Generating sufficient matrices

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Abstract. The class of sufficient matrices (**SU**) are important in the theory and solvability of the linear complementarity problems (LCP) as it was proven that SU-LCPs can be solved in polynomial number of iterations using interior point algorithms (IPA) that depends on the size of problem n, bit length L and the value $\kappa \geq 0$ that characterise the matrix property. Furthermore, the SU-matrices are the wider class of matrices for which criss-cross algorithms (CCA) solves the problem in finite number of iterations. Important deficiency of the published IPAs for SU-LCPs is that in most publications there are no numerical examination at all. Main reason for this might lie in the fact that only few SU matrices are known that does not fall into the classes of **PSD** and **P** matrices. Our goal is to generate different **SU** (but not **PSD** or **P**) matrices and test problems on which the different IPAs can be tested and the results can be compared.

Keywords: sufficient matrices · linear complementarity problems · interior-point algorithms

1 Introduction

The class of $\mathbf{P}_0$ matrices and its subclasses play an important role in the theory of the LCP. We say that a matrix is in $\mathbf{P}_0$ if all the principal minors are nonnegative. Couple of decades ago two subclass were defined: the class of **SU** matrices in 1989 by Cottle et al. [2] and the class of $\mathbf{P}_*(\kappa)$ matrices in 1991 by Kojima, et al. [1]. Kojima, et al. [1], Guu and Cottle [3] and Valiaho [4] in a series of publication proved that these matrix classes are equivalent, i.e. $\mathbf{P}_* = \mathbf{SU}$.

For different variants of CCAs and IPAs for SU-LCPs good survey can be found in Csizmadia [7] and M. Nagy [8], respectively.

Our goal is to generate sufficient matrices, therefore we need to find different ways to construct **SU**-matrices. By building a set of **SU**-matrices, test set problems for SU-LCPs can be defined, thus practical, computational performance of the IPAs can be tested. Important definitions, some lemmas and some constructions for **SU**-matrices are summarized. Finally, we illustrate a way we generated **SU**-matrices using the discussed lemmas and constructions.

In this paper we omit the proofs of the known lemmas and Construction 1, because those can be found in the literature. For our own, new results, sketch of the proofs are included for most of the cases.

We distinguish between the *scalar product* ($\mathbf{x}^T\mathbf{y} \in \mathbb{R}$), and the *Hadamard (coordinate-wise) product* ($\mathbf{x}\,\mathbf{y} \in \mathbb{R}^n$) of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

2 Definitions, lemmas, constructions

Definition 1. *A matrix $A \in \mathbb{R}^{n\times n}$ is called a* ***PSD****-*matrix *if for every vector $\boldsymbol{x} \in \mathbb{R}^n$*

$$\boldsymbol{x}^T A\, \boldsymbol{x} \geq 0. \tag{1}$$

Now, we are ready to introduce the classes $\mathbf{P}_*(\kappa)$ and $\mathbf{P}_*$ as a generalization of **PSD**-matrices (for details see Kojima et al. [1]).

Definition 2. *A matrix $A \in \mathbb{R}^{n\times n}$ is called a $\boldsymbol{P}_*(\kappa)$-*matrix *(for some $\kappa \geq 0$) if for every vector $\boldsymbol{x} \in \mathbb{R}^n$*

$$(1+4\kappa) \sum_{i \in I_+(\boldsymbol{x})} x_i\, y_i + \sum_{i \in I_-(\boldsymbol{x})} x_i\, y_i \geq 0 \tag{2}$$

where $\boldsymbol{y} = A\,\boldsymbol{x}$, $I_+(\boldsymbol{x}) = \{i : \, x_i\, y_i > 0\}$ and $I_-(\boldsymbol{x}) = \{i : \, x_i\, y_i < 0\}$.

If $\kappa = 0$ we get back the definition of **PSD**-matrices. Now, we can introduce

$$\mathbf{P}_* = \cup_{\kappa \geq 0} \mathbf{P}_*(\kappa). \tag{3}$$

The classes **CSU**, **RSU** and **SU** were defined by Cottle et al. [2].

Definition 3. *A matrix $A \in \mathbb{R}^{n\times n}$ is called* column sufficient *matrix (or belongs to the* ***CSU*** *class of matrices) if for every vector $\boldsymbol{x} \in \mathbb{R}^n$ it satisfies the following condition*

$$\boldsymbol{x}\,\boldsymbol{y} \leq \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{x}\,\boldsymbol{y} = \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{y} = A\,\boldsymbol{x}$.

It is easy to see that a matrix is a sufficient matrix, if $I_+(\mathbf{x}) = \emptyset$ implies that $I_-(\mathbf{x}) = \emptyset$. Furthermore, any sufficient matrix has the property that if $\exists i \in I_-(\mathbf{x})$ then there should be another index $j \in I_+(\mathbf{x})$.

Definition 4. *A matrix $A \in \mathbb{R}^{n\times n}$ is called* row sufficient *matrix (or belongs to the* ***RSU*** *class of matrices) if $A^T \in$* ***CSU****.*

A matrix $A \in \mathbb{R}^{n\times n}$ is called sufficient *matrix (or belongs to the* ***SU*** *class of matrices) if $A \in$* ***CSU*** *$\cap$* ***RSU****.*

In most cases the complexity of the IPAs depends on the handicap of the matrix, therefore testing the algorithm on a matrix with zero handicap is not appropriate, thus our goal is to generate sufficient matrices with positive handicap.

Definition 5. *$A \in \mathbb{R}^{n\times n}$, $\boldsymbol{x} \in \mathbb{R}^n$ where $\boldsymbol{x}^T A \boldsymbol{x} < 0$ and let us define the following function*

$$F_A(\boldsymbol{x}) = -\frac{\boldsymbol{x}^T A \boldsymbol{x}}{\sum_{i \in I_+(\boldsymbol{x})} x_i (A\boldsymbol{x})_i}. \tag{5}$$

The handicap of a **SU** matrix A is denoted by $\kappa(A)$, and

$$\kappa(A) = \begin{cases} 0 & \text{if } A \in \mathbf{PSD} \\ \frac{1}{4}\sup\{F_A(\mathbf{x}) | \mathbf{x}^T A\mathbf{x} < 0\} & \text{otherwise} \end{cases} \tag{6}$$

If $A \notin \mathbf{PSD}$ then there exists a vector $\mathbf{x}$ for which $\mathbf{x}^T A\mathbf{x} < 0$ therefore $\kappa(A)$ is well defined. The handicap of a **SU** matrix A is basically the smallest κ for which $A \in \mathbf{P}_*(\kappa)$.

Definition 6. *The* principal pivot operation *(PPO) transforms the equation system $\boldsymbol{y} = A\boldsymbol{x}$ ($A \in \mathbb{R}^{n\times n}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$) into an equivalent one, where the variables x_i and y_i are exchanged for certain indices $i \in R$.*

If $R = \{1, 2, ..., j\}$ for some $j \in \{1, 2, ..., n\}$, then the coefficient matrix of the new equation system is:

$$\mathcal{P}_R A = \begin{pmatrix} A^{-1}_{RR} & -A^{-1}_{RR}A_{R\overline{R}} \\ A_{\overline{R}R}A^{-1}_{RR} & A_{\overline{R}\overline{R}} - A_{\overline{R}R}A^{-1}_{RR}A_{R\overline{R}} \end{pmatrix}. \tag{7}$$

The Lemmas 1-5 and Construction 1 can be found in Cottle, Pang and Stone [5].

Lemma 1. *Every principal submatrix of a sufficient matrix is also sufficient.*

Lemma 2. *If $A \in \mathbb{R}^{n\times n}$, $P = diag(p_1, ..., p_n)$, $Q = diag(q_1, ..., q_n)$, where $p_i q_i > 0$ for all i, and $B = PAQ$, then*

1. *If $A \in$ **SU** then so is B.*
2. *If $A \in \boldsymbol{P}_*(\kappa)$ for some $\kappa \geq 0$, then $B \in \boldsymbol{P}_*(\kappa')$, where $\kappa' \geq \kappa$ is such that*

$$\frac{1 + 4\kappa'}{1 + 4\kappa} = \frac{\max_i(p_i/q_i)}{\min_i(p_i/q_i)}. \tag{8}$$

Lemma 3. *The matrix classes **CSU**, **RSU**, **SU**, $\boldsymbol{P}_*(\kappa)$, $\boldsymbol{P}_*$ are closed under the PPO and the principal permutations of rows and columns.*

Lemma 4. *The handicap of a sufficient matrix is at least as large as the handicap of any of its principal submatrix.*

Lemma 5. *The handicap is invariant under the PPO.*

Lemma 6 (Construction 1). *If $A \in \mathbb{R}^{n\times n}$ is in **SU** then so is the following matrix*

$$\begin{pmatrix} A & I \\ -I & D \end{pmatrix}, \tag{9}$$

where $I, D \in \mathbb{R}^{n\times n}$, and I is the identity matrix and D is a diagonal matrix with nonnegative elements.

Now, we summarize two of our constructions which were used during the sufficient matrix generation process. From now on let $\mathcal{I} = \{1, 2, \ldots, n\}$ and $\mathcal{J}_k = \{n + 1, n + 2, \ldots, n + k\}$ be set of indices.

Lemma 7 (Construction 2). *Let a sufficient matrix $A \in \mathbb{R}^{n\times n}$ be given. Let us define the matrix $C \in \mathbb{R}^{(n+k)\times(n+k)}$ in the following way*

$$c_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i, j \le n \\ 1 & \text{if } i = 1 \text{ and } j \in \mathcal{J}_k \\ -1 & \text{if } j = 1 \text{ and } i \in \mathcal{J}_k \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $k \in \mathbb{N}$ is arbitrary. Then the matrix C is sufficient.

Proof. First we prove the column sufficiency using the definition. Let $\mathbf{x} \in \mathbb{R}^{n+k}$, $\tilde{\mathbf{x}} = \mathbf{x}_{\mathcal{I}}$, $\mathbf{y} = C\mathbf{x}$ and $\tilde{\mathbf{y}} = A\tilde{\mathbf{x}}$. The Hadamard product

$$\mathbf{xy} = \begin{pmatrix} x_1\tilde{y}_1 + x_1 \sum_{i\in\mathcal{J}_k} x_i \\ x_2\tilde{y}_2 \\ \vdots \\ x_n\tilde{y}_n \\ -x_{n+1}\, x_1 \\ \vdots \\ -x_{n+k}\, x_1 \end{pmatrix}. \tag{11}$$

If $-x_i\, x_1 > 0$ for some $i \in \mathcal{J}_k$ then $I_+(\mathbf{x}) \neq \emptyset$. Otherwise $-x_i\, x_1 \le 0$ for all $i \in \mathcal{J}_k$ so $\sum_{i\in\mathcal{J}_k} x_i x_1 = x_1 \sum_{i\in\mathcal{J}_k} x_i \ge 0$. In this case (as $A \in \mathbf{SU}$) we know that among the first n coordinate of the vector $\mathbf{xy}$ there must be at least one positive, or every coordinate is zero. This proves the column sufficiency, and the row sufficiency can be proved exactly in the same way. □

Note, that if $A \notin \mathbf{PSD}$ then $C \notin \mathbf{PSD}$.

Lemma 8. *The matrix $E \in \mathbb{R}^{n\times n}$ of ones is sufficient.*

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} = E\mathbf{x}$ and the corresponding Hadamard product

$$\mathbf{xy} = \begin{pmatrix} x_1 \sum_{i=1}^n x_i \\ \vdots \\ x_n \sum_{i=1}^n x_i \end{pmatrix}. \tag{12}$$

If $\sum_{i=1}^n x_i = 0$ then $\mathbf{xy} = \mathbf{0}$. If $\sum_{i=1}^n x_i$ is positive (negative) then $\exists i \in \mathcal{I}$ for which x_i is positive (negative) so $i \in I_+(\mathbf{x})$, therefore $I_+(\mathbf{x}) \neq \emptyset$. □

Previous lemma gives us a useful tool in proving the following statement.

Lemma 9 (Construction 3). *Let a matrix $D \in \mathbb{R}^{2n\times 2n}$ is defined as follows*

$$d_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{I}\times\mathcal{I} \cup \mathcal{J}_n \times \mathcal{J}_n \cup (n, n+1) \\ -1 & \text{if } (i,j) = (n+1, n) \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Then D is sufficient matrix.

Proof. Let $\mathbf{x} \in \mathbb{R}^{2n}$, $\mathbf{y} = D\mathbf{x}$ and the corresponding Hadamard product

$$\mathbf{xy} = \begin{pmatrix} x_1 \sum_{i=1}^{n} x_i \\ \vdots \\ x_{n-1} \sum_{i=1}^{n} x_i \\ x_n \sum_{i=1}^{n} x_i + x_n x_{n+1} \\ x_{n+1} \sum_{i=n+1}^{2n} x_i - x_n x_{n+1} \\ x_{n+2} \sum_{i=n+1}^{2n} x_i \\ \vdots \\ x_{2n} \sum_{i=n+1}^{2n} x_i \end{pmatrix}. \tag{14}$$

If $x_n x_{n+1} = 0$ then D is sufficient because of Lemma 8. Considering Lemma 8 we can also see that if $x_n x_{n+1}$ is positive (negative) there must be a positive element among the first (second) n coordinate of the vector $\mathbf{xy}$, and this is exactly what we needed. Again, the row sufficiency can be proved exactly the same way so we omit that. $\square$

This construction can be generalized: if $A, B \in \mathbb{R}^{n \times n}$ are in $\mathbf{P}_0$ of rank 1, and $F = diag(A, B)$, then $f_{n,n+1}$ and $f_{n+1,n}$ can be chosen such that F is sufficient (and $f_{n,n+1}\, f_{n+1,n} < 0$).

3 Example: building a sufficient matrix

Matrices of order 1 are sufficient if the (only) element is nonnegative. Sufficient matrices of order 2 were characterized by Guu and Cottle in [3]. Deciding whether a matrix of order 3 is in $\mathbf{SU}$ can be calculated on paper, or even in head quite fast, using the lemmas in [6]. (It takes less than a minute after some practice.) We calculated several sufficient matrices of order 3, and then we applied the mentioned lemmas and constructions to increase its size and to hide the original structure.

Let us illustrate the construction of a larger sufficient matrix from smaller ones using Constructions 1-3. and some of Lemmas 1-5. Let us start with a given sufficient matrix A. (Sufficiency of A can be checked using the definition.)

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & -2 & 1 & 1 \\ -1 & 1 & -3 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & -2 & 1 & 1 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

The sufficient matrix B can be obtained from A by applying Construction 2. From matrix B, the sufficient matrix C can be built by using Construction 1.

$$D = \begin{pmatrix} 1 & -2 & -3 & -1 & 1 & -1 & 2 \\ 1 & 1 & 5 & -1 & 1 & -1 & -1 \\ 3 & -3 & 0 & -3 & 3 & -3 & 3 \\ -1 & 2 & 3 & 1 & -1 & 1 & -2 \\ 2 & -4 & -6 & -2 & 2 & -2 & 4 \\ -1 & 2 & 3 & 1 & -1 & 1 & -2 \\ -1 & -1 & -5 & 1 & -1 & 1 & 10 \end{pmatrix}$$

Applying PPO (Lemma 3) to matrix C and some scaling (Lemma 2) the resulting matrix D is sufficient matrix, as well. Due to the scaling, the handicap of matrices C and D, might be different.

All our sufficient matrix examples can be downloaded from the internet. Currently there are 10 pieces of matrices of order 10 and 20, and one matrix of order 700. As every principal submatrix of a sufficient matrix is sufficient (Lemma 1), the 700×700 matrix grants us an immense amount of sufficient matrices. By the time of the Vocal conference we are going to choose additional test examples, so the IPAs can be tested uniformly.

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