NORTH-HOLLAND

Criteria for Sufficient Matrices

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ABSTRACT

Column sufficient, row sufficient, and sufficient matrices have recently arisen in connection with the linear complementarity problem. We review and supplement the basic theory of these matrix classes, propose new criteria for identifying them, and compare these criteria with the existing ones. Our main mathematical tool is principal pivoting.

1. INTRODUCTION

The classes of column sufficient, row sufficient, and sufficient matrices have recently arisen in connection with the linear complementarity problem (LCP); see [6]. A matrix $A \in \mathbb{R}^{n\times n}$ is *column sufficient* if for all $x \in \mathbb{R}^n$

$$x_i(Ax)_i \leqslant 0, \quad i = 1,\dots,n \quad \Rightarrow \quad x_i(Ax)_i = 0, \quad i = 1,\dots,n, \quad (1.1)$$

and *row sufficient* if A^T is column sufficient. A is *sufficient* if it is both row and column sufficient. These matrix classes have an intrinsic role in the theory of the LCP [6]: row sufficient matrices are linked to the existence of solutions, and column sufficient matrices are associated with the convexity of the solution set. Sufficient matrices also have algorithmic significance for the LCP. Nondegenerate LCPs with row sufficient matrices can be processed by Lemke's method [12, 6] and by the principal pivoting method of Dantzig and

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Cottle [4, 2]. Any sufficient LCP can be solved by the principal pivoting method and the criss-cross method [11, 17]. Some parametric LCPs with sufficient matrices can also be solved, see [17].

The theory of sufficient matrices has been studied in [2, 3, 5, 6]. We review and supplement the basic theory of sufficient matrices, develop new criteria for identifying them, and compare these criteria with the existing ones. Our main mathematical tool is principal pivoting.

The organization of the paper is as follows. After some preliminaries, in Section 3 we review and supplement the basic theory of sufficient matrices. We show, for example, that if a column (row) sufficient matrix A is of rank r and some collection of r columns (rows) is linearly independent, then the corresponding principal submatrix is nonsingular. As a consequence, if some columns (rows) of A are linearly independent, then so are the corresponding rows (columns). In Section 4 we develop criteria for sufficient matrices. A matrix A is called (row, column) sufficient of order k if all its $k \times k$ principal submatrices are (row, column) sufficient. We show that $A \in \mathbb{R}^{n \times n}$ of rank $r < n$ is (row, column) sufficient if it is (row, column) sufficient of order $r + 1$. As sharpenings of this result we state two conditions under which A, of rank $r < n$ and (row, column) sufficient of order r, is (row, column) sufficient. We show also that a matrix $A \in \mathbb{R}^{n \times n}$ with positive determinant is (row, column) sufficient if it is (row, column) sufficient of order $k < n$ and A^{-1} is (row, column) sufficient of order $n - k$. Moreover, we give necessary and sufficient conditions for $A \in \mathbb{R}^{n \times n}$, (row, column) sufficient of order $n - 1$, to be (row, column) sufficient. This result is crucial for constructing practical tests for sufficiency, to be recorded in Section 5. Our criteria are inductive: to test $A \in \mathbb{R}^{n \times n}$ for (row, column) sufficiency we check sequentially for $k = 2, 3, \ldots$ whether A (or A^{-1}), (row, column) sufficient of order $k - 1$, is (row, column) sufficient of order k. We propose four criteria, two of which apply for nonsingular matrices only, and compare them with those due to Cottle and Guu [5]. All these criteria, both ours and Cottle and Guu's, are combinatorially explosive, so they are practicable for small matrices only. In such cases our best criteria are more efficient than the tests of Cottle and Guu.

2. PRELIMINARIES

If $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ (A is a real $m \times n$ matrix), we write A^T for its transpose. If $R \subset \{1, \ldots, m\}$ and $S \subset \{1, \ldots, n\}$, we denote the submatrix of A induced by rows $i \in R$ and columns $j \in S$ by A_{RS}, the row submatrix of A consisting of rows $i \in R$ by $A_{R\cdot}$, and the column submatrix of A

consisting of columns $j \in S$ by $A_{\cdot S}$, letting h stand for the singleton $\{h\}$. If A is square, we write det A or $|A|$ for its determinant, adj A for its adjoint, and $A^{[k]}$ for its leading $k \times k$ principal submatrix. A diagonal matrix $D \in \mathbb{R}^{n \times n}$ with the diagonal elements $d_1, \ldots, d_n$ is denoted by $D = \operatorname{diag}(d_1, \ldots, d_n)$. By a *principal permutation* of a square matrix we mean simultaneous permutation of the rows and the columns. In particular, we write $\mathscr{C}_{rs}$ for the principal permutation interchanging rows and columns r and s. Any vector $x \in \mathbb{R}^n$ is interpreted as an $n \times 1$ matrix and denoted by $x = [x_1, \ldots, x_n]^T$ or, for simplicity, by $x = (x_1, \ldots, x_n)$. We write x_R for the subvector of x consisting of components $i \in R$. If $x, y \in \mathbb{R}^n$, their *Hadamard product* $x * y \in \mathbb{R}^n$ is defined by

$$(x * y)_i = x_i y_i, \qquad i = 1, \ldots, n.$$

We define $N = \{1, \ldots, n\}$ and denote the complement of a set $R \subset N$ with respect to N by $\bar{R}$. We abbreviate $R + s = R \cup \{s\}$, $R - r = R \setminus \{r\}$, and $R + s - r = R \cup \{s\} \setminus \{r\}$. The cardinality of a set R is denoted by $|R|$, and the empty set by $\varnothing$. We say that $x, y \in \mathbb{R}$ have the *same sign* if $x = y = 0$ or $xy > 0$, and *opposite signs* if $x = y = 0$ or $xy < 0$. If $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ stand for the greatest integer less than or equal to x. We use the symbol $:=$ for definition.

If $A \in \mathbb{R}^{m \times n}$ and $R \subset \{1, \ldots, m\}$, $S \subset \{1, \ldots, n\}$ with $|R| = |S|$, the *pivotal operation* $\mathscr{P}_{RS}$ transforms the table

$$\mathsf{A}: \quad y = \begin{array}{c} x \\ \boxed{A} \end{array}$$

into an equivalent table $\hat{\mathsf{A}}$ (containing a matrix $\hat{A}$) in which the variables y_R and x_S have been exchanged; see e.g. [2, 14, 16]. We denote $\hat{A} = \mathscr{P}_{RS} A$ whether A and $\hat{A}$ are tables or matrices (here the elements of R and S refer to the rows and columns of A, respectively). $\mathscr{P}_{RS}$ is defined if and only if the *pivot* A_{RS} of the operation is nonsingular; see [14, Theorem 1.1]. If $R = S = \varnothing$, then $\mathscr{P}_{RS}$ is defined as the identity operation. The *principal pivotal operation* $\mathscr{P}_{RR}$ is abbreviated as $\mathscr{P}_R$. Any matrix obtained from a square matrix A by means of a principal pivotal operation followed by a principal permutation is called a *principal transform* of A. If $A = [A_{ij}] \in \mathbb{R}^{m \times n}$ is a block matrix, we let $\mathscr{P}_{(rs)}$ stand for the *block pivotal operation* with the pivot A_{rs}, abbreviating $\mathscr{P}_{(rr)}$ by $\mathscr{P}_{(r)}$. The *single pivotal operation* with the pivot a_{rs} is denoted by $\mathscr{P}_{rs}$, and the *single principal pivotal operation* with the pivot a_{rr} by $\mathscr{P}_r$. If, in $A \in \mathbb{R}^{n \times n}$, we have $a_{rr} a_{ss} = 0 \neq a_{rs} a_{sr}$, then

$\mathscr{P}_{\{r,s\}}A = \mathscr{C}_{rs}\mathscr{P}_{sr}\mathscr{P}_{rs}A$. We write $\mathscr{P}^*_{RS}$ for the *pivotal condensation*, i.e., the operation $\mathscr{P}_{RS}$ followed by the deletion of rows $i \in R$ and columns $j \in S$. In connection with $\mathscr{P}^*_{RS}$ the original numbering of the remaining rows and columns is retained. *Schur's determinantal formula* [10] is as follows:

$$\det A = \det A_{RR} \det(\mathscr{P}^*_R A), \qquad \det A_{RR} \neq 0. \tag{2.1}$$

Moreover, for any $A \in \mathbb{R}^{n\times n}$, we have by [13, (8.17)]

$$\operatorname{rank} A = \operatorname{rank} A_{RR} + \operatorname{rank}(\mathscr{P}^*_R A), \qquad \det A_{RR} \neq 0. \tag{2.2}$$

The pivotal theory applies to dual linear relations also. So we may treat the systems $y = Ax$ and $v = -A^T u$ in a *double pivoting scheme* [1], writing the former system by rows and the latter one by columns. We have, for example,

$$\begin{array}{rc|cc|cc} & & x^1 & x^2 & & \\ \hline y^1 = & & A_{11} & A_{12} & -u^1 & \mathscr{P}_{(1)} \\ y^2 = & & A_{21} & A_{22} & -u^2 & \leftrightarrow \\ \hline & & v^1 = & v^2 = & & \end{array}$$

$$\begin{array}{r|cc|c} & y^1 & x^2 & \\ \hline x^1 = & A_{11}^{-1} & -A_{11}^{-1}A_{12} & -v^1 \\ y^2 = & A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} & -u^2 \\ \hline & u^1 = & v^2 = & \end{array} \tag{2.3}$$

We shall have occasion to apply this double scheme later on.

The elements of a matrix obtained from $A \in \mathbb{R}^{m\times n}$ by means of a pivotal operation can be expressed as ratios of determinants.

THEOREM 2.1. *Let*

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \hat{G} = \mathscr{P}_{(1)}G = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix},$$

where A is nonsingular. Then

(i) $\hat{a}_{ij} = A_{ji} \div \det A$,
(ii) $\hat{b}_{ij} = \det E_{ij} \div \det A$,

(iii) $\hat{c}_{ij} = \det F_{ij} \div \det A$,

(iv) $\hat{d}_{ij} = \begin{vmatrix} A & B_{\cdot j} \\ C_{i\cdot} & d_{ij} \end{vmatrix} \div \det A$,

where

A_{ji} *is the cofactor of* a_{ji} *in* A,
E_{ij} *is obtained from* A *by replacing column* i *with* $-B_{\cdot j}$,
F_{ij} *is obtained from* A *by replacing row* j *with* $C_{i\cdot}$.

Proof. (i): Obvious.

(ii): Interpret $\hat{B}_{\cdot j} = -A^{-1}B_{\cdot j}$ as the solution to the equation $Ax = -B_{\cdot j}$, and use Cramer's rule.

(iii): Interpret $(\hat{C}_{i\cdot})^T = (C_{i\cdot}A^{-1})^T = (A^T)^{-1}C_{i\cdot}^T$ as the solution to the equation $A^T x = C_{i\cdot}^T$, and use Cramer's rule.

(iv): Note that $\hat{d}_{ij} = d_{ij} - C_{i\cdot}A^{-1}B_{\cdot j}$, and apply (2.1) to the block determinant. ■

A matrix $A \in \mathbb{R}^{n\times n}$ is called a $\mathbf{P}_0$-*matrix* ($\mathbf{P}$-*matrix*) if all its principal minors are nonnegative (positive), and a $\mathbf{P}_1$-*matrix* if all its principal minors are positive except one, which is zero. All the $\mathbf{P}_0$-matrices ($\mathbf{P}$-matrices) form the class $\mathbf{P}_0$ ($\mathbf{P}$).

THEOREM 2.2 [9, 8]. *$A \in \mathbb{R}^{n\times n}$ is a $\mathbf{P}_0$-matrix ($\mathbf{P}$-matrix) if and only if for every nonzero vector $x \in \mathbb{R}^n$ there exists an index k such that $x_k \neq 0$ and $x_k y_k \geqslant 0$ (> 0), where $y = Ax$.*

In this study we shall be concerned with column sufficient, row sufficient, and sufficient matrices, as defined in Section 1. In addition, a vacuous matrix is defined to be (row, column) sufficient. Note that (1.1) is equivalent to

$$x * (Ax) \leqslant 0 \quad \Rightarrow \quad x * (Ax) = 0.$$

Examples of sufficient matrices are nonnegative definite matrices and $\mathbf{P}$-matrices. There are, however, also sufficient matrices not belonging to these classes.

We say that $A \in \mathbb{R}^{n\times n}$ is sufficient (row sufficient, column sufficient, $\mathbf{P}_0$, $\mathbf{P}$) of *order* k ($0 \leqslant k \leqslant n$) if every $k \times k$ principal submatrix of it belongs to the class in question.

For most results we shall verify the column sufficient case only, the proofs of the other cases being analogous.

3. BASIC THEORY

We shall review and supplement the basic theory of sufficient matrices developed in [2, 3, 5, 6].

THEOREM 3.1 [2]. *If $A \in \mathbb{R}^{n\times n}$ is (row, column) sufficient, then so is* (i) *any principal permutation of A,* (ii) *any principal submatrix of A, and* (iii) *any principal transform of A.*

LEMMA 3.1. *If $A \in \mathbb{R}^{n\times n}$ is (row, column) sufficient of order k and $B = \mathscr{P}_R A$ with $R \subset N$, $|R| = h < k$, then B is (row, column) sufficient of order $k - h$.*

Proof. (The column sufficient case). Consider any submatrix B_{SS} of B where $S \subset N$, $|S| \leqslant k - h$. Because $|R \cup S| \leqslant k$, $A_{R\cup S, R\cup S}$ is column sufficient, and so are $B_{R\cup S, R\cup S}$ and B_{SS}. ■

THEOREM 3.2 [6]. *If $A \in \mathbb{R}^{n\times n}$ is (row, column) sufficient, then $A \in \mathbf{P}_0$. In particular, A has nonnegative diagonal elements.*

THEOREM 3.3 [2]. *Let $A \in \mathbb{R}^{n\times n}$ with $a_{kk} = 0$. Then:*

(i) *If A is column sufficient, then $a_{ik} = 0$ or $a_{ik}a_{ki} < 0$ for all $i \neq k$.*
(ii) *If A is row sufficient, then $a_{ki} = 0$ or $a_{ik}a_{ki} < 0$ for all $i \neq k$.*
(iii) *If A is sufficient, then $a_{ik} = a_{ki} = 0$ or $a_{ik}a_{ki} < 0$ for all $i \neq k$.*

THEOREM 3.4. *Let $A \in \mathbb{R}^{n\times n}$, of rank $r < n$, be column (row) sufficient, and let $R \subset N$, $|R| = r$, be such that columns (rows) $i \in R$ are linearly independent. Then A_{RR} is nonsingular.*

Proof. (The column sufficient case). By induction on r. If $r = 0$, there is nothing to prove. Then assume that the theorem holds for matrices of rank $< r$, and let A be of rank $r \geqslant 1$. Let R be as stated in the theorem, and assume, without loss of generality, that $R = \{1, \ldots, r\}$. Assume, on the contrary, that rank $A_{RR} = k < r$. Let $S \subset R$, $|S| = k$, be such that columns $j \in S$ of A_{RR} are linearly independent. Then, by the induction hypothesis,

A_{SS} is nonsingular. The matrix $B = \mathscr{P}_S^* A$ is of the form

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where the zero block is of order $r - k$ and rank $B_{21} = r - k$. Theorem 3.3(i) implies that $B_{12} \neq 0$. So

$$\text{rank } A = k + \text{rank } B \geqslant k + \text{rank } B_{21} + \text{rank } B_{12} > r$$

[see (2.2)], a contradiction. ■

An analogous result holds for sufficient matrices.

THEOREM 3.5. *If $A \in \mathbb{R}^{n \times n}$ is column (row) sufficient and columns (rows) $i \in S$ of A are linearly independent, then so are rows (columns) $i \in S$ of A. If A is sufficient, then rows $i \in S$ are linearly independent if and only if columns $i \in S$ are.*

Proof. (The column sufficient case). Let A be of rank r, and let R, $S \subset R \subset N$, $|R| = r$, be such that columns $i \in R$ of A are linearly independent. Then A_{RR} is nonsingular. Hence rows $i \in R$ of A are linearly independent, and so are rows $i \in S$ of A. ■

THEOREM 3.6. *Let $A \in \mathbb{R}^{n \times n}$ be column sufficient, let columns $j \in S$ of A be linearly independent, and let $A_{k\cdot} = v^T A_{S\cdot}$, where $k \notin S$. Then there exists a u such that $A_{\cdot k} = A_{\cdot S} u$, where necessarily $u_i = 0$ or $u_i v_i > 0$ for all i.*

Proof. Let A be of rank r, let $|S| = h$, and let R, $S \subset R \subset N$, $|R| = r$, be such that columns $i \in R$ of A are linearly independent. Without loss of generality assume that $S = \{1, \ldots, h\}$, $R = \{1, \ldots, r\}$, $k = n$. We solve the equations $A_{\cdot n} = A_{\cdot S} u$ and $A_{n\cdot} = v^T A_{S\cdot}$ using the double pivoting scheme; cf. (2.3). The initial table is

$$A: \quad \begin{array}{r|cccc|c} & u & 0 & 0 & -1 & \\ \hline 0 = & A_{11} & A_{12} & A_{13} & A_{14} & -v \\ 0 = & A_{21} & A_{22} & A_{23} & A_{24} & 0 \\ 0 = & A_{31} & A_{32} & A_{33} & A_{34} & 0 \\ 0 = & A_{41} & A_{42} & A_{43} & a_{nn} & 1 \\ \hline & 0 = & 0 = & 0 = & 0 = & \end{array}$$

where A_{11} and A_{22} are of orders h and $r-h$, respectively. The operation $\mathscr{P}_R$ leads from A to the table

$$B:\quad \begin{array}{r|cccc|c} & 0 & 0 & 0 & -1 & \\ \hline u= & B_{11} & B_{12} & B_{13} & B_{14} & 0 \\ 0= & B_{21} & B_{22} & B_{23} & B_{24} & 0 \\ 0= & B_{31} & B_{32} & 0 & 0 & 0 \\ 0= & B_{41} & B_{42} & 0 & 0 & 1 \\ \hline & v= & 0= & 0= & 0= & \end{array}$$

Here $B_{42}=0$ (because $A_{n\cdot}=v^TA_{S\cdot}$ has a solution), implying $B_{24}=0$ (because B is column sufficient). So $A_{\cdot n}=A_{\cdot S}u$ has the unique solution $u=-B_{14}$. The unique solution of the equation $A_{n\cdot}=v^TA_{S\cdot}$ is $v=B_{41}^T$. Finally, it follows from the column sufficiency of B that $u_i=0$ or $u_iv_i>0$ for all i. ■

Analogous results hold for sufficient and row sufficient matrices.

4. CRITERIA FOR SUFFICIENT MATRICES

We shall develop criteria for (row, column) sufficient matrices. We begin with by giving two basic results.

THEOREM 4.1 [5]. *$A\in\mathbb{R}^{n\times n}$ is (row, column) sufficient if and only if every matrix obtained from it by means of a principal pivotal operation is (row, column) sufficient of order two.*

REMARK 4.1. There is a minor defect in the proof of [5, Theorem 1]. In case I, the authors have not noted that, in their M, column k may be zero while row k is nonzero. This defect is easily remedied by replacing x_k with zero.

THEOREM 4.2. *If $A\in\mathbb{R}^{n\times n}$ is (row, column) sufficient of order $n-1$ and* $\det A>0$, *then A is (row, column) sufficient.*

Proof. (The column sufficient case). Note that $A\in\mathbf{P}_0$. Assume, on the contrary, that there is an $x\in\mathbb{R}^n$ such that $x*y\leqslant 0$ and $x*y\neq 0$ where

$y = Ax$. We have two cases:

(i) $x_k = 0$ *for some* k. Then $x_{N-k} * y_{N-k} \leqslant 0$ and $x_{N-k} * y_{N-k} \neq 0$, contradicting the column sufficiency of $A_{N-k,N-k}$.

(ii) $x_i \neq 0$ *for all* i. Without loss of generality assume that $x > 0$, $y \leqslant 0$ (if $x_i < 0$, multiply x_i, y_i, row i of A, and column i of A by -1). Letting $z = -A^{-1}e$ with $e = (1, \dots, 1) \in \mathbb{R}^n$ and choosing a real number $c > 0$ so small that $\bar{x} := x + cz > 0$, we have $\bar{y} := A\bar{x} = y - ce < 0$, contradicting Theorem 2.2. ■

In order to apply Theorem 4.1 it is necessary to be able to identify 2×2 (row, column) sufficient matrices; see [2, Propositions 8 and 9] and [5, Lemmas 1, 2, and 5]. We given another characterization.

THEOREM 4.3. *Let* $A \in \mathbb{R}^{2\times 2}$ *have nonnegative diagonal. If* $\det A > 0$, *then* A *is sufficient. If* $\det A = 0$, *then*

(i) *A is column sufficient if and only if* $a_{ii} = 0 \Rightarrow A_{\cdot i} = 0$ *for* $i = 1, 2$;

(ii) *A is row sufficient if and only if* $a_{ii} = 0 \Rightarrow A_{i\cdot} = 0$ *for* $i = 1, 2$;

(iii) *A is sufficient if and only if* $a_{ii} = 0 \Rightarrow A_{i\cdot} = 0$ *and* $A_{\cdot i} = 0$ *for* $i = 1, 2$.

In particular, any 2×2 $\mathbf{P}_0$*-matrix with positive diagonal is sufficient.*

Proof. (The column sufficient case). If $\det A > 0$, use Theorem 4.2. If $\det A = 0$, use [5, Lemma 1], considering separately the following three cases: (a) $a_{11} = a_{22} = 0$, (b) exactly one of a_{11}, a_{22} is positive, and (c) $a_{11}a_{22} = a_{12}a_{21} > 0$. ■

In the following two theorems we give some consequences of Theorems 4.1 and 4.2 for singular and nonsingular matrices.

THEOREM 4.4. *$A \in \mathbb{R}^{n\times n}$ of rank $r < n$ is (row, column) sufficient if and only if it is (row, column) sufficient of order $r + 1$.*

Proof. (The column sufficient case). *Necessity*: Obvious.

Sufficiency: We apply Theorem 4.1. Let $B = \mathscr{P}_R A$, $R \subset N$, and let $S \subset N$, $|S| = 2$. Clearly, $|R| \leqslant r$. We have to show that B_{SS} is column sufficient. There are two cases:

(i) $|R \cup S| \leqslant r + 1$. Then $A_{R\cup S, R\cup S}$ is column sufficient, and so are $B_{R\cup S, R\cup S}$ and B_{SS}.

(ii) $|R| = r$, $R \cap S = \varnothing$. Now $B_{SS} = 0$; see (2.2). ■

THEOREM 4.5. *If* $A \in \mathbb{R}^{n\times n}$ *with* $\det A > 0$ *is (row, column) sufficient of order* k $(1 \leqslant k \leqslant n-1)$ *and* A^{-1} *is (row, column) sufficient of order* $n-k$, *then* A *is (row, column) sufficient.*

Proof. (The column sufficient case). If $k = 1$ or $n-1$, apply Theorem 4.2 to A^{-1} or A. Assume then that $2 \leqslant k \leqslant n-2$. We shall apply Theorem 4.1. Let $C = A^{-1}$, $B = \mathscr{P}_R A = \mathscr{P}_{\bar{R}} C$ with $R \subset N$, and $S \subset N$ with $|S| = 2$. We have to show that B_{SS} is column sufficient. We have three cases.

Case I: $|R \cup S| \leqslant k$. Now $A_{R\cup S, R\cup S}$ is column sufficient, and so are $B_{R\cup S, R\cup S}$ and B_{SS}.

Case II: $|R \cup S| \geqslant k+2$. We have

$$|R \cup S| + |\bar{R} \cup S| = |R| + |\bar{R} \cap S| + |\bar{R}| + |R \cap S| = n+2,$$

whence $|\bar{R} \cup S| = n + 2 - |R \cup S| \leqslant n-k$. Now $C_{\bar{R}\cup S, \bar{R}\cup S}$ is column sufficient, and so are $B_{\bar{R}\cup S, \bar{R}\cup S}$ and B_{SS}.

Case III: $|R \cup S| = k+1$. We have three subcases:

(i) $|R| = k+1$, $S \subset R$. Now the nonsingular matrix A_{RR}, of order $k+1$, is column sufficient of order k. Moreover,

$$\det A_{RR} = \det A \div \det B_{\bar{R}\bar{R}} = \det A \det C_{\bar{R}\bar{R}} > 0.$$

Thus A_{RR} is column sufficient (see Theorem 4.2), and so are B_{RR} and B_{SS}.

(ii) $|R| = k-1$, $S \subset \bar{R}$. Apply (i) with the roles of A, R and A^{-1}, $\bar{R}$ interchanged.

(iii) $|R| = k$, $S = \{s, t\}$ with $s \in R$, $t \in \bar{R}$. Now B_{RR} and $B_{\bar{R}\bar{R}}$ are column sufficient. If $b_{ss} > 0$, apply (ii) with R replaced by $R - s$. If $b_{tt} > 0$, apply (i) with R replaced by $R + t$. There remains the case $b_{ss} = b_{tt} = 0$. Assume that B_{SS} is not column sufficient. There are two possibilities. Firstly, if $b_{ts} \neq 0$, $b_{st} b_{ts} \geqslant 0$, then there is an $h \in R$ such that $b_{hs} \neq 0$ (then $b_{hs} b_{sh} < 0$). Consider the matrix $D := \mathscr{C}_{hs} \mathscr{P}_{sh} \mathscr{P}_{hs} B = \mathscr{P}_{R\setminus\{h,s\}} A$, which is column sufficient of order two (see Lemma 3.1), contradicting the fact that $d_{hh} = 0 \neq d_{th}$ and $d_{ht} d_{th} \geqslant 0$. Secondly, if $b_{ts} = 0 \neq b_{st}$, apply the preceding case with the roles of A, R, s and A^{-1}, $\bar{R}$, t interchanged. ∎

In Theorems 4.7–4.8 below we sharpen Theorem 4.4. As a preliminary result we state the following.

THEOREM 4.6. *The matrix*

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where the zero blocks are square, is (row, column) sufficient if and only if all the corresponding minors of B and $-C^T$ have the same sign.

Proof. (The column sufficient case). Let B be of order $m \times (n-m)$, and let the columns of B and the rows of C be numbered $m+1, \dots, n$.

Necessity: Let $R \subset M := \{1, \dots, m\}$ and $S \subset \overline{M}$ with $|R| = |S|$. By Theorem 3.2, $\det A_{R \cup S, R \cup S} = \det B_{RS} \det(-C_{SR}) \geqslant 0$. Moreover,

$$\det B_{RS} \neq 0$$

$$\Rightarrow \text{columns } i \in S \text{ of } A_{R \cup S, R \cup S} \text{ are linearly independent}$$

$$\Rightarrow \text{rows } i \in S \text{ of } A_{R \cup S, R \cup S} \text{ are linearly independent (by Theorem 3.5)}$$

$$\Rightarrow \det C_{SR} \neq 0.$$

Analogously, $\det C_{SR} \neq 0 \Rightarrow \det B_{RS} \neq 0$.

Sufficiency: We apply Theorem 4.1. If $D = \mathscr{P}_T A$, $T \subset N$, then T is of the form $T = R \cup S$, where $R \subset M$, $S \subset \overline{M}$, $|R| = |S|$, and $A_{RS} = B_{RS}$ and $A_{SR} = C_{SR}$ are nonsingular. So D is a principal permutation of $E := \mathscr{P}_{SR}\mathscr{P}_{RS} A$; see [14, Lemma 1.1]. Here $\mathscr{P}_{RS}$ affects only B and $\mathscr{P}_{SR}$ only C. It suffices to show that e_{ij} and e_{ji} have opposite signs for all $(i, j) \in M \times \overline{M}$. This follows, however, from Theorem 2.1. ■

THEOREM 4.7. *Let $A \in \mathbb{R}^{n \times n}$ be of rank r $(1 \leqslant r \leqslant n-1)$. Then:*

(i) *If A is column (row) sufficient of order r, then it is column (row) sufficient if and only if for any $R \subset N$ with $|R| = r$,*

$$\det A_{RR} = 0 \quad \Rightarrow \quad \textit{the columns (rows) } i \in R \textit{ of } A \textit{ are linearly dependent.}$$

(ii) *If A is sufficient of order r, then it is sufficient if and only if, for any $R \subset N$ with $|R| = r$,*

$$\det A_{RR} = 0 \quad \Rightarrow \quad \textit{the rows and columns } i \in R \textit{ of } A \textit{ are linearly dependent.}$$

Proof. (The column sufficient case). *Necessity*: See Theorem 3.4.

Sufficiency: Assume that A is not column sufficient. In view of Theorem 4.1 and the proof of Theorem 4.4 we have the following two possibilities:

(a) There is an $R \subset N$, $|R| = r-1$, and an $S = \{s, t\} \subset \overline{R}$ with $s \neq t$ such that, in $B := \mathscr{P}_R A$, the submatrix B_{SS} is not column sufficient. B_{SS}

must be singular, because otherwise we would have rank $A > r$; see (2.2). Without loss of generality we may assume that $b_{ss} = b_{st} = 0 \neq b_{ts}$ (see Theorem 4.3). But then $A_{R+s,\,R+s}$ is singular (because $b_{ss} = 0$), and columns $i \in R + s$ of A are linearly independent (because $b_{ts} \neq 0$), a contradiction.

(b) There is an $R \subset N$, $|R| = r$, and an $S = \{s, t\}$ with $s \in R$, $t \in \overline{R}$ such that, in $B := \mathscr{P}_R A$, the submatrix B_{SS} is not column sufficient. Note that $b_{tt} = 0$. If $b_{ss} > 0$, apply (a) with R replaced by $R - s$. Assume then that $b_{ss} = 0$. The case $b_{ts} \neq 0$, $b_{st} b_{ts} \geqslant 0$ is impossible; cf. the proof of Theorem 4.5, case III(iii). In the remaining case $b_{ts} = 0 \neq b_{st}$ we choose an $h \in R$ such that $b_{hs} \neq 0$ (then $b_{sh} \neq 0$). In the matrix $C := \mathscr{P}_{R \setminus \{h,s\}} A = \mathscr{P}_{\{h,s\}} B = \mathscr{C}_{hs} \mathscr{P}_{sh} \mathscr{P}_{hs} B$ we have $c_{hh} = c_{th} = 0$, $c_{ht}, c_{sh} \neq 0$. So A_{TT} with $T = R + t - s$ is singular while the columns $j \in T$ of A are linearly independent, a contradiction. ■

COROLLARY 4.1. *Let $A \in \mathbb{R}^{n \times n}$ be of rank one and have nonnegative diagonal. Then:*

(i) *A is column sufficient if and only if $a_{ii} = 0 \Rightarrow A_{\cdot i} = 0$ for all $i \in N$.*
(ii) *A is row sufficient if and only if $a_{ii} = 0 \Rightarrow A_{i \cdot} = 0$ for all $i \in N$.*
(iii) *A is sufficient if and only if $a_{ii} = 0 \Rightarrow A_{i \cdot} = 0$ and $A_{\cdot i} = 0$ for all $i \in N$.*

In particular, A is sufficient if it has positive diagonal.

COROLLARY 4.2. *Let $x, y \in \mathbb{R}^n \setminus \{0\}$ and $A = yx^T \in \mathbb{R}^{n \times n}$. Then:*

(i) *A is column sufficient if and only if $x_i = 0$ or $x_i y_i > 0$ for all $i \in N$.*
(ii) *A is row sufficient if and only if $y_i = 0$ or $x_i y_i > 0$ for all $i \in N$.*
(iii) *A is sufficient if and only if $x_i = y_i = 0$ or $x_i y_i > 0$ for all $i \in N$.*

COROLLARY 4.3. *If $A \in \mathbb{R}^{n \times n}$, of rank r, is* **P** *of order r, then it is sufficient.*

COROLLARY 4.4 (see [5, p. 66]). *Any* $\mathbf{P}_1$*-matrix is sufficient.*

Proof. See [7, p. 211] and apply Corollary 4.3. ■

THEOREM 4.8. *Let $A \in \mathbb{R}^{n \times n}$ be of rank $r < n$ and column (row) sufficient of order r, and let columns (rows) $i \in R \subset N$ with $|R| = r$ of A be linearly independent. If A_{RR} is singular, then A is not column (row) sufficient. Otherwise, denoting $B = \mathscr{P}_R A$, A is column (row) sufficient if and only if any nonzero minor of $B_{R\overline{R}}$ ($B_{\overline{R}R}$) and the corresponding minor of $-B_{\overline{R}R}^T$ ($-B_{R\overline{R}}^T$) have the same sign.*

Proof. (The column sufficient case). For the case $\det A_{RR} = 0$ refer to Theorem 3.4. Then assume that $\det A_{RR} > 0$.

Necessity is established like $\det B_{RS} \neq 0 \Rightarrow \det C_{RS} \neq 0$ in the proof of the necessity part of Theorem 4.6.

Sufficiency: We use Theorem 4.7 (i). Now $B_{\bar{R}\bar{R}} = 0$ and, by [15, Theorem 1],

$$\begin{aligned}&\{H \subset N \,|\, |H| = r, \det A_{HH} = 0\}\\ &\qquad = \{R \setminus S \cup T \,|\, S \subset R, T \subset \bar{R}, |S| = |T|, \det B_{ST} = 0\}. \quad (4.1)\end{aligned}$$

Take any sets S, T satisfying the conditions in (4.1). We have to show that columns $i \in R \setminus S \cup T$ of A are linearly dependent or, equivalently, that the equation $y = Ax$ with $y = 0$, $x_S = 0$, $x_{\bar{R} \setminus T} = 0$ has a solution $x \neq 0$. Now $y = Ax$ is equivalent to $w = Bz$, where w is obtained from y by replacing y_R with x_R and z is obtained from x by replacing x_R with y_R. It is easy to see that this equation has a solution $(w, z) \neq 0$: Take $\bar{x}_T \neq 0$ such that $B_{ST}\bar{x}_T = 0$; then z with $z_T = x_T = \bar{x}_T$, $z_{\bar{T}} = 0$ yields a w with $w_{R \setminus S} = x_{R \setminus S} = B_{R \setminus S, T}\bar{x}_T$, $w_{\bar{R} \cup S} = 0$. So the equation $Ax = 0$ has a nontrivial solution. ∎

An analogous result holds for sufficient matrices.

COROLLARY 4.5. *Let $A \in \mathbb{R}^{n \times n}$ be of rank $n - 1$, and let $B = \mathscr{P}_{N-k} A$. Then*:

(i) *If A is column (row) sufficient of order $n - 1$, then it is column (row) sufficient if and only if $b_{ik} = 0$ ($b_{ki} = 0$) or $b_{ik}b_{ki} < 0$ for all $i \neq k$.*

(ii) *If A is sufficient of order $n - 1$, then it is sufficient if and only if $b_{ik} = b_{ki} = 0$ or $b_{ik}b_{ki} < 0$ for all $i \neq k$.*

THEOREM 4.9. *Let $A \in \mathbb{R}^{n \times n}$ be column (row) sufficient of order $n - 1$. Then it is not column (row) sufficient if and only if either* (i) $\det A < 0$ *or* (ii) rank $A = n - 1$ *and there is a $k \in N$ such that $A_{N-k, N-k}$ is singular while columns (rows) $i \in N - k$ are linearly independent.*

Proof. *Sufficiency*: See Theorems 3.2 and 3.4.

Necessity: If $\det A \geqslant 0$, then A is singular by Theorem 4.2. If rank $A < n - 1$, then A is column (row) sufficient by Theorem 4.4. So rank $A = n - 1$. The existence of a k as mentioned in the theorem is guaranteed by Theorem 4.7. ∎

An analogous result holds for sufficient matrices.

5. PRACTICALITIES

Finding out whether a given matrix $A \in \mathbb{R}^{n\times n} \setminus \{0\}$ is (row, column) sufficient may be very laborious even for moderate values of n. If n is small, one may use Theorems 4.2–4.5, 4.7, and 4.8, and Corollaries 4.1 and 4.5. For example:

$n = 2$. Use Theorem 4.3.

$n = 3$. If $\det A > 0$, use Theorem 4.2. If $\operatorname{rank} A = 2$, use Theorem 4.7. If $\operatorname{rank} A = 1$, apply Corollary 4.1.

$n = 4$. If $\det A > 0$, use Theorem 4.2 or 4.5 (with $k = 2$); if A is singular, use Theorem 4.8.

$n = 5$. If $\det A > 0$, use Theorem 4.5 (with $k = 3$); if A is singular, use Theorem 4.8.

When testing $A \in \mathbb{R}^{n\times n}$ for (row, column) sufficiency, we check sequentially for $k = 2, 3, \dots$ whether A (or A^{-1}), (row, column) sufficient of order $k - 1$, is (row, column) sufficient of order k. We shall develop a procedure for accomplishing this. The procedure will be based on Corollary 4.5 and the following lemma.

LEMMA 5.1. *Let $A \in \mathbb{R}^{n\times n}$, of rank r, be (row, column) sufficient of order $n - 1$. Assume that single or double principal pivotal condensations are applied to A as long as possible. Then:*

(i) *If $r \neq n - 1$, the process can be continued until a zero matrix (possibly vacuous) of order $n - r$ ensues.*

(ii) *If $r = n - 1$, the process can be continued until one of the following three kinds of matrices ensues:*

$$[0], \qquad \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}, \quad a, b \neq 0. \tag{5.1}$$

Proof. By induction on n. The cases $n = 1$ and $n = 2$ are trivial. Assume then that the lemma holds for all orders $< n$, and let $A \in \mathbb{R}^{n\times n}$, $n \geqslant 3$, satisfy the assumption. If $A = 0$, there is nothing to prove. If $A \neq 0$ and the diagonal of A is nonzero, choose a $k \in N$ such that $a_{kk} > 0$ and apply the induction hypothesis to $B := \mathscr{P}_k^* A$; see (2.2) and Lemma 3.1. If $A \neq 0$ and $a_{ii} = 0$ for all $i \in N$, choose $(h, k) \in N \times N$ such that $a_{hk} \neq 0$. Then $a_{kh} \neq 0$ because A is (row, column) sufficient of order two. Finally, apply the induction hypothesis to $B := \mathscr{P}_{\{h,k\}}^* A$. ∎

REMARK 5.1. Consider Lemma 5.1. In (i), A is (row, column) sufficient if and only if $\det A \geqslant 0$. If, in case (ii), the terminal matrix is 2×2, then A

is not (row, column) sufficient. If the terminal matrix is [0], let $k \in N$ be such that $\mathscr{P}^*_{N-k} A = [0]$, and let $B = \mathscr{P}_{N-k} A$. Then Corollary 4.5 tells us whether A is (row, column) sufficient or not.

PROCEDURE 5.1 [Checking whether $A \in \mathbb{R}^{n \times n}$, column (row) sufficient of order $n-1$, is column (row) sufficient].

S1: Set $B = A$, $k = n$, $K = N$.
S2: If $B_{KK} \neq 0$ go to S3.
If $k \geqslant 2$, stop; A is column (row) sufficient.
If $k = 1$, stop; letting $K = \{h\}$, A is column (row) sufficient if and only if $b_{ih} = 0$ ($b_{hi} = 0$) or $b_{ih} b_{hi} < 0$ for all $i \in N - h$.
S3: If $k > 1$, go to S4. If $k = 1$, stop; letting $K = \{h\}$, A is column (row) sufficient if and only if $b_{hh} > 0$.
S4: If B_{KK} has zero diagonal, go to S5. Otherwise choose an $i \in K$ such that $b_{ii} > 0$, set $B \leftarrow \mathscr{P}_i B$, $K \leftarrow K - i$, $k \leftarrow k - 1$, and go to S2.
S5: If $k = 2$, stop; A is column (row) sufficient if and only if $\det B_{KK} > 0$.
If $k > 2$, choose $(i, j) \in K \times K$ such that $b_{ij} \neq 0$, set $B \leftarrow \mathscr{P}_{\{i,j\}} B$, $K \leftarrow K \setminus \{i, j\}$, $k \leftarrow k - 2$, and go to S2.

With a minor modification in step S2, case $k = 1$, Procedure 5.1 checks whether $A \in \mathbb{R}^{n \times n}$, sufficient of order $n - 1$, is sufficient. In case A is nonsingular, the last single or double principal pivot is omitted in Procedure 5.1. So the procedure requires at most $n - 1$ single pivots, i.e., at most $n^2(n-1)$ operations (multiplications or divisions). Consequently, checking whether $A \in \mathbb{R}^{n \times n}$ which is (row, column) sufficient of order $k - 1$ is (row, column) sufficient of order k requires at most $k^2(k-1)\binom{n}{k}$ operations, where $\binom{n}{k}$ is a binomial coefficient.

Procedure 5.1 can be improved by replacing full pivotal operations with pivotal condensations. We shall need the following result.

THEOREM 5.1. *Let $A \in \mathbb{R}^{n \times n}$, of rank $n - 1$, be (row, column) sufficient of order $n - 1$, and let $x, y \in \mathbb{R}^n$ be the right and left eigenvectors, respectively, associated with the zero eigenvalue of A. Then A is (row, column) sufficient if and only if yx^T or $-yx^T$ is.*

Proof. (The column sufficient case). We have three cases according to the terminal matrices given in (5.1):

(i) *The process mentioned in Lemma* 5.1 *terminates in* [0]. Without loss of generality assume that $A^{[n-1]}$ is nonsingular. We determine x and y using the double pivoting scheme [cf. (2.3)]:

$$\begin{array}{rcc|r} & x^1 & x_n & \\ 0 = & A_{11} & A_{12} & -y^1 \\ 0 = & A_{21} & a_{nn} & -y_n \\ & 0 = & 0 = & \end{array} \quad \xrightarrow{\mathscr{P}_{(1)}} \quad \begin{array}{rcc|r} & 0 & x_n & \\ x^1 = & B_{11} & B_{12} & 0 \\ 0 = & B_{21} & 0 & -y_n \\ & y^1 = & 0 = & \end{array} \tag{5.2}$$

Choosing x_n, y_n so that $x_n y_n > 0$, we obtain, for example, $x^T = [B_{12}^T, 1]$, $y^T = [-B_{21}, 1]$. By Corollaries 4.2 and 4.5, A is column sufficient if and only if yx^T is. If x_n, y_n are chosen so that $x_n y_n < 0$, then A is column sufficient if and only if $-yx^T$ is.

(ii) *The process mentioned in Lemma* 5.1 *terminates in the second matrix of* (5.1). Without loss of generality assume that $A^{[n-2]}$ is nonsingular. To determine x and y we pass from the first table of (5.2) to

$$\begin{array}{rccc|r} & 0 & x_{n-1} & x_n & \\ x^2 = & B_{11} & B_{12} & B_{13} & 0 \\ 0 = & B_{21} & 0 & a & -y_{n-1} \\ 0 = & B_{31} & 0 & 0 & -y_n \\ & y^2 = & 0 = & 0 = & \end{array}$$

where $x^2 = (x_1, \ldots, x_{n-2})$, $y^2 = (y_1, \ldots, y_{n-2})$, and $a \neq 0$. We must choose $x_n = 0 \neq x_{n-1}$ and $y_{n-1} = 0 \neq y_n$. But then none of A, yx^T and $-yx^T$ is column sufficient.

(iii) *The process mentioned in Lemma* 5.1 *terminates in the third matrix of* (5.1). This case is analogous to (ii). ■

COROLLARY 5.1. *If* $A \in \mathbb{R}^{n\times n}$, *of rank* $n-1$, *is (row, column) sufficient, then* adj A *is (column, row) sufficient.*

Proof. (The column sufficient case). Let x and y be, respectively, the right and left eigenvectors of A associated with the zero eigenvalue. Without loss of generality assume that $\det A^{[n-1]} > 0$ and $x_n = y_n = 1$ (see the proof of Theorem 5.1). Because $A(\operatorname{adj} A) = A^T(\operatorname{adj} A)^T = 0$, the columns (rows) of adj A are multiples of x (y^T), whence $\operatorname{adj} A = \alpha xy^T = \alpha(yx^T)^T$, where $\alpha = \det A^{[n-1]}$. Finally apply Theorem 5.1. ■

We return to the improvement of Procedure 5.1. In view of Remark 5.1, it is sufficient to use pivotal condensations instead of pivotal operations except in case (ii) of Lemma 5.1, as the matrix [0] ensues. In this case we apply Theorem 5.1. The vectors x and y can be determined with the aid of Gaussian elimination. To illustrate this we give a small numerical example.

EXAMPLE 5.1. Check whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & -2 \\ 1 & -1 & 1 & -3 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

is (row, column) sufficient. A is of rank 3 and sufficient of order 3, and $A^{[3]}$ is nonsingular. We determine the right eigenvector x and the left eigenvector y of A associated with the zero eigenvalue using pivotal condensations in the double scheme (the pivots are in boldface):

$$A:\quad \begin{array}{r|cccc|l} & x_1 & x_2 & x_3 & x_4 & \\ \hline 0 = & \mathbf{1} & 1 & 1 & -1 & -y_1 \\ 0 = & 1 & 1 & 2 & -2 & -y_2 \\ 0 = & 1 & -1 & 1 & -3 & -y_3 \\ 0 = & 0 & 2 & 1 & 1 & -y_4 \\ \hline & 0 = & 0 = & 0 = & 0 = & \end{array}$$

$$\rightarrow \quad \overline{B}:\quad \begin{array}{r|ccc|l} & x_2 & x_3 & x_4 & \\ \hline 0 = & 0 & \mathbf{1} & -1 & -y_2 \\ 0 = & -2 & 0 & -2 & -y_3 \\ 0 = & 2 & 1 & 1 & -y_4 \\ \hline & 0 = & 0 = & 0 = & \end{array}$$

$$\rightarrow \quad \overline{C}:\quad \begin{array}{r|cc|l} & x_2 & x_4 & \\ \hline 0 = & \mathbf{-2} & -2 & -y_3 \\ 0 = & 2 & 2 & -y_4 \\ \hline & 0 = & 0 = & \end{array}$$

$$\rightarrow \quad \overline{D}:\quad \begin{array}{r|c|l} & x_4 & \\ \hline 0 = & 0 & -y_4 \\ \hline & 0 = & \end{array}$$

We determine x and y by back substitution, taking $x_4 = y_4 = 1$. From the pivot row and column of table $\overline{C}$ we obtain $x_2 = -1$, $y_3 = 1$. Similarly, table $\overline{B}$ yields $x_3 = 1$, $y_2 = -1$ (at this point we may check that $x_3 y_3 > 0$, $x_2 y_2 > 0$). Finally, from table A we obtain $x_1 = 1$, $y_1 = 0$. So the matrix A is row but not column sufficient.

Think of using the improved method to test whether $A \in \mathbb{R}^{n \times n}$, (row, column) sufficient of order $n-1$, is (row, column) sufficient. Then, in the forward phase, at most $n(n^2-1)/3$ operations are needed. In the possible back substitution one may compare the signs of x_i and y_i for any i as soon as x_i and y_i have been generated. So one may find that A is not row or column sufficient before the whole vectors x and y are at hand; see Theorem 5.1 and Corollary 4.2. Calculating the first, second, third,... component of x to be determined in the back substitution requires at most 0, 1, 2,... operations. So, to compute x at most $n(n-1)/2$ operations are needed. The same holds for y. It follows that determining x and y requires altogether at most $n(n-1)(n+4)/3$ operations. This is to be compared with the upper bound $n^2(n-1)$ in Procedure 5.1. The memory requirements in the improved procedure are not essentially greater than in the original procedure. The following example illustrates this.

EXAMPLE 5.2. In Example 5.1 the tables A, $\overline{B}$, $\overline{C}$, and $\overline{D}$ need not be saved separately for back substitution, because it is possible to write table $\overline{B}$ over A, table $\overline{C}$ over $\overline{B}$, and table $\overline{D}$ over $\overline{C}$. To see this, define $B = \mathscr{P}_1 A$, $C = \mathscr{P}_{23} B$, $D = \mathscr{P}_{32} C$. When proceeding, write the elements of A, B, C, D to be needed in the back substitution in the same table as follows:

a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	b_{22}	b_{23}	b_{24}
a_{31}	c_{32}	b_{33}	c_{34}
a_{41}	c_{42}	b_{43}	d_{44}

With the aid of appropriate bookkeeping, one may determine x and y from this table using back substitution.

Finally we list various criteria for (row, column) sufficiency and compare them with each other. We denote their worst-case operation counts by $\nu(n)$

TABLE 1

WORST-CASE OPERATION COUNTS OF VARIOUS CRITERIA FOR SUFFICIENCY

Criterion	Operation Count				
	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 30$
$\nu_1(n)$	14.1×10^2	19.4×10^4	14.3×10^6	8.2×10^8	1.9×10^{12}
$\nu_2(n)$	5.6×10^2	13.8×10^4	14.6×10^6	11.0×10^8	3.7×10^{12}
$\nu_3(n)$	4.0×10^2	7.7×10^4	7.2×10^6	5.0×10^8	1.6×10^{12}
$\nu_4(n)$	2.9×10^2	7.5×10^4	9.2×10^6	7.3×10^8	2.7×10^{12}
$\nu_5(n)$	2.5×10^2	4.8×10^4	4.9×10^6	3.5×10^8	1.2×10^{12}

when applied to an $n \times n$ matrix. Some numerical values are recorded in Table 1.

Test 1. Application of Theorem 4.1 (due to Cottle and Guu [5]). We have

$$\nu_1(n) = 2^n n(2n - 1) - n^2.$$

Test 2. Our inductive test using full pivotal operations. We check, for $k = 2, \dots, n$, whether $A \in \mathbb{R}^{n \times n}$, (row, column) sufficient of order $k - 1$, is (row, column) sufficient of order k. The general step, accomplished by $\binom{n}{k}$ applications of Procedure 5.1, requires at most $k^2(k-1)\binom{n}{k}$ operations. So

$$\nu_2(n) = 2^{n-3} n(n - 1)(n + 2).$$

Test 3. Our improved inductive test using pivotal condensations. For it,

$$\nu_3(n) = \tfrac{1}{3} 2^{n-3} n(n - 1)(n + 10).$$

Test 4 (for nonsingular matrices only). Using Theorem 4.5 with $k = \lfloor n/2 \rfloor$ (test 2, with interruption, is applied to A and A^{-1}). We have

$$\nu_4(n) = \nu_2(n) + n^2 - \tfrac{1}{4} n(n^2 - \delta)\binom{n}{m}$$

where $m = \lfloor n/2 \rfloor$ and $\delta = 0$ or 1 according as n is even or odd.

Test 5 (for nonsingular matrices only). As in test 4 but using an interrupted test 3 instead of an interrupted test 2. Defining m and δ as for test 4, we have

$$\nu_5(n) = \nu_3(n) + n^2 - \tfrac{1}{12}(n+4)(n^2-\delta)\binom{n}{m}.$$

Test 6. The inductive test for column sufficiency due to Cottle and Guu [5]. The general step is the same as in test 2. However, its accomplishment requires in the worst case solving $2^k\binom{n}{k}$ linear programs in k variables (in [5], the factor $\binom{n}{k}$ has inadvertently been omitted). So this test is much less efficient than the preceding ones.

The working space required by test 1 is about n^3 memory places, and that required by tests 2–5 is about n^2 memory places. All the above tests are combinatorially explosive and thus practicable for small matrices only. In such cases tests 3 and 5 are more efficient than test 1; see Table 1.

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