SOLUTIONS OF DIFFERENTIAL GEOMETRY 1 MID-TERM 2, MAY 21ST, 2024

(1) Let us consider the parameterized surface

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} \cos^3\theta\cos^3\varphi \\ \cos^3\theta\sin^3\varphi \\ \sin^3\theta \end{pmatrix}$$

Show that $\vec{r}$ is regular over $(\theta, \varphi) \in (0, \pi/2)^2$. For any $(\theta, \varphi) \in (0, \pi/2)^2$, let the affine tangent space of $\vec{r}$ at (θ, φ) intersect the axes in points

$$\begin{pmatrix} x(\theta,\varphi) \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y(\theta,\varphi) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ z(\theta,\varphi) \end{pmatrix}$$

Show that then one has

$$x(\theta,\varphi)^2 + y(\theta,\varphi)^2 + z(\theta,\varphi)^2 = 1.$$

Solution:
We have

$$\partial_\theta\vec{r} = \begin{pmatrix} -3\cos^2\theta\sin\theta\cos^3\varphi \\ -3\cos^2\theta\sin\theta\sin^3\varphi \\ 3\sin^2\theta\cos\theta \end{pmatrix}, \quad \partial_\varphi\vec{r} = \begin{pmatrix} -3\cos^3\theta\cos^2\varphi\sin\varphi \\ 3\cos^3\theta\sin^2\varphi\cos\varphi \\ 0 \end{pmatrix}$$

These vectors are linearly independent if and only if their cross product vanishes. To compute their cross product, we may replace them by parallel vectors

$$\begin{pmatrix} -\cos\theta\cos^3\varphi \\ -\cos\theta\sin^3\varphi \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} -\cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

as long as the parallelism ratios

$$3\cos\theta\sin\theta, \quad 3\cos^3\theta\cos\varphi\sin\varphi$$

are nonzero, which is the case over the domain in question. Now, we have

$$\begin{pmatrix} -\cos\theta\cos^3\varphi \\ -\cos\theta\sin^3\varphi \\ \sin\theta \end{pmatrix} \times \begin{pmatrix} -\cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = -\begin{pmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ \cos\theta\cos\varphi\sin\varphi \end{pmatrix}$$

This vector is nonzero on the domain in question, so $\vec{r}$ is regular there. Moreover, an equation of the affine tangent space at (θ, φ) reads as

$$(x - \cos^3\theta\cos^3\varphi)\sin\theta\sin\varphi + (y - \cos^3\theta\sin^3\varphi)\sin\theta\cos\varphi + (z - \sin^3\theta)\cos\theta\cos\varphi\sin\varphi = 0.$$

Plugging $y = 0 = z$ into this equation yields

$$\begin{aligned} x(\theta, \varphi) &= \cos^3\theta\cos^3\varphi + \cos^3\theta\sin^2\varphi\cos\varphi + \sin^2\theta\cos\theta\cos\varphi \\ &= \cos^3\theta(\cos^3\varphi + \sin^2\varphi\cos\varphi) + \sin^2\theta\cos\theta\cos\varphi \\ &= \cos^3\theta\cos\varphi + \sin^2\theta\cos\theta\cos\varphi \\ &= \cos\theta\cos\varphi \end{aligned}$$

Similarly, plugging $x = 0 = z$ into the equation of the affine tangent space gives after algebraic manipulations

$$y(\theta, \varphi) = \cos\theta\sin\varphi,$$

and plugging $x = 0 = y$ gives

$$z(\theta, \varphi) = \sin\theta.$$

The desired identity now follows trivially.

(2) Consider the surface

$$z = x^2 + 4y^2.$$

Using Meusnier's theorem, find its normal curvature at the point $(x_0, y_0, z_0) = (1, 0, 1)$ in the direction of the tangent vector $(0, 1, 0)$.

Solution: A parameterization is

$$\vec{r}(x, y) = \begin{pmatrix} x \\ y \\ x^2 + 4y^2 \end{pmatrix}$$

and the partial differentials are

$$\partial_x\vec{r} = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \quad \partial_y\vec{r} = \begin{pmatrix} 0 \\ 1 \\ 8y \end{pmatrix}$$

Plugging $x = x_0, y = y_0$ gives

$$\partial_x\vec{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \partial_y\vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Their cross product is

$$\partial_x\vec{r} \times \partial_y\vec{r} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

so the unit normal vector at the given point is

$$\vec{N} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

The intersection of the surface with the plane $z = 1$ is an ellipse whose axes are the coordinate axes. The second Frenet vector of this ellipse at (x_0, y_0, z_0) is

$$\vec{t}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

We get

$$\langle \vec{t}_2, \vec{N} \rangle = \frac{2}{\sqrt{5}}.$$

We now compute the curvature of the ellipse with parameterization

$$\gamma(t) = \begin{pmatrix} \cos t \\ \frac{1}{2} \sin t \end{pmatrix}$$

at $t = 0$. We find

$$\dot{\gamma}(t) = \begin{pmatrix} -\sin t \\ \frac{1}{2} \cos t \end{pmatrix}, \quad \ddot{\gamma}(t) = \begin{pmatrix} -\cos t \\ -\frac{1}{2} \sin t \end{pmatrix}$$

so

$$|\dot{\gamma}(0)| = \frac{1}{2}$$

and

$$\det(\dot{\gamma}(0), \ddot{\gamma}(0)) = \det \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2}.$$

For the curvature of the ellipse we find

$$\kappa = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^3} = 4.$$

For the given normal curvature we get

$$k = \kappa \langle \vec{t}_2, \vec{N} \rangle = \frac{8}{\sqrt{5}}.$$

(3) Let

$$\vec{r} \colon \Omega \to \mathbb{R}^n$$

be a regular parameterized hypersurface, $\vec{N}$ its unit normal vector field and $\mathcal{G} = (g_{ij})_{i,j=1}^{n-1}$ the Gram matrix of its first fundamental form written in Gauss frame. For fixed $\varepsilon > 0$ consider the surface

$$\vec{r}(\varepsilon) \colon \vec{u} \mapsto \vec{r}(\vec{u}) + \varepsilon \vec{N}(\vec{u}).$$

Let $\mathcal{G}(\varepsilon) = (g_{ij}(\varepsilon))_{i,j=1}^{n-1}$ be the Gram matrix of the first fundamental form of $\vec{r}(\varepsilon)$. Show that we have

$$\det(\mathcal{G}(\varepsilon)) = (1 - 2\varepsilon(n-1)H + O(\varepsilon^2)) \det(\mathcal{G}),$$

where H is the mean curvature of $\vec{r}$ and $O(\varepsilon^2)$ stands for terms of higher degree in ε.

Solution: We have

$$\vec{r}(\varepsilon)_i = \vec{r}_i + \varepsilon \vec{N}_i = \vec{r}_i - \varepsilon L_{\vec{p}}(\vec{r}_i),$$

therefore

$$\begin{aligned} g_{ij}(\varepsilon) &= \langle \vec{r}(\varepsilon)_i, \vec{r}(\varepsilon)_j \rangle \\ &= \langle \vec{r}_i, \vec{r}_j \rangle - \varepsilon \langle L_{\vec{p}}(\vec{r}_i), \vec{r}_j \rangle - \varepsilon \langle \vec{r}_i, L_{\vec{p}}(\vec{r}_j) \rangle + O(\varepsilon^2) \\ &= \langle \vec{r}_i, \vec{r}_j \rangle - 2\varepsilon \langle L_{\vec{p}}(\vec{r}_i), \vec{r}_j \rangle + O(\varepsilon^2) \end{aligned}$$

where the last line follows because $L_{\vec{p}}$ is self-adjoint. This shows that

$$\mathcal{G}(\varepsilon) = \mathcal{G} - 2\varepsilon \mathcal{B} + O(\varepsilon^2) = (1 - 2\varepsilon \mathcal{L} + O(\varepsilon^2))\mathcal{G}.$$

Taking determinant, we find

$$\begin{aligned} \det(\mathcal{G}(\varepsilon)) &= \det(1 - 2\varepsilon \mathcal{L} + O(\varepsilon^2)) \det(\mathcal{G}) \\ &= (1 - 2\varepsilon \operatorname{tr} \mathcal{L} + O(\varepsilon^2)) \det(\mathcal{G}) \\ &= (1 - 2\varepsilon(n-1)H + O(\varepsilon^2)) \det(\mathcal{G}). \end{aligned}$$