SOLUTION SHEET OF DIFFERENTIAL GEOMETRY 1, 2'ND MID-TERM, MAY 17TH, 2022

(1) Determine the values $t_0 \in \mathbb{R}$ such that the normal plane of the curve

$$\gamma(t) = \begin{pmatrix} \sin^2 t \\ \sin t \cos t \\ \cos^2 t \end{pmatrix}$$

at t_0 passes through the origin.

Solution: A normal vector of the normal plane at t_0 is given by

$$\dot{\gamma}(t_0) = \begin{pmatrix} \sin(2t_0) \\ \cos(2t_0) \\ -\sin(2t_0) \end{pmatrix}.$$

The normal plane at t_0 has equation

$$\langle \dot{\gamma}(t_0), \vec{x} - \gamma(t_0) \rangle = 0.$$

Therefore, the normal plane at t_0 contains the origin if and only if

$$\langle \dot{\gamma}(t_0), \gamma(t_0) \rangle = 0.$$

This is the case if and only if the equation

$$\sin(4t_0) = 0$$

holds, whose solutions are $t_0 = \mathbb{Z} \cdot \frac{\pi}{4}$.

(2) Consider the moment curve $u \mapsto (u, u^2, u^3)$ defined for $u > 0$. Show that the regular part of the tangent surface of the moment curve has empty intersection with the moment curve.

Solution: The parameterization of the regular part of the tangent surface is

$$\vec{r}(u, v) = \begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix} + v \begin{pmatrix} 1 \\ 2u \\ 3u^2 \end{pmatrix}$$

for $v \in \mathbb{R} \setminus \{0\}$. The existence of a value $t \in \mathbb{R}_+$ such that $\vec{r}(u, v) = (t, t^2, t^3)$ is equivalent to the matrix equation

$$\begin{pmatrix} u & 1 & t \\ u^2 & 2u & t^2 \\ u^3 & 3u^2 & t^3 \end{pmatrix} \begin{pmatrix} 1 \\ v \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix of this system is $u^2t(u-t)^2$, so it vanishes if and only if $t = u$. In this case, the solution of the linear system is $v = 0$, which is excluded by the regularity assumption.

(3) Let $\vec{r}$ be the parameterization of a regular surface $M \subset \mathbb{R}^3$. Fix a constant $d \in \mathbb{R} \setminus \{0\}$, and let $\vec{\rho}$ be the parameterization of the parallel surface M_d of M by distance d. Assume that M_d is regular. Show that $\vec{p} = \vec{r}(u_0, v_0)$ is an umbilical point of M with mean curvature κ if and only if $\vec{q} = \vec{\rho}(u_0, v_0)$ is an umbilical point of M_d with mean curvature $\frac{\kappa}{1-d\kappa}$.

Solution: By definition, the point $\vec{p}$ is umbilic for M if the Weingarten map $L_{\vec{p}}$ of M is a constant κ multiple of the identity, and in this case its mean curvature is equal to κ. Assume this is the case. Let $\vec{N}$ denote the unit normal field of M. For M_d, we have

$$\vec{\rho}_u = \vec{r}_u + d\vec{N}_u = (1 - d\kappa)\vec{r}_u,$$

and similarly

$$\vec{\rho}_v = (1 - d\kappa)\vec{r}_v.$$

The regularity assumption on M_d then implies that $1 - d\kappa \neq 0$. It follows that

$$\vec{\rho}_u \times \vec{\rho}_v = (1 - d\kappa)^2 \vec{r}_u \times \vec{r}_v,$$

so $\vec{N}$ is the unit normal field of M_d too. Consider the Weingarten map $L_{\vec{q}}$ of M_d. We then find

$$L_{\vec{q}}(\vec{\rho}_u) = -\vec{N}_u = L_{\vec{p}}(\vec{r}_u) = \kappa \vec{r}_u = \frac{\kappa}{1 - d\kappa}\vec{\rho}_u,$$

and similarly

$$L_{\vec{q}}(\vec{\rho}_v) = \frac{\kappa}{1 - d\kappa}\vec{\rho}_v.$$

By the regularity assumption, these two vectors generate $T_{\vec{q}}M_d$, therefore

$$L_{\vec{q}} = \frac{\kappa}{1 - d\kappa}\mathrm{I},$$

so $\vec{q}$ is umbilic for M_d with mean curvature $\frac{\kappa}{1-d\kappa}$. For the converse direction, apply the same argument replacing d by $-d$ and exchanging the roles of M and M_d, and observing that

$$\frac{\frac{\kappa}{1-d\kappa}}{1 + d\frac{\kappa}{1-d\kappa}} = \kappa.$$

(4) Let $\vec{r}$ be the parameterization of a regular surface in $\mathbb{R}^3$, and $\vec{p}$ a point on the surface. Assume that the Gaussian curvature of the surface at $\vec{p}$ is non-zero. We say that a non-zero tangent vector $\vec{v}$ at $\vec{p}$ is isotropic if the normal curvature of the surface in direction $\vec{v}$ vanishes. Show that the following conditions are equivalent: (a) any two isotropic tangent vectors at $\vec{p}$ are either parallel or perpendicular to each other, (b) the equality $\kappa_1 = -\kappa_2$ holds.

Solution: According to the Principal Axis Theorem, the tangent space at $\vec{p}$ of the surface admits an orthonormal basis $\vec{x}_1, \vec{x}_2$ with respect to which the Gram matrix of II_p is diagonal. Said differently, for $\vec{v} = v_1\vec{x}_1 + v_2\vec{x}_2$ we have

$$II_p(\vec{v}, \vec{v}) = \kappa_1 v_1^2 + \kappa_2 v_2^2.$$

By the assumption on K, we have $\kappa_1 \neq 0 \neq \kappa_2$. Then, $\vec{v}$ is isotropic if and only if

$$\frac{v_2}{v_1} = \pm\sqrt{-\frac{\kappa_1}{\kappa_2}}.$$

Therefore, a necessary condition of the existence of isotropic vectors is

$$\kappa_1 < 0 < \kappa_2.$$

The two different isotropic directions are spanned by

$$\sqrt{\kappa_2}\vec{x}_1 \pm \sqrt{-\kappa_1}\vec{x}_2.$$

They are perpendicular to each other if and only if

$$0 = \langle \sqrt{\kappa_2}\vec{x}_1 + \sqrt{-\kappa_1}\vec{x}_2, \sqrt{\kappa_2}\vec{x}_1 - \sqrt{-\kappa_1}\vec{x}_2 \rangle = \kappa_2 + \kappa_1.$$

This proves the desired assertion.