, then $F(x^0) \leqslant 4\hat{\kappa}(A_{N-h,N-h})$. If $x_i^0 \neq 0$ for all $i \in N$ and column $h \in N$ of A is zero, then row h of A is zero also (see Theorem 2.1) and x_h^0 may be set to zero without affecting $F(x^0)$; so we have the preceding case. In the sequel we assume that $x_i^0 \neq 0$ for all $i \in N$ and that all columns of A are nonzero. Then, by Theorem 2.1, all rows of A are nonzero, too.

If $y_h^0 = 0$ for some $h \in N$, there are two possibilities. First, if $a_{hh} > 0$, then $F(x^0) \leqslant 4\hat{\kappa}(B_{N-h,N-h})$ where $B = \mathscr{P}_h A$. Second, if $a_{hh} = 0$, then $a_{hk}a_{kh} < 0$ for some $k \in N$, see Theorem 2.1. So $F(x^0) \leqslant 4\hat{\kappa}(B_{N-h,N-h})$ where $B = \mathscr{P}_{\{h,k\}}A$.

Finally assume that $y_i^0 \neq 0$ for all $i \in N$. Because $F(\lambda x) = F(x)$ for any $\lambda \neq 0$, we may assume without loss of generality that $x_n^0 = 1$. In what follows, Σ shall mean summing over the set $N - n$, Σ_+ over the set $I_+(x^0) - n$, and Σ_- over the set $I_-(x^0) - n$. In addition, we let $\gamma = 0$ or 1 according as $x_n^0 y_n^0 = y_n^0 = -\Sigma\delta_i x_i^0$ is negative or positive. We evaluate $F(x)$ at the point $x(t) = (x_1^0, \dots, x_{n-1}^0, t)$:

$$\bar{F}(t) = \frac{-\hat{x}^{0T}\hat{A}\,\hat{x}^0}{\Sigma_+\hat{x}_i^0(\hat{y}_i^0 + t\delta_i) - \gamma t\Sigma\delta_i x_i^0} \tag{3.2}$$

where $\hat{y}^0 = \hat{A}\hat{x}^0$. This function and its derivative

$$\bar{F}'(t) = \frac{(\hat{x}^{0T}\hat{A}\,\hat{x}^0)(\Sigma_+\delta_i x_i^0 - \gamma\Sigma\delta_i x_i^0)}{\left[\Sigma_+\hat{x}_i^0(\hat{y}_i^0 + t\delta_i) - \gamma t\Sigma\delta_i x_i^0\right]^2}$$

are defined for all values of t for which the denominator in (3.2) differs from zero. We have four cases.

Case I: $\gamma = 0$, $\Sigma_+\delta_i x_i^0 < 0$. Then $\delta_i = 1$, $x_i^0 < 0$, $y_i^0 < 0$ for some $i \in I_+(x^0) - n$. Note that $y_i(t) := \hat{y}_i^0 + t$ changes sign in the interval $(1, \infty)$. We increase t from 1 until, for the first time, $y_h(t)$ for some $h \in N - n$ becomes zero at t_0, say. For all $t \in [1, t_0]$, $\bar{F}(t)$ and $\bar{F}'(t)$ are defined and $\bar{F}'(t) > 0$. Therefore $F(x^0) < \bar{F}(t_0)$, and the present case reduces to the case $y_h^0 = 0$ above.

Case II: $\gamma = 0$, $\Sigma_+\delta_i x_i^0 \geqslant 0$. We diminish t from 1 until, for the first time, it or $y_h(t)$ for some $h \in N - n$ becomes zero at $t = t_0$, say. All the time, $\bar{F}(t)$ and $\bar{F}'(t)$ are defined, and $\bar{F}'(t) \leqslant 0$. So $F(x^0) \leqslant \bar{F}(t_0)$, and the present case reduces to the case $x_n^0 = 0$ or to the case $y_h^0 = 0$ according as $t_0 = 0$ or $t_0 > 0$.

Case III: $\gamma = 1$, $\Sigma_+\delta_i x_i^0 - \gamma\Sigma\delta_i x_i^0 = -\Sigma_-\delta_i x_i^0 < 0$. Then $\delta_i = 1$, $x_i^0 > 0$, $y_i^0 < 0$ for some $i \in I_-(x^0) - n$. Continue as in case I.

Case IV: $\gamma = 1$, $\Sigma_+\delta_i x_i^0 - \gamma\Sigma\delta_i x_i^0 = -\Sigma_-\delta_i x_i^0 \geqslant 0$. Continue as in case II.

Summarizing the above discussion, we have shown that if the sufficient matrix A of (3.1) does not belong to **PSD**, then $\hat{\kappa}(A) \leqslant c$, where

$$c := \max\{\hat{\kappa}(B_{N-i,N-i}) \mid B \text{ is a principal transform of } A \text{ and } i \in N\},$$

which is finite by the induction hypothesis. [In fact, $\hat{\kappa}(A) = c$ must hold, because otherwise we would have $\hat{\kappa}(B) = \hat{\kappa}(A) < \hat{\kappa}(B_{N-i,N-i})$ for some principal transform B of A and for some $i \in N$, which is impossible.] ∎

The author is indebted to Professor Richard W. Cottle for proposing this research problem to him and for giving constructive comments on a preliminary version of the paper.

REFERENCES

1 R. W. Cottle, The principal pivoting method revisited, *Math. Programming* 48:369–385 (1990).
2 R. W. Cottle and Y.-Y. Chang, Least-index resolution of degeneracy in linear complementarity problems with sufficient matrices, *SIAM J. Matrix Anal. Appl.* 13:1131–1141 (1992).
3 R. W. Cottle and S.-M. Guu, Two characterizations of sufficient matrices, *Linear Algebra Appl.* 170:65–74 (1992).
4 R. W. Cottle and S.-M. Guu, Are $\mathbf{P}_*$-matrices just sufficient?, presented at the 36th Joint National Meeting of Operations Research Society of America and The Institute of Management Science, Phoenix, AZ, 1 Nov. 1993.
5 R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Academic, Boston, 1992.
6 R. W. Cottle, J.-S. Pang, and V. Venkateswaran, Sufficient matrices and the linear complementarity problem, *Linear Algebra Appl.* 114/115:231–249 (1989).
7 S.-M. Guu and R. W. Cottle, On a subclass of $\mathbf{P}_0$, *Linear Algebra Appl.*, 223/224:325–335 (1995).
8 M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Springer-Verlag, Berlin, 1991.
9 H. Väliaho, Criteria for sufficient matrices, *Linear Algebra Appl.*, to appear.

Received 8 June 1994; final manuscript accepted 18 July 1994