NORTH-HOLLAND

Determining the Handicap of a Sufficient Matrix

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ABSTRACT

Any linear complementarity problem with a sufficient matrix can be solved by means of the unified interior point method. The complexity bound of the method is the better the smaller the so-called handicap of the matrix is. We propose a method for determining the handicap of a sufficient matrix and show that a sufficient matrix and its transpose have the same handicap. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

The class **SU** of sufficient matrices was recently identified by Cottle, Pang, and Venkateswaran [4] in connection with the linear complementarity problem. A matrix $A \in \mathbb{R}^{n \times n}$ is *column sufficient* if for all $x \in \mathbb{R}^n$

$$x_i(Ax)_i \leqslant 0, \quad i = 1, \dots, n \Rightarrow x_i(Ax)_i = 0, \quad i = 1, \dots, n,$$

and *row sufficient* if A^T is column sufficient. A is *sufficient* if it is both row and column sufficient. It is well known that $\mathbf{P} \subset \mathbf{SU} \subset \mathbf{P}_0$, where $\mathbf{P}_0$ ($\mathbf{P}$) is the class of matrices with nonnegative (positive) principal minors.

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It has been shown in [8, 7, 9] that $A \in \mathbb{R}^{n \times n}$ is sufficient if and only if there is a $\kappa \geqslant 0$ such that

$$(1 + 4\kappa) \sum_{i \in I_+(x)} x_i y_i + \sum_{i \in I_-(x)} x_i y_i \geqslant 0 \qquad \text{for all} \quad x \in \mathbb{R}^n, \quad (1.1)$$

where $y = Ax$ and

$$I_+(x) = \{i \mid x_i y_i > 0\} \quad \text{and} \quad I_-(x) = \{i \mid x_i y_i < 0\}.$$

For a fixed κ, the class of all matrices satisfying (1.1) will be denoted by **SU**(κ); this class is the same as the class $\mathbf{P}_*(\kappa)$ in [8]. Note that **SU**(0) = **PSD**, the class of positive semidefinite (psd) matrices.

Any sufficient linear complementarity problem can be solved by means of the unified interior point method [8]. The smaller κ in (1.1) can be chosen, the better the complexity bound of the method is. Therefore the smallest κ for which (1.1) holds is of importance. This value is called the *handicap* of the sufficient matrix A and denoted by $\hat{\kappa}(A)$. If $x \in \mathbb{R}^n$ and $I_-(x) \neq \varnothing$, then $I_+(x) \neq \varnothing$, and the ratio

$$F_A(x) := \frac{-x^T A x}{\sum_{i \in I_+(x)} x_i (Ax)_i} \qquad (1.2)$$

is well defined. We have

$$\hat{\kappa}(A) := \begin{cases} 0 & \text{if} \quad A \in \mathbf{PSD}, \\ \frac{1}{4} \sup\{F_A(x) \mid x^T A x < 0\} & \text{otherwise.} \end{cases}$$

Note that $F_A(\lambda x) = F_A(x)$ for any $\lambda \neq 0$.

The organization of the paper is as follows. After some preliminaries we shall, in Section 3, recall and supplement the basic theory of the classes **SU** and **SU**(κ). Then, in Section 4, we derive a general expression for the handicap of a sufficient indefinite matrix of order two (for the part of non-**P**-matrices, this result has earlier been established by Guu and Cottle [7]). Section 5 is devoted to determining the handicaps of **P**-matrices. We give a numerical example to illustrate the method. In Section 6 we show that, for $n \geqslant 3$, determining the handicap of a sufficient matrix $A \in \mathbb{R}^{n \times n}$, not in **P**, can be reduced to determining handicaps of **P**-matrices of order less than

n and those of sufficient matrices of order two. Finally, in Section 7, we show that the handicaps of a sufficient matrix and its transpose are equal.

2. PRELIMINARIES

If $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ (A is a real $m \times n$ matrix), we write A^T for its transpose. If $R \subset \{1,\dots,m\}$ and $S \subset \{1,\dots,n\}$, we denote the submatrix of A induced by rows $i \in R$ and columns $j \in S$ by A_{RS}. We let $A_{i.}$ stand for the ith row of A, and $A_{.j}$ for the jth column of A. A diagonal matrix $D \in \mathbb{R}^{n\times n}$ with the diagonal elements $d_1,\dots,d_n$ is denoted by $D = \operatorname{diag}(d_1,\dots,d_n)$. This convention generalizes to block diagonal matrices; then the diagonal elements d_i are replaced by diagonal blocks D_i. The class of positive definite (pd) matrices will be denoted by **PD**. By a *principal permutation* of a square matrix we mean simultaneous permutation of the rows and the columns. In particular, we write $\mathscr{C}_{rs}$ for the principal permutation interchanging rows and columns r and s. Any vector $x \in \mathbb{R}^n$ is interpreted as an $n \times 1$ matrix and denoted by $x = [x_1,\dots,x_n]^T$ or, for simplicity, by $x = (x_1,\dots,x_n)$. We write x_R for the subvector of x consisting of components $i \in R$ and let e_i stand for the ith coordinate vector. For vectors we shall use the Euclidean norm $\|\cdot\|$. If $x, y \in \mathbb{R}^n$, their *Hadamard product* $x * y \in \mathbb{R}^n$ is defined by $(x * y)_i = x_i y_i$, $i = 1,\dots,n$. Moreover, we define $N = \{1,\dots,n\}$, denote the empty set by $\emptyset$ and the cardinality of a set R by $|R|$, and abbreviate $R - r = R \setminus \{r\}$. The symbol $:=$ will be used for definition.

If $A \in \mathbb{R}^{n\times n}$, $R \subset N$, and A_{RR} is nonsingular, the *principal pivotal operation* $\mathscr{P}_R$ transforms the equation $y = Ax$ into an equivalent equation in which the variables y_R and x_R have been exchanged; see e.g. [3, pp. 68–78]. If the matrix of the latter equation is $\hat{A}$, we write $\hat{A} = \mathscr{P}_R A$ (in the case $R = \emptyset$ we have $\hat{A} = \mathscr{P}_\emptyset A = A$). We call $\hat{A}$ a *principal pivotal transform* of A, and any principal permutation of $\hat{A}$ a *principal transform* of A. The single principal pivotal operation $\mathscr{P}_{\{r\}}$ will simply be denoted by $\mathscr{P}_r$. We shall need the following result.

THEOREM 2.1 [1]. *Let $A \in \mathbb{R}^{n\times n}$ have the nonsingular principal submatrix A_{RR}. Then*

$$\mathscr{P}_R A^T = D(\mathscr{P}_R A)^T D$$

where $D = \mathrm{diag}(d_1, \ldots, d_n)$ *such that for all* $i \in N$,

$$d_i = \begin{cases} 1, & i \in R, \\ -1, & i \notin R. \end{cases} \tag{2.1}$$

As an application of principal pivoting we determine the unique global minimum of a strictly convex quadratic function $q(x) = q_0 + c^T x + \frac{1}{2} x^T D x^T$ where $D = D^T \in \mathbb{R}^{n \times n}$:

$$\begin{array}{rc} & \begin{array}{cc} x & 1 \end{array} \\ \begin{array}{r} 0 = \\ 2q = \end{array} & \boxed{\begin{array}{cc} D & c \\ c^T & 2q_0 \end{array}} \end{array} \quad \overset{\mathscr{P}_{N-n}}{\to} \quad \begin{array}{rc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{r} x = \\ 2q = \end{array} & \boxed{\begin{array}{cc} D^{-1} & -D^{-1}c \\ c^T D^{-1} & 2q_0 - c^T D^{-1} c \end{array}} \end{array} \tag{2.2}$$

Here $-D^{-1}c$ is the unique global minimum $\hat{x}$, and $2q_0 - c^T D^{-1} c$ is double the optimal value $\hat{q}$ of q.

Next we recall a well-known result on **P**-matrices.

THEOREM 2.2 [6]. *$A \in \mathbb{R}^{n \times n}$ is a **P**-matrix if and only if for every $x \in \mathbb{R}^n \setminus \{0\}$ there exists an index k such that $x_k(Ax)_k > 0$.*

From this theorem we deduce that if $A \in \mathbf{P} \cap \mathbb{R}^{n \times n}$, then the F_A of (1.2) is defined for all $x \in \mathbb{R}^n \setminus \{0\}$. We have the following.

THEOREM 2.3. *If $A \in \mathbf{P} \cap \mathbb{R}^{n \times n}$ is not pd, then there exists an $\hat{x} \in \mathbb{R}^n \setminus \{0\}$ such that $\hat{\kappa}(A) = \frac{1}{4} F_A(\hat{x})$.*

Proof. In case $A \in \mathbf{PSD}$ the result is obvious. Assume then that $A \notin \mathbf{PSD}$. Defining $G = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, we have that

$$\hat{\kappa}(A) = \tfrac{1}{4} \sup\{F_A(x) \mid x \in G\}.$$

The denominator of F_A is positive and continuous in the compact set G and hence attains its smallest value in G. It follows that F_A is continuous in G and attains its largest value in G. ∎

REMARK 2.1. Note that $\hat{\kappa}(A)$ for $A \in \mathbf{P}$ is a continuous function of the elements of A. This is because a function which is continuous in a compact set is uniformly continuous in this set.

3. BASIC THEORY

In this section we review and supplement the basic theory of the classes **SU** and $\mathbf{SU}(\kappa)$; cf. [1, 2, 7, 8].

THEOREM 3.1 [1]. *If* $A \in \mathbb{R}^{n\times n}$ *is sufficient and* $a_{kk} = 0$, *then* $a_{ik} = a_{ki} = 0$ *or* $a_{ik}a_{ki} < 0$ *for all* $i \neq k$.

THEOREM 3.2. *If* $A \in \mathbb{R}^{n\times n}$ *belongs to* $\mathbf{SU}(\kappa)$, *then so does* (i) *any principal submatrix of* A, (ii) *any principal permutation of* A, *and* (iii) *any principal pivotal transform of* A.

THEOREM 3.3. *Let* $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$. *Then*:

(i) *The handicaps of* A *and all its principal transforms are the same.*
(ii) *The handicap of* A *is at least as large as that of any of its proper principal submatrices.*

THEOREM 3.4 (Cf. [8]). *Let* $A \in \mathbf{SU}(\kappa) \cap \mathbb{R}^{n\times n}$, $P = \operatorname{diag}(p_1, \dots, p_n)$, $Q = \operatorname{diag}(q_1, \dots, q_n)$, *where* $\kappa \geqslant 0$ *and* $p_i q_i > 0$ *for all* $i \in N$. *Then* $B := PAQ \in \mathbf{SU}(\kappa')$, *where* $\kappa' \geqslant \kappa$ *is such that*

$$\frac{1 + 4\kappa'}{1 + 4\kappa} = \frac{\max_{i \in N}(p_i/q_i)}{\min_{i \in N}(p_i/q_i)}.$$

In particular, if the diagonal elements of a diagonal matrix D *are nonzero, then* $\hat{\kappa}(DAD) = \hat{\kappa}(A)$.

THEOREM 3.5. *Let* $A = \operatorname{diag}(A_1, A_2)$. *Then* $\hat{\kappa}(A) = \max\{\hat{\kappa}(A_1), \hat{\kappa}(A_2)\}$.

Proof. Omitted. ■

THEOREM 3.6. *Let $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$, and let $D \in \mathbb{R}^{n\times n}$ be a nonnegative diagonal matrix. Then $\hat{\kappa}(A+D) \leqslant \hat{\kappa}(A)$.*

Proof. Let $\hat{\kappa} = \hat{\kappa}(A)$, $D = \operatorname{diag}(d_1, \dots, d_n)$, and define

$$I'_+(x) = \{i \in N \mid x_i[(A+D)x]_i > 0\} = \{i \in N \mid x_i[Ax]_i + d_i x_i^2 > 0\},$$

$$I_+(x) = \{i \in N \mid x_i[Ax]_i > 0\}.$$

Then, because $I_+(x) \subset I'_+(x)$,

$$x^T(A+D)x + 4\hat{\kappa} \sum_{i \in I'_+(x)} \left(x_i[Ax]_i + d_i x_i^2\right)$$

$$\geqslant x^T Ax + 4\hat{\kappa} \sum_{i \in I_+(x)} x_i[Ax]_i \geqslant 0. \qquad \blacksquare$$

The following theorem is a consequence of [8, Lemma 5.3], Theorem 3.6, and Theorem 3.3(ii).

THEOREM 3.7. *Let $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$, let $D \in \mathbb{R}^{n\times n}$ be a nonnegative diagonal matrix, and let*

$$A' = \begin{bmatrix} A & I \\ -I & D \end{bmatrix}.$$

Then $\hat{\kappa}(A') = \hat{\kappa}(A)$.

COROLLARY 3.1. *Let $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$, let $d \geqslant 0$, and let*

$$A' = \begin{bmatrix} A & -e_1 \\ e_1^T & d \end{bmatrix}.$$

Then $\hat{\kappa}(A') = \hat{\kappa}(A)$.

THEOREM 3.8. *Assume that $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$ and that $a_{ik} = a_{ih}$, $a_{ki} = a_{hi}$ for all $i \in N - k$ and $a_{kk} \geqslant a_{hh}$. Then $\hat{\kappa}(A) = \hat{\kappa}(A_{N-k,N-k})$.*

Proof. Assume without loss of generality that $h = 1$, $k = n$. If $A_{1.} = 0$, then $A_{.1} = 0$, and Theorem 3.5 applies. Then assume that $A_{1.} \neq 0$. If $a_{11} > 0$, set $R = \{1\}$. If $a_{11} = 0$, then there is a $p \in \{2, \dots, n-1\}$ such that $a_{1p} a_{p1} < 0$; then define $R = \{1, p\}$. In both cases let $B = \mathscr{P}_R A$, having $b_{1n} = -1$, $b_{n1} = 1$, $b_{nn} \geqslant 0$, and $b_{in} = b_{ni} = 0$, $i = 2, \dots, n-1$. Finally, by Theorem 3.3(i) and Corollary 3.1,

$$\hat{\kappa}(A) = \hat{\kappa}(B) = \hat{\kappa}(B_{N-n, N-n}) = \hat{\kappa}(A_{N-n, N-n}). \qquad \blacksquare$$

4. HANDICAPS OF SUFFICIENT MATRICES OF ORDER TWO

The following theorem contains complete information about the handicap of a sufficient indefinite matrix of order two.

THEOREM 4.1. *Assume that $A \in \mathbb{R}^{2\times 2}$ is sufficient but not psd. Then*

$$1 + 4\hat{\kappa}(A) = \frac{\max\{a_{12}^2, a_{21}^2\}}{\left(\sqrt{a_{11}a_{22}} + \sqrt{\det A}\right)^2}. \tag{4.1}$$

If $A \notin \mathbf{P}$, then, more simply,

$$1 + 4\hat{\kappa}(A) = \max\left\{\left|\frac{a_{12}}{a_{21}}\right|, \left|\frac{a_{21}}{a_{12}}\right|\right\}. \tag{4.2}$$

If $a_{11} = a_{22} = 0$, then the supremum of $F(x) := F_A(x)$ is reached for any vector $x = (x_1, x_2)$ with $\operatorname{sgn} x_1 x_2 = -\operatorname{sgn}(a_{12} + a_{21})$.

If $A \notin \mathbf{P}$ and $a_{11} + a_{22} > 0$, then the supremum of $F(x)$ cannot be reached. A value of $F(x)$ arbitrarily close to the supremum can be obtained for the following choices where $\epsilon > 0$ is arbitrarily small:

Case	x_1	x_2
$a_{11} > 0 = a_{22}$	$-\epsilon \operatorname{sgn}(a_{12} + a_{21})$	1
$a_{11} = 0 < a_{22}$	1	$-\epsilon \operatorname{sgn}(a_{12} + a_{21})$
$a_{11}a_{22} = a_{12}a_{21} > 0$	$-a_{12} + \epsilon \operatorname{sgn}(a_{12} - a_{21})$	a_{11}

If $A \in \mathbf{P}$, then the supremum of $F(x)$ is reached at the following points:

Case	x_1	x_2
$\lvert a_{12}\rvert > \lvert a_{21}\rvert$	$-a_{12}\sqrt{a_{22}}$	$\sqrt{a_{11}}(\sqrt{a_{11}a_{22}} + \sqrt{\det A})$
$\lvert a_{12}\rvert < \lvert a_{21}\rvert$	$\sqrt{a_{22}}(\sqrt{a_{11}a_{22}} + \sqrt{\det A})$	$-a_{21}\sqrt{a_{11}}$

Proof. Assume first that $A \notin \mathbf{P}$.

Case I. $a_{11} \geqslant 0 = a_{22}$. Taking $x_2 = 1$, we have that $x^TAx < 0 \Rightarrow (a_{12} + a_{21})x_1 < 0$. There are two subcases.

(i) $|a_{12}| > |a_{21}|$. Then

$$(a_{12} + a_{21})x_1 < 0 \Rightarrow 0 < a_{21}x_1 < -a_{12}x_1$$
$$\Rightarrow x_2y_2 = a_{21}x_1 > 0 \Rightarrow x_1y_1 < 0.$$

So

$$\bar{F}(x_1) = -1 + \frac{-a_{11}x_1^2 - a_{12}x_1}{a_{21}x_1} \leqslant -1 + \frac{-a_{12}}{a_{21}}.$$

We see that $\sup F(x) = -1 - a_{12}/a_{21}$. If $a_{11} = 0$, then the supremum is reached for all vectors with $\operatorname{sgn} x_1 = \operatorname{sgn} a_{21}$. If $a_{11} > 0$, then the supremum cannot be reached; a value of $F(x)$ arbitrarily close to the supremum is obtained by taking $x = (\epsilon \operatorname{sgn} a_{21}, 1)$, where $\epsilon > 0$ is arbitrarily small.

(ii) $|a_{12}| < |a_{21}|$. Now $0 < a_{12}x_1 < -a_{21}x_1$, $x_2y_2 > 0$, $x_1y_1 > 0$, whence

$$\bar{F}(x_1) = -1 + \frac{-a_{21}x_1}{a_{11}x_1^2 + a_{12}x_1} \leqslant -1 + \frac{-a_{21}}{a_{12}}.$$

We note that $\sup F(x) = -1 - a_{21}/a_{12}$. If $a_{11} = 0$, then the supremum is reached for all vectors with $\operatorname{sgn} x_1 = \operatorname{sgn} a_{12}$. If $a_{11} > 0$, then the supremum cannot be reached; a value of $F(x)$ arbitrarily close to the supremum is obtained by taking $x = (\epsilon \operatorname{sgn} a_{12}, 1)$, where $\epsilon > 0$ is arbitrarily small.

Case II. $a_{11} = 0 \leqslant a_{22}$. This case is reduced to case I by defining $u = (x_2, x_1)$, $v = (y_2, y_1)$, $B = \mathscr{C}_{12}A$.

Case III. $a_{11}a_{22} = a_{12}a_{21} > 0$. We reduce this case to case I by defining $u = (y_1, x_2)$, $v = (x_1, y_2)$, $B = \mathscr{P}_1 A$. We note that (4.2) holds. A value of $F(x)$ arbitrarily close to the supremum is obtained by taking $u = (-\epsilon \operatorname{sgn}(b_{12} + b_{21}), 1)$, where $\epsilon > 0$ is arbitrarily small; then $x_2 = u_2 = 1$ and

$$x_1 = v_1 = b_{11}u_1 + b_{12}u_2 = \epsilon a_{11}^{-1} \operatorname{sgn}(a_{12} - a_{21}) - a_{11}^{-1}a_{12}.$$

Then assume that $A \in \mathbf{P}$.

Case I. $|a_{12}| > |a_{21}|$. There are three subcases.

(i) $a_{12} > 0 \geqslant a_{21}$, $a_{12} + a_{21} > 0$. Taking $x_2 = 1$, we obtain

$$x^T Ax < 0 \Rightarrow x_1 < 0 \Rightarrow x_2 y_2 = a_{21}x_1 + a_{22} > 0$$
$$\Rightarrow x_1 y_1 < 0 \Rightarrow y_1 > 0 \Rightarrow -a_{12}/a_{11} < x_1 < 0.$$

Now,

$$\bar{F}(x_1) = -1 + \frac{-a_{11}x_1^2 - a_{12}x_1}{a_{21}x_1 + a_{22}},$$

$$\bar{F}'(x_1) = \frac{-a_{11}a_{21}x_1^2 - 2a_{11}a_{22}x_1 - a_{12}a_{22}}{(a_{21}x_1 + a_{22})^2}.$$

It is easy to show that the global maximum of $\bar{F}(x_1)$ in the interval $x_1 \in (-a_{12}/a_{11}, 0)$ is

$$\hat{x}_1 = \frac{-a_{12}\sqrt{a_{22}}}{\sqrt{a_{11}}\left(\sqrt{a_{11}a_{22}} + \sqrt{\det A}\right)}$$

(this expression is valid in the case $a_{21} = 0$ too). Simple calculations lead to (4.1).

(ii) $a_{12} < 0 \leqslant a_{21}$, $a_{12} + a_{21} < 0$. We reduce this case to (i) by defining $u = (-x_1, x_2)$, $v = (-y_1, y_2)$, $B = DAD$, where $D = \operatorname{diag}(-1, 1)$.

(iii) $a_{12}a_{21} > 0$, $|a_{12}| > |a_{21}|$. This case is reduced to (i)–(ii) above by defining $u = (y_1, x_2)$, $v = (x_1, y_2)$, $B = \mathscr{P}_1 A$. The supremum of $F(x)$ is reached by taking

$$u = \left(-b_{12}\sqrt{b_{22}}\,,\sqrt{b_{11}}\left(\sqrt{b_{11}b_{22}} + \sqrt{\det B}\right)\right)$$

$$= a_{11}^{-3/2}\left(a_{12}\sqrt{\det A}\,, \sqrt{a_{11}a_{22}} + \sqrt{\det A}\right),$$

i.e., $x_2 = u_2$ and

$$x_1 = v_1 = b_{11}u_1 + b_{12}u_2 = -a_{11}^{-2}a_{12}\sqrt{a_{22}}\,.$$

Case II. $a_{12}a_{21} > 0$, $|a_{12}| < |a_{21}|$. We reduce this case to case I by defining $u = (x_2, x_1)$, $v = (y_2, y_1)$, $B = \mathscr{C}_{12}A$. ∎

REMARK 4.1. Equation (4.2) is essentially due to Guu and Cottle [7]. Their proof differs somewhat from ours.

REMARK 4.2. It follows from Theorem 4.1 that, for $A \in \mathbf{SU} \cap \mathbb{R}^{2\times 2}$,

(i) $\hat{\kappa}(A) = \hat{\kappa}(A^T)$;
(ii) $\hat{\kappa}(A)$ is a continuous function of the elements of A.

REMARK 4.3. If $A \in \mathbf{P} \cap \mathbb{R}^{2\times 2}$, then $F_A(x)$ is not necessarily concave in the set $\{x \in \mathbb{R}^2 \mid x^TAx < 0\}$. To see this, let $a_{11} = a_{22} = 1$, $a_{12} = 8$, $a_{21} = -1$; then $\partial^2 F_A(1, -1)/\partial x_2^2 > 0$.

5. HANDICAPS OF P-MATRICES

Let $A \in \mathbb{R}^{n\times n}$ be a **P**-matrix but not psd. We shall determine the handicap of A by calculating the handicaps of all principal submatrices of order k of all principal pivotal transforms of A sequentially for $k = 2, \dots, n$.

Assume that

$$\hat{\kappa}_{n-1} := \max\{\hat{\kappa}(B_{N-i}) \mid B \text{ is a principal pivotal transform of } A \text{ and } i \in N\} \tag{5.1}$$

is known. We shall derive a necessary condition for $\hat{\kappa}(A) > \hat{\kappa}_{n-1}$ to hold. So assume that

$$\hat{\kappa}(A) = \tfrac{1}{4}F_A(\hat{x}) > \hat{\kappa}_{n-1}, \tag{5.2}$$

see Theorem 2.3. Here $\hat{x}_i \neq 0$ for all $i \in N$, because otherwise we would have $\hat{\kappa}(A) = \hat{\kappa}_{n-1}$. We show that also $\hat{y}_i := A_{i\cdot}\hat{x} \neq 0$ for all $i \in N$. Assume, on the contrary, that $\hat{y}_k = 0$ for some $k \in N$. Then $\hat{\kappa}(\mathscr{P}_k A)$ is attained at a point whose kth component equals zero. But then $\hat{\kappa}(A) = \hat{\kappa}(\mathscr{P}_k A) = \hat{\kappa}_{n-1}$, a contradiction. Because $\hat{x}_i \neq 0$ for all $i \in N$, we may assume without loss of generality tht $\hat{x}_n = 1$; so $\hat{x} = (\hat{x}^1, 1)$ with $\hat{x}^1 \in \mathbb{R}^{n-1}$. There are two cases.

Case I. $n \in I_+(\hat{x})$. We define

$$R = I_-(\hat{x}), \qquad S = I_+(\hat{x}) - n,$$

$$G_1 = \left\{x^1 \in \mathbb{R}^{n-1} \mid x_R * y_R < 0,\, x_S * y_S > 0\right\}, \tag{5.3}$$

where $y = Ax$. We have that

$$\hat{t} := 4\hat{\kappa}(A) + 1 = -\frac{f(\hat{x}^1)}{g(\hat{x}^1)}, \tag{5.4}$$

where

$$f(x^1) = \tfrac{1}{2}x_R^T\left(A_{RR} + A_{RR}^T\right)x_R + x_R^T A_{RS} x_S + x_R^T A_{Rn}, \tag{5.5}$$

$$g(x^1) = x_R^T A_{SR}^T x_S + \tfrac{1}{2}x_S^T\left(A_{SS} + A_{SS}^T\right)x_S + x_S^T A_{Sn}$$
$$+ A_{nR}x_R + A_{nS}x_S + a_{nn}. \tag{5.6}$$

Clearly, $\hat{t} > 1$. By [5], the nonlinear program

$$\min\left\{h(x^1, t) := tg(x^1) + f(x^1) \mid x^1 \in G_1\right\} \tag{5.7}$$

with $t = \hat{t}$ has the minimum value zero. This minimum value is attained in the interior of G_1 but not on its boundary. We show that the quadratic function $h(x^1, \hat{t})$ is by necessity strictly convex. First, if $h(x^1, \hat{t})$ is indefinite, it cannot have a minimum in the interior of G_1. Second, if $h(x^1, \hat{t})$ is convex but not strictly, it has the value zero on a whole line in $\mathbb{R}^{n-1}$. But any line in $\mathbb{R}^{n-1}$ intersects the boundary of G_1. So also this case is impossible.

We have shown that solving (5.7) with $t = \hat{t}$ amounts to finding the unique global minimum of the strictly convex quadratic function $h(x^1, \hat{t})$ in $\mathbb{R}^{n-1}$. Recalling (2.2), this can be accomplished by performing $\mathscr{P}_{N-n}$ to the table

$$
B(t):\quad
\begin{array}{r|ccc|}
 & x_R & x_S & 1 \\
\hline
0 = & A_{RR} + A_{RR}^T & A_{RS} + tA_{SR}^T & A_{Rn} + tA_{nR}^T \\
0 = & A_{RS}^T + tA_{SR} & t(A_{SS} + A_{SS}^T) & t(A_{nS}^T + A_{Sn}) \\
2h = & A_{Rn}^T + tA_{nR} & t(A_{nS} + A_{Sn}^T) & 2ta_{nn} \\
\hline
\end{array}
\tag{5.8}
$$

where $t = \hat{t}$. Equivalently, one can perform the sequence $\mathscr{P}_1, \dots, \mathscr{P}_{n-1}$ of single principal pivots to the table (5.8); the pivots in this sequence are positive. In the resulting table, $\hat{x}_R$ and $\hat{x}_S$ are in positions (i, n), $i = 1, \dots, n-1$, and the element (n, n), containing twice the optimal value of h, is zero. Denote the matrix contained in the table (5.8) by $B(t)$; then $B(\hat{t})$ is singular, because the product of it and a nonzero vector equals zero.

We continue by showing that all proper principal submatrices of $B(\hat{t})$ are pd. Assume, on the contrary, that $B_{HH}(\hat{t})$ with $H \subset N$, $H \ni n$, $|H| \leqslant n-1$ is singular. Then the element (n, n) of $\mathscr{P}_{H-n}B(\hat{t})$ is zero. Because $\mathscr{P}_{N-n}B(\hat{t}) = \mathscr{P}_{N \setminus H}\mathscr{P}_{H-n}B(\hat{t})$, this implies that either the minimum value of the problem (5.7) with $t = \hat{t}$ is negative or that $\hat{x}_i = 0$ for all $i \in N \setminus H$, a contradiction.

Case II: $n \in I_-(\hat{x})$. We define

$$R = I_+(\hat{x}), \qquad S = I_-(\hat{x}) - n,$$

$$G_2 = \{x^1 \in \mathbb{R}^{n-1} \mid x_R * y_R > 0,\, x_S * y_S < 0\}. \tag{5.9}$$

We have that

$$\hat{t} := 4\hat{\kappa}(A) + 1 = -\frac{g(\hat{x}^1)}{f(\hat{x}^1)} \tag{5.10}$$

or, equivalently,

$$\hat{t}^{-1} = -\frac{f(\hat{x}^1)}{g(\hat{x}^1)},$$

where $f(x^1)$ and $g(x^1)$ are as in (5.5)–(5.6). We see that the developments in case I are now valid when replacing $\hat{t}$ by $\hat{t}^{-1}$ and G_1 by G_2. So, for example, $h(x^1, \hat{t}^{-1})$ is a strictly convex quadratic function.

We have attained the following. For $\hat{x} \in \mathbb{R}^n$ with $\hat{x}_n = 1$ to satisfy (5.2) it is necessary that there is $\varnothing \neq R \subset N - n$ such that, with $\hat{t} = 4\hat{\kappa}(A) + 1$ and $S = (N - n) \setminus R$, either

(i) $n \in I_+(\hat{x})$, all proper principal submatrices of $B(\hat{t})$ are pd, $\det B(\hat{t}) = 0$, and

$$\begin{bmatrix} \hat{x}_R \\ \hat{x}_S \end{bmatrix} = -B^{-1}_{N-n,N-n}(\hat{t})\, B_{N-n,n}(\hat{t}), \tag{5.11}$$

so that $\hat{x}^1 \in G_1$, or

(ii) $n \in I_-(\hat{x})$, all proper principal submatrices of $B(\hat{t}^{-1})$ are pd, $\det B(\hat{t}^{-1}) = 0$, and the point $\hat{x}^1$ obtained from (5.11) with $\hat{t}$ replaced by $\hat{t}^{-1}$ belongs to G_2.

Based on the above developments, we now construct a recursive procedure for determining $\hat{\kappa}(A)$. Assume that the $\hat{\kappa}_{n-1}$ of (5.1) has already been calculated. Select $t_0 := 4\hat{\kappa}_{n-1} + 1$ as the first candidate of $\hat{t}$. We go through all the nonempty sets $R \subset N - n$ as follows. For a selected R construct the matrix $B(t)$ of (5.8). If A_{RR} or A_{SS} is not pd, select a new set R. Otherwise determine all the roots of the equation $\det B(t) = 0$ which lie in the intervals $(0, t_0^{-1})$ and (t_0, ∞) (an algebraic equation of degree $\leqslant n$ has to be solved by means of some numerical method). If some root $t > t_0$ yields a pd $B_{N-n,N-n}(t)$ and $\hat{x}^1 \in G_1$, set $t_0 \leftarrow t$. Likewise, if some positive root $t < t_0^{-1}$ yields a pd $B_{N-n,N-n}(t)$ and $\hat{x}^1 \in G_2$, set $t_0 \leftarrow t^{-1}$. [In both cases, $\hat{x}^1$ is obtained from (5.11) with $\hat{t}$ replaced by t. Note that for a given G_1 or G_2 there cannot be more than one candidate of $\hat{t}$.] Then select a new set R, etc. After going through all the sets R we have $\hat{\kappa}(A) = \frac{1}{4}(t_0 - 1)$ (in fact, this holds only approximatively, because the candidates are determined numerically).

The method is very laborious, so it is practicable for small **P**-matrices only. Below we illustrate the method in the case of a **P**-matrix of order three.

EXAMPLE 5.1. We determine the handicap of the **P**-matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & -2 \\ -4 & 3 & 1 \end{bmatrix}.$$

All the proper principal submatrices of all principal pivotal transforms of A are psd, whence $\hat{\kappa}_2 = 0$. There are three cases. We give the equation $\det B(t) = 0$ in each case below and summarize the solution of the example in Table 1. From this table it appears that $\hat{t} = 0.862468^{-1} = 1.159464$, implying $\hat{\kappa}(A) = \frac{1}{4}(\hat{t} - 1) = 0.0398659$.

Case I. $R = \{1\}$, $S = \{2\}$ $(x_1 * y_1 < 0,\ x_2 * y_2 > 0,\ x_3 * y_3 > 0$, or these inequalities reversed):

$$\det B(t) = \begin{vmatrix} 8 & 1+2t & 2-4t \\ 1+2t & 2t & t \\ 2-4t & t & 2t \end{vmatrix} = 0 \Leftrightarrow t(28t^2 - 24t + 3) = 0.$$

Case II. $R = \{2\}$, $S = \{1\}$ $(x_1 * y_1 > 0,\ x_2 * y_2 < 0,\ x_3 * y_3 > 0$, or these inequalities reversed):

$$\det B(t) = \begin{vmatrix} 2 & 2+t & -2+3t \\ 2+t & 8t & -2t \\ -2+3t & -2t & 2t \end{vmatrix} = 0 \Leftrightarrow t(43t^2 - 48t + 12) = 0.$$

TABLE 1
SOLVING EXAMPLE 5.1

R	t	x	y	Obstacle
{1}	0.151930	(−0.308, 0.821, 1)	(1.590, −1.795, 4.694)	$x_3y_3 > 0$
	0.705213	(0.522, −1.392, 1)	(2.696, −2.348, −5.265)	$x_1y_1 > 0$
{2}	0.378001	(−1.392, 2.088, 1)	(−1.481, −2.696, 12.834)	$x_3y_3 > 0$
	0.738278	(0.821, −1.231, 1)	(4.052, −1.590, −5.977)	$x_1y_1 > 0$
{1, 2}	0.421623	(−0.405, 0.975, 1)	(1.356, −1.835, 5.542)	$x_3y_3 > 0$
	0.862468	(0.666, −1.293, 1)	(3.371, −1.961, −5.542)	

Case III. $R = \{1,2\}$, $S = \varnothing$ ($x_1 * y_1 < 0$, $x_2 * y_2 < 0$, $x_3 * y_3 > 0$, or these inequalities reversed):

$$\det B(t) = \begin{vmatrix} 8 & 3 & 2-4t \\ 3 & 2 & -2+3t \\ 2-4t & -2+3t & 2t \end{vmatrix} = 0 \Leftrightarrow 88t^2 - 113t + 32 = 0.$$

6. HANDICAPS OF SUFFICIENT MATRICES NOT IN P

In this section we shall show that, for $n \geqslant 3$, the handicap of $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$ not in **P** is equal to the maximum over the handicaps of the proper principal submatrices of all principal pivotal transforms of A. So determining the handicap of $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$ not in **P** can be reduced to determining handicaps of **P**-matrices of order less than n and those of sufficient matrices of order two. We begin with an auxiliary result.

THEOREM 6.1. *Let* $A \in \mathbb{R}^{n\times n}$ *with* $A_{N-n,N-n} = 0$ *be sufficient. Then*

$$1 + 4\hat{\kappa}(A) = \frac{\max_{i\in N}|a_{ni}/a_{in}|}{\min_{i\in N}|a_{ni}/a_{in}|}, \tag{6.1}$$

where $0/0$ *is defined to be equal to one.*

Proof. In view of Theorem 3.5 we may assume that $a_{in}a_{ni} < 0$ for all $i \in N - n$. Let B be the matrix obtained from A by replacing a_{in} with $\operatorname{sgn} a_{in}$ and a_{ni} with $\operatorname{sgn} a_{ni}$, $i = 1, \dots, n-1$. Then $B \in \mathbf{PSD}$, whence $\hat{\kappa}(B) = 0$. By Theorem 3.4, (6.1) holds with $=$ replaced by $\leqslant$ (take $p_i = |a_{in}|$, $q_i = |a_{ni}|$ for all $i \in N - n$ and $p_n = q_n = 1$). We show that the reverse inequality holds too. We let $h, k \in N - n$ be such that

$$\left|\frac{a_{nh}}{a_{hn}}\right| = \min_{i\in N-n}\left|\frac{a_{ni}}{a_{in}}\right|, \qquad \left|\frac{a_{nk}}{a_{kn}}\right| = \max_{i\in N-n}\left|\frac{a_{ni}}{a_{in}}\right|. \tag{6.2}$$

There are three cases.

(i) $|a_{nh}/a_{hn}| \geqslant 1$. Letting $R = \{k, n\}$, we have

$$1 + 4\hat{\kappa}(A) \geqslant 1 + 4\hat{\kappa}(A_{RR}) = |a_{nk}/a_{kn}|.$$

(ii) $|a_{nk}/a_{kn}| \leqslant 1$. Letting $R = \{h, n\}$, we have

$$1 + 4\hat{\kappa}(A) \geqslant 1 + 4\hat{\kappa}(A_{RR}) = |a_{hn}/a_{nh}| = 1 \div |a_{nh}/a_{hn}|.$$

(iii) $|a_{nk}/a_{kn}| > 1 > |a_{nh}/a_{hn}|$. Letting $C = \mathscr{P}_{\{h,n\}}A$ and $R = \{h, k\}$, we have

$$1 + 4\hat{\kappa}(A) \geqslant 1 + 4\hat{\kappa}(C_{RR}) = |a_{nk}/a_{kn}| \div |a_{nh}/a_{hn}|. \qquad \blacksquare$$

The following theorem is an extension of [9, Theorem 3.1].

THEOREM 6.2. *Assume that $A \in \mathbb{R}^{n\times n}$ with $n \geqslant 3$ is sufficient but not a* **P**-*matrix. Then*

$$\hat{\kappa}(A) = \max\{\hat{\kappa}(B_{N-i,N-i}) \mid B \text{ is a principal pivotal transform of } A \text{ and } i \in N\}. \quad (6.3)$$

Proof. It follows from Theorem 3.3 that $\geqslant$ holds in (6.3). We show that the reverse inequality holds too. There is a principal transform of A whose trailing diagonal element is zero. Hence we may assume without loss of generality that $a_{nn} = 0$. If $A_{\cdot h} = 0$ ($A_{h\cdot} = 0$) for some $h \in N$, then $A_{h\cdot} = 0$ ($A_{\cdot h} = 0$) too, and the result follows from Theorem 3.5. So we assume in the sequel that all rows and columns of A are nonzero. For simplicity, we denote $F_A(x)$ by $F(x)$. Consider any $x^0 \in \mathbb{R}^n$ such that $x^{0T}Ax^0 < 0$ and let $y^0 = Ax^0$. If $x_h^0 = 0$ for some $h \in N$, then $F(x^0) \leqslant 4\hat{\kappa}(A_{N-h,N-h})$. In what follows we assume that $x_i^0 \neq 0$ for all $i \in N$.

If $y_h^0 = 0$ for some $h \in N$, then there are two possibilities. First, if $a_{hh} > 0$, then $F(x^0) \leqslant 4\hat{\kappa}(B_{N-h,N-h})$, where $B = \mathscr{P}_h A$. Second, if $a_{hh} = 0$, then $a_{hk}a_{kh} < 0$ for some $k \in N$; see Theorem 3.1. So $F(x^0) \leqslant 4\hat{\kappa}(B_{N-h,N-h})$, where $B = \mathscr{P}_{\{h,k\}}A$.

Finally, assume that $y_i^0 \neq 0$ for all $i \in N$. Without loss of generality we may assume that $x_n^0 = 1$. In the sequel Σ will mean summing over the set $N - n$, Σ_+ over the set $I_+(x^0) - n$, and Σ_- over the set $I_-(x^0) - n$. In addition, we let $\delta = 0$ or 1 according as $x_n^0 y_n^0 = y_n^0 = \Sigma a_{ni} x_i^0$ is negative or positive. Defining $\hat{A} = A_{N-n,N-n}$, $\hat{x}^0 = (x_1^0, \ldots, x_{n-1}^0)$, $\hat{y}^0 = \hat{A}\hat{x}^0$, we evaluate $F(x)$ at the point $x(t) = (x_1^0, \dots, x_{n-1}^0, t)$:

$$\bar{F}(t) = \frac{-\hat{x}^{0T}\hat{A}\hat{x}^0 - t\Sigma(a_{in} + a_{ni})x_i^0}{\Sigma_+ x_i^0(\hat{y}_i^0 + a_{in}t) + \delta t y_n^0}. \tag{6.4}$$

This function and its derivative

$$\bar{F}'(t) = \frac{(\hat{x}^{0T}\hat{A}\hat{x}^0)[\Sigma_+ a_{in}x_i^0 + \delta y_n^0] - (\Sigma_+ x_i^0\hat{y}_i^0)\Sigma(a_{in} + a_{ni})x_i^0}{[\Sigma_+ x_i^0(\hat{y}_i^0 + a_{in}t) + \delta t y_n^0]^2}$$

are defined for all values of t for which the denominator in (6.4) is nonzero. Note that for such values of t the sign of $\bar{F}'(t)$ is independent of t. We have two cases.

Case I. $\bar{F}'(1) \leqslant 0$. Diminish t from one until (i) for the first time $y_h(t) := \hat{y}_h^0 + a_{hn}t$ for some $h \in N - n$ becomes zero at $t = t_0 > 0$ (say), or (ii) t tends to zero. We consider (i) and (ii) separately.

(i) $\bar{F}(t)$ and $\bar{F}'(t)$ are defined for all $t \in [t_0, 1]$. This is seen as follows. Assume, on the contrary, that at t_0 the denominator in (6.4) equals zero. This can occur only if $x_i^0 y_i(t_0) \leqslant 0$ for all $i \in N - n$ and $\delta = 0$ (implying $t_0 y_n^0 < 0$). This is, however, impossible, because A is sufficient. Because $\bar{F}'(t) \leqslant 0$ for all $t \in [t_0, 1]$, the present case reduces to the case $y_h^0 = 0$ above.

(ii) We have $x_i^0\hat{y}_i^0 \geqslant 0$ for all $i \in I_+(x^0) - n$ and $x_i^0\hat{y}_i^0 \leqslant 0$ for all $i \in I_-(x^0) - n$. If $x_i^0\hat{y}_i^0 > 0$ for some $i \in I_+(x^0) - n$, then the denominator in (6.4) is positive for $t = 0$, and the present case is reduced to the case $x_n^0 = 0$. Otherwise we have $x_i^0\hat{y}_i^0 \leqslant 0$ for all $i \in N - n$, implying $x_i^0\hat{y}_i^0 = 0$ for all $i \in N - n$ and $\hat{x}^{0T}\hat{A}\hat{x}^0 = 0$ (because $\hat{A}$ is sufficient). So, for all $t > 0$, $\bar{F}(t)$ equals

$$\frac{-\Sigma(a_{in} + a_{ni})x_i^0}{\Sigma_+ a_{in}x_i^0 + \delta y_n^0}. \tag{6.5}$$

Because $a_{in}x_i^0 = x_i^0(\hat{y}_i^0 + a_{in}) = x_i^0 y_i^0$ for all $i \in N - n$, the ratio (6.5) equals the ratio $F_B(x^0)$ where

$$B = \begin{bmatrix} 0 & \cdots & 0 & a_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix}. \tag{6.6}$$

So $F(x^0) \leqslant \hat{\kappa}(B)$. But, defining h and k as in (6.2), $\hat{\kappa}(B)$ equals $\hat{\kappa}(A_{RR})$ where $R = \{k, n\}$ or $\{h, n\}$, or $\hat{\kappa}(C_{RR})$ where $C = \mathscr{P}_{\{h,n\}}A$ and $R = \{h, k\}$.

Case II. $F'(1) > 0$. We increase t from one until (i) for the first time $y_h(t)$ for some $h \in N - n$ becomes zero at some point $t = t_0$ (say), or (ii) t tends to ∞. We consider (i) and (ii) separately.

(i) $\bar{F}(t)$ and $\bar{F}'(t)$ are defined for all $t \in [1, t_0]$. This is shown in the same way as in case I(i) above. So the present case reduces to the case $y_h^0 = 0$.

(ii) Now, $a_{in}x_i^0 \geqslant 0$ for all $i \in I_+(x^0) - n$ and $a_{in}x_i^0 \leqslant 0$ for all $i \in I_-(x^0) - n$. We show that the denominator in (6.5) is positive. Assume, on the contrary, that the denominator is zero. Then $a_{in}x_i^0 \leqslant 0$ for all $i \in N - n$ and $\delta = 0$, i.e., $\sum a_{ni}x_i^0 = y_n^0 = x_n^0 y_n^0 < 0$. This is, however, impossible because the B of (6.6) is sufficient. It follows that $\lim_{t\to\infty} \bar{F}(t)$ exists and is given by (6.5). Because $\bar{F}'(t) > 0$ for all $t \geqslant 1$, we have $F(x^0) \leqslant \hat{\kappa}(B)$ where B is taken from (6.6). The rest is as in case I(ii) above. ■

REMARK 6.1. It may be conjectured that $\hat{\kappa}(A)$ for $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$ is a continuous function of the elements of A; cf. Remarks 2.1 and 4.2.

7. INVARIANCE OF THE HANDICAP UNDER TRANSPOSITION

By Remark 4.2, $\hat{\kappa}(A) = \hat{\kappa}(A^T)$ for any two by two sufficient matrix A. This result generalizes to the whole class **SU**.

THEOREM 7.1. *If* $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$, *then* $\hat{\kappa}(A) = \hat{\kappa}(A^T)$.

Proof. By induction on n. The case $n = 1$ is trivial; for the case $n = 2$ refer to Remark 4.2. Then assume that the theorem is true for the order $n - 1$, and consider any $A \in \mathbf{SU} \cap \mathbb{R}^{n\times n}$ where $n \geqslant 3$. We show first that the $\hat{\kappa}_{n-1}$ of (5.1) equals the corresponding quantity $\hat{\kappa}'_{n-1}$ for A^T. There are an $R \subset N$ and a $k \in N$ such that $\hat{\kappa}_{n-1} = \hat{\kappa}(B_{N-k,N-k})$ with $B = \mathscr{P}_R A$. By

Theorem 2.1, $C := \mathscr{P}_R A^T = DB^T D$, where D is as in (2.1). It follows that $C_{N-k,N-k} = D_{N-k,N-k} B^T_{N-k,N-k} D_{N-k,N-k}$, whence, using Theorem 3.4 and the induction hypothesis,

$$\hat{\kappa}'_{n-1} \geqslant \hat{\kappa}(C_{N-k,N-k}) = \hat{\kappa}\left(B^T_{N-k,N-k}\right) = \hat{\kappa}(B_{N-k,N-k}) = \hat{\kappa}_{n-1}.$$

The reverse inequality $\hat{\kappa}_{n-1} \geqslant \hat{\kappa}'_{n-1}$ can be established by interchanging the roles of A and A^T above. So $\hat{\kappa}_{n-1} = \hat{\kappa}'_{n-1}$.

If $A \in \mathbf{PSD}$, then $\hat{\kappa}(A) = \hat{\kappa}(A^T) = 0$. In the case $A \notin \mathbf{P}$ we have by Theorem 6.2 that $\hat{\kappa}(A) = \hat{\kappa}_{n-1} = \hat{\kappa}'_{n-1} = \hat{\kappa}(A^T)$. Then let $A \in \mathbf{P} \setminus \mathbf{PSD}$. We show first that $\hat{\kappa}(A^T) \geqslant \hat{\kappa}(A)$. If $\hat{\kappa}(A) = \hat{\kappa}_{n-1}$, then $\hat{\kappa}(A) = \hat{\kappa}_{n-1} = \hat{\kappa}'_{n-1} \leqslant \hat{\kappa}(A^T)$. Then assume that (5.2) holds. Consider first the case $n \in I_+(\hat{x})$ (case I in Section 5). We adopt the notation in (5.3)–(5.6) and construct the table (5.8) with $t = \hat{t}$. We show that the point $\hat{x}'$ with $\hat{x}'_R = \hat{t}^{-1}\hat{x}_R$, $\hat{x}'_S = \hat{x}_S$, $\hat{x}'_n = 1$ yields $\hat{\kappa}(A)$ as a candidate of $\hat{\kappa}(A^T)$. Now $\hat{y} = A\hat{x}$, $\hat{y}' = A^T\hat{x}'$ whence

$$\begin{bmatrix} \hat{y}_R \\ \hat{y}_S \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} A_{RR} & A_{RS} & A_{Rn} \\ A_{SR} & A_{SS} & A_{Sn} \\ A_{nR} & A_{nS} & a_{nn} \end{bmatrix} \begin{bmatrix} \hat{x}_R \\ \hat{x}_S \\ 1 \end{bmatrix}, \tag{7.1}$$

$$\begin{bmatrix} \hat{y}'_R \\ \hat{y}'_S \\ \hat{y}'_n \end{bmatrix} = \begin{bmatrix} A^T_{RR} & A^T_{SR} & A^T_{nR} \\ A^T_{RS} & A^T_{SS} & A^T_{nS} \\ A^T_{Rn} & A^T_{Sn} & a_{nn} \end{bmatrix} \begin{bmatrix} \hat{t}^{-1}\hat{x}_R \\ \hat{x}_S \\ 1 \end{bmatrix}, \tag{7.2}$$

implying

$$\begin{bmatrix} \hat{y}_R + \hat{t}\hat{y}'_R \\ \hat{t}(\hat{y}_S + \hat{y}'_S) \\ \hat{t}(\hat{y}_n + \hat{y}'_n) \end{bmatrix} = B(\hat{t}) \begin{bmatrix} \hat{x}_R \\ \hat{x}_S \\ 1 \end{bmatrix} = 0 \tag{7.3}$$

[see (5.8)] and further $\hat{y}'_R = -\hat{t}^{-1}\hat{y}_R$, $\hat{y}'_S = -\hat{y}_S$, $\hat{y}'_n = -\hat{y}_n$. So

$$\hat{t}' := -\frac{(\hat{x}'_S)^T \hat{y}'_S + \hat{y}'_n}{(\hat{x}'_R)^T \hat{y}'_R} = -\frac{\hat{x}_S^T \hat{y}_S + \hat{y}_n}{\hat{t}^{-2}(\hat{x}_R)^T \hat{y}_R} = \hat{t}$$

is a candidate of $4\hat{\kappa}(A^T) + 1$ or, equivalently, $\hat{\kappa}(A)$ is a candidate of $\hat{\kappa}(A^T)$.

The case $n \in I_{-}(\hat{x})$ (case II in Section 5) is analogous. Adopt the notation in (5.9)–(5.10) and (5.5)–(5.6), and construct the table (5.8) with $t = \hat{t}^{-1}$. Then the point $\hat{x}'$ with $\hat{x}'_R = \hat{t}\hat{x}_R$, $\hat{x}'_S = \hat{x}_S$, $\hat{x}'_n = 1$ yields $\hat{y}'$ with $\hat{y}'_R = -\hat{t}\hat{y}_R$, $\hat{y}'_S = -\hat{y}_S$, $\hat{y}'_n = -\hat{y}_n$ [now (7.1)–(7.3) hold with $\hat{t}$ replaced by $\hat{t}^{-1}$]. So

$$\hat{t}' := -\frac{(\hat{x}'_R)^T \hat{y}'_R}{(\hat{x}'_S)^T \hat{y}'_S + \hat{y}'_n} = -\frac{\hat{t}^2 \hat{x}_R^T \hat{y}_R}{\hat{x}_S^T \hat{y}_S + \hat{y}_n} = \hat{t}$$

is a candidate of $4\hat{\kappa}(A^T) + 1$ or, equivalently, $\hat{\kappa}(A)$ is a candidate of $\hat{\kappa}(A^T)$.

We have shown that $\hat{\kappa}(A^T) \geqslant \hat{\kappa}(A)$. The reverse inequality can be established by interchanging the roles of A and A^T. ■

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