SOLUTION SHEET OF DIFFERENTIAL GEOMETRY 1, RE-TAKE OF 2'ND MID-TERM , MAY 27, 2022

(1) Let $\gamma\colon (a,b) \to \mathbb{R}^3$ be a regular curve. Suppose that the origin is incident to all normal planes of γ. Show that then γ lies on a sphere centered at the origin.

Solution: The equation of the normal plane of γ at $\gamma(t_0)$ is

$$\langle \dot\gamma(t_0), \vec{x} - \gamma(t_0)\rangle = 0.$$

If $\vec{x} = \vec{0}$ satisfies this equation for every value of t_0, then

$$\langle \dot\gamma, \gamma\rangle \equiv 0,$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\langle \gamma, \gamma\rangle \equiv 0.$$

By the Fundamental Theorem of Calculus, we then have

$$|\gamma|^2 = \langle \gamma, \gamma\rangle \equiv c$$

for some constant $c \in \mathbb{R}$. The possibility $c < 0$ is clearly excluded. The regularity assumption rules out $c = 0$. Hence, γ takes values in the sphere of radius $\sqrt{c}$ centered at $\vec{0}$.

(2) Using Meusnier's theorem, compute the normal curvature of the surface defined by $z = e^{-(x^2+y^2)}$ at the point $x_0 = 1, y_0 = 0, z_0 = e^{-1}$, in the direction $\vec{v} = (0,1,0)^T$.

Solution: A possible parameterization of the surface is

$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ e^{-(x^2+y^2)} \end{pmatrix}$$

with $x, y \in \mathbb{R}$. We then have

$$\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ -2xe^{-(x^2+y^2)} \end{pmatrix} \qquad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ 2ye^{-(x^2+y^2)} \end{pmatrix}.$$

Their evaluations are

$$\vec{r}_x(x_0,y_0,z_0) = \begin{pmatrix} 1 \\ 0 \\ -2e^{-1} \end{pmatrix} \qquad \vec{r}_y(x_0,y_0,z_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The normal vector $\vec{N}$ is parallel to

$$\vec{r}_x(x_0, y_0, z_0) \times \vec{r}_y(x_0, y_0, z_0) = \begin{pmatrix} 2e^{-1} \\ 0 \\ 1 \end{pmatrix}.$$

It is therefore given by

$$\vec{N} = \frac{1}{\sqrt{4e^{-2} + 1}} \begin{pmatrix} 2e^{-1} \\ 0 \\ 1 \end{pmatrix}.$$

The intersection of the surface with the plane $z = z_0$ is a circle of radius 1 (and thus, of curvature $\kappa = 1$), and it has tangent vector $\vec{v}$ at $(x_0, y_0, z_0)^T$, and second Frenet vector $\vec{t}_2 = (-1, 0, 0)^T$. By Meusnier's theorem, for the normal curvature we find

$$k(\vec{v}) = \langle \vec{t}_2, \vec{N} \rangle \kappa = -\frac{2e^{-1}}{\sqrt{4e^{-2} + 1}} = -\frac{1}{\sqrt{1 + \frac{e^2}{4}}}.$$

(3) Determine all spherical umbilical points of the surface of revolution obtained by rotating the curve $y(x) = \sin x$ around the x-axis.

Solution: The domain of definition of the surface of revolution is the set where the values of sin are positive, i.e. the union of intervals $(2k\pi, (2k+1)\pi)$ for all $k \in \mathbb{Z}$. By the formulas for the principal curvatures of a surface of revolution, the umbilical points are the solutions of the equation

$$-y''(x)y(x) = 1 + (y'(x))^2,$$

i.e.

$$-\sin^2(x) = 1 + \cos^2(x).$$

Rearranging terms, we are lead to

$$\cos(2x) = -1,$$

whose solutions are $x = \frac{\pi}{2} + 2k\pi$, where $k \in \mathbb{Z}$ (the same expression with $2k$ replaced by $2k + 1$ would give points that do not belong to the domain of definition.) Plugging these values into the formulas, one sees that both curvatures are then equal to 1, so all the umbilical points that we have found are spherical.

(4) Let $p, q > 0$ be fixed. Compute the Gaussian curvature at an arbitrary point of the surface defined by $z = \frac{x^2}{2p} + \frac{y^2}{2q}$.

Solution: A parameterization of the surface is given by

$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ \frac{x^2}{2p} + \frac{y^2}{2q} \end{pmatrix}$$

with $x, y \in \mathbb{R}$. We then have

$$\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ \frac{x}{p} \end{pmatrix} \qquad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ \frac{y}{q} \end{pmatrix}.$$

We deduce that

$$\mathcal{G} = \begin{pmatrix} 1 + \frac{x^2}{p^2} & \frac{xy}{pq} \\ \frac{xy}{pq} & 1 + \frac{y^2}{q^2} \end{pmatrix}.$$

The normal vector $\vec{N}$ is parallel to

$$\vec{r}_x \times \vec{r}_y = \begin{pmatrix} -\frac{x}{p} \\ -\frac{y}{q} \\ 1 \end{pmatrix},$$

thus it is given by

$$\vec{N} = \frac{1}{\sqrt{\frac{x^2}{p^2} + \frac{y^2}{q^2} + 1}} \begin{pmatrix} -\frac{x}{p} \\ -\frac{y}{q} \\ 1 \end{pmatrix}.$$

We further have

$$\vec{r}_{xx} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{p} \end{pmatrix} \qquad \vec{r}_{xy} = \vec{0} \qquad \vec{r}_{yy} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{q} \end{pmatrix}.$$

We find

$$\mathcal{B} = \frac{1}{\sqrt{\frac{x^2}{p^2} + \frac{y^2}{q^2} + 1}} \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & \frac{1}{q} \end{pmatrix}.$$

To sum up, we get

$$K = \frac{\det(\mathcal{B})}{\det(\mathcal{G})} = \frac{1}{pq\left(\frac{x^2}{p^2} + \frac{y^2}{q^2} + 1\right)^2}$$