SOLUTION SHEET OF DIFFERENTIAL GEOMETRY MID-TERM 1, APRIL 9TH, 2024

(1) Consider the plane curve

$$\delta(t) = e^{it} + \frac{1}{2}e^{-2it} = \begin{pmatrix} \cos t + \frac{1}{2}\cos 2t \\ \sin t - \frac{1}{2}\sin 2t \end{pmatrix}.$$

Show that δ parameterizes the path of a peripheral point on a circle of radius $1/2$ rolling without sliding in the interior of a fixed circle of radius $3/2$ centered at the origin.

Solution: Let the angular speed of the center of the rolling circle be 1, then its position is $C(t) = e^{it}$. The peripheral point then moves with angular speed -2 with respect to $C(t)$ because the ratio of the radii is 2 and its direction is opposite to that of $C(t)$. As the radius of the rolling circle is $1/2$, the relative motion of $P(t)$ about $C(t)$ is $\frac{1}{2}e^{-2it}$. Adding the two motions, we get the desired result.

(2) Compute the length of δ over $t \in [0, 2\pi/3]$.

Solution: We have

$$\dot{\delta}(t) = ie^{it} - ie^{-2it},$$

thus

$$|\dot{\delta}(t)|^2 = (e^{it} - e^{-2it})(e^{-it} - e^{2it}) = 2 - 2\cos 3t = 4\sin^2\frac{3t}{2}.$$

Over the domain of definition, $\sin\frac{3t}{2} > 0$. For the length we find

$$\int_0^{2\pi/3} 2\sin\frac{3t}{2}\,\mathrm{d}\,t = \left[-\frac{4}{3}\cos\frac{3t}{2}\right]_{t=0}^{2\pi/3} = \frac{8}{3}.$$

(3) Determine the distinguished Frenet frame of δ, and compute its curvature using Frenet's formulas.

Solution: We have

$$\vec{t}(t) = \frac{\dot{\delta}(t)}{|\dot{\delta}(t)|} = i\frac{e^{it} - e^{-2it}}{2\sin\frac{3t}{2}} = -e^{-it/2}.$$

It follows that

$$\vec{n}(t) = i\vec{t}(t) = -ie^{-it/2}.$$

On the other hand, we have

$$\frac{\mathrm{d}\,\vec{t}}{\mathrm{d}\,t} = \frac{i}{2}e^{-it/2} = -\frac{1}{2}\vec{n}(t).$$

According to Frenet's formulas, we have

$$|\dot{\delta}(t)|\kappa(t) = -\frac{1}{2},$$

and

$$\kappa(t) = -\frac{1}{2|\dot{\delta}(t)|} = -\frac{1}{4\sin\frac{3t}{2}}.$$

(4) Determine the evolute of δ, and show that it is similar to a reparameterization of δ. What is the similarity ratio?

Solution: We have

$$\begin{aligned}\tilde{\delta}(t) &= e^{it} + \frac{1}{2}e^{-2it} - 4\sin\frac{3t}{2}\cdot(-ie^{-it/2}) \\ &= e^{it} + \frac{1}{2}e^{-2it} + 2(e^{it} - e^{-2it}) \\ &= 3\left(e^{it} - \frac{1}{2}e^{-2it}\right) \\ &= -3\left(e^{i(t+\pi)} + \frac{1}{2}e^{-2i(t+\pi)}\right).\end{aligned}$$

This holds on $\mathbb{R}^3 \setminus (\mathbb{Z}\cdot 2\pi/3)$. The reparameterization is $t \mapsto t+\pi$, the similarity ratio is -3.

(5) Determine the involute of δ over $[0,\pi/3]$ corresponding to the value $l = \frac{4}{3}$. Show that it is similar to a reparameterization of δ and find the similarity ratio.

Solution: We have

$$s(t) = \left[-\frac{4}{3}\cos\frac{3\tau}{2}\right]_{\tau=0}^{t} = \frac{4}{3}\left(1-\cos\frac{3t}{2}\right),$$

so

$$\frac{4}{3} - s(t) = \cos\frac{3t}{2}.$$

We find

$$\begin{aligned}\hat{\delta}(t) &= e^{it} + \frac{1}{2}e^{-2it} - \frac{4}{3}\cos\frac{3t}{2}e^{-it/2} \\ &= e^{it} + \frac{1}{2}e^{-2it} - \frac{2}{3}(e^{3it/2} + e^{-3it/2})e^{-it/2} \\ &= e^{it} + \frac{1}{2}e^{-2it} - \frac{2}{3}(e^{it/2} + e^{-2it/2}) \\ &= \frac{1}{3}e^{it} - \frac{1}{6}e^{-2it} \\ &= -\frac{1}{3}\left(e^{i(t+\pi)} + \frac{1}{2}e^{-2i(t+\pi)}\right).\end{aligned}$$

The reparameterization is $t \mapsto t + \pi$, the similarity ratio is $-1/3$.