# Perversity equals weight for Painlevé systems

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# Outline

Hodge theory, Riemann–Hilbert

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Filtrations

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Dolbeault side

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Example: PIII(D7)

# Wild non-abelian Hodge theory

Simpson '90, Biquard–Boalch '04: fix

- $C$: smooth projective curve over $\mathbb{C}$
- $r \geq 2$ rank
- $p_1, \ldots, p_n \in C$ irregular singularities
- a flag type and parabolic weights at each $p_j$
- an irregular type and an adjoint orbit of the residue at each $p_j$.

The space of solutions of Hitchin's equations

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection $D$ on a rank $r$ smooth Hermitian vector bundle $(V, h)$ and a field $\theta : V \to V \otimes \Omega^{1,0}_C$ having prescribed singular behaviour near $p_j \rightsquigarrow$ hyper-Kähler moduli space $\mathcal{M}_{\text{Hod}}$.

# de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\mathrm{Hod}}$ with a geometric meaning:

- de Rham: $\mathcal{M}_{\mathrm{dR}}$ parameterising poly-stable parabolic connections with irregular singularities
- Dolbeault: $\mathcal{M}_{\mathrm{Dol}}$ parameterising poly-stable parabolic Higgs bundles with higher-order poles.

By non-abelian Hodge theory, $\mathcal{M}_{\mathrm{dR}}$ and $\mathcal{M}_{\mathrm{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\mathrm{Hod}}$).

# IRREGULAR RIEMANN–HILBERT CORRESPONDENCE

- Mebkhout, Kashiwara, Birkhoff, Jurkat, Deligne–Malgrange: equivalence between the categories of irregular connections and Stokes-filtered local systems.
- Boalch '07: algebraic construction of wild character varieties $\mathcal{M}_{\mathsf{B}}$ parameterising Stokes data.
- Irregular Riemann–Hilbert correspondence: bi-analytic map

$$\mathsf{RH} : \mathcal{M}_{\mathsf{dR}} \to \mathcal{M}_{\mathsf{B}}.$$

Conclusion: $\mathcal{M}_{\mathsf{dR}}$, $\mathcal{M}_{\mathsf{Dol}}$ and $\mathcal{M}_{\mathsf{B}}$ are all diffeomorphic to each other, in particular they have the same cohomology spaces.

# Painlevé spaces

From now on, we set $C = \mathbb{C}P^1$ and we assume $r = 2$ and $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Hod}} = 4$. There exists a finite list

$$PI, PII, PIII(D6), PIII(D7), PIII(D8), PIV, PV_{\mathrm{deg}}, PV, PVI$$

of irregular types with this property, called Painlevé cases. From now on, we let

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

and we write $PX$ to refer to one of the above Painlevé cases. We therefore have smooth non-compact Kähler surfaces

$$\mathcal{M}_{\mathrm{dR}}^{PX}, \quad \mathcal{M}_{\mathrm{Dol}}^{PX}, \quad \mathcal{M}_{\mathrm{B}}^{PX}$$

diffeomorphic to each other for any fixed $X$.

# Middle perversity $t$-structure

Given an algebraic variety $Y$, consider the derived category

$$D^b(Y, \mathbb{Q})$$

of bounded complexes of $\mathbb{Q}$-vector spaces $K$ on $Y$ with constructible cohomology sheaves of finite rank.
Beilinson–Bernstein–Deligne '82: truncation functors

$$^{\mathfrak{p}}\tau_{\leq i} : D^b(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leq i}(Y, \mathbb{Q})$$

encoding the support condition for the middle perversity function, giving rise to a system of truncations

$$0 \to \cdots \to {}^{\mathfrak{p}}\tau_{\leq -p} K \to {}^{\mathfrak{p}}\tau_{\leq -p+1} K \to \cdots \to K$$

# Perverse filtration on Dolbeault spaces

Hitchin '87: for $\mathcal{M}_{\mathrm{Dol}}$ a Dolbeault moduli space there exists a surjective map

$$h : \mathcal{M}_{\mathrm{Dol}} \to Y = \mathbb{C}^N.$$

Consider

$$K = \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}} \in D^b(Y, \mathbb{Q}).$$

The perverse filtration $P$ on

$$\mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

is defined as

$$P^p \mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = \mathrm{Im}(\mathbf{H}^*(Y, {}^{\mathfrak{p}}\tau_{\leq -p} \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}})).$$

We define the perverse Hodge polynomial of $\mathcal{M}_{\mathrm{Dol}}$ by

$$PH(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) q^i t^k.$$

# Flag filtration on Dolbeault spaces

For an affine variety $Y$ of dimension $n$ consider a generic flag

$$Y_{-n} \subset \cdots \subset Y_{-1} \subset Y_0 = Y,$$

where $Y_p$ are the intersections of $Y$ with a fixed generic linear flag under a fixed projective embedding. Given any $K$ one may consider the sequence of complexes

$$0 \subseteq K_{Y \setminus Y_{-1}} \subseteq \cdots \subseteq K_{Y \setminus Y_{-n}} \subseteq K.$$

It gives rise to the flag filtration $F$ defined by

$$F^i H^l(Y, K) = \mathrm{Ker}(H^l(Y, K) \to H^l(Y_{i-1}, K|_{Y_{i-1}})).$$

**Theorem (de Cataldo–Migliorini '10)**

*For $Y$ affine we have*

$$P^p H^l(Y, K) = F^{p+l} H^l(Y, K).$$

# Weight filtration on Betti spaces

As $\mathcal{M}_{\mathrm{B}}$ is an affine algebraic variety, Deligne's Hodge II. ('71) shows that $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ carries a weight filtration $W$. We derive a polynomial

$$WH(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}_{2i}^{W} H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{C}) q^i t^k.$$

Hausel–Rodriguez-Villegas '08: $WH$ is indeed a polynomial.

# $P = W$ conjecture

### Theorem (de Cataldo–Hausel–Migliorini '12)

*For rank 2 Dolbeault and Betti spaces corresponding to each other under non-abelian Hodge theory and the Riemann–Hilbert correspondence, we have*

$$PH(q,t) = WH(q,t).$$

### Conjecture (de Cataldo–Hausel–Migliorini '12)

*The same assertion holds for any rank $r$.*

# $P = W$ in the Painlevé cases

Let us set

$$PH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) q^i t^k,$$

$$WH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}_{2i}^W H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) q^i t^k.$$

## Theorem (Sz '18)

*For each*

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

*we have*

$$PH^{PX}(q,t) = q^{-1} WH^{PX}(q,t).$$

## Hitchin fibration

Irregular Hitchin map

$$h : \mathcal{M}_{\text{Dol}}^{PX} \to Y = \mathbb{C}.$$

**Theorem (Ivanics–Stipsicz–Szabó '17)**

*There exists an embedding*

$$\mathcal{M}_{\text{Dol}}^{PX} \hookrightarrow E(1) = \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P}^2$$

*and an elliptic fibration*

$$\tilde{h} : E(1) \to \mathbb{C}P^1$$

*extending h.*

Denote by $F_{\infty}^{PX}$ the non-reduced curve $E(1) \setminus \mathcal{M}_{\text{Dol}}^{PX} = \tilde{h}^{-1}(\infty)$.

# Euler characteristic and perverse polynomial

## Proposition

*We have*

$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-3} H^0(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 1$$
$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 1$$
$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-2} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 10 - \chi(F^{PX}_{\infty}).$$

*In particular, we have*

$$PH^{PX}(q,t) = q^{-1} + (10 - \chi(F^{PX}_{\infty}))q^{-2}t^2 + q^{-3}t^2.$$

# Table of perverse polynomials

| $X$ | $F_\infty^{PX}$ | $PH^{PX}(q,t)$ |
|---|---|---|
| $VI$ | $D_4^{(1)}$ | $q^{-1}+4q^{-2}t^2+q^{-3}t^2$ |
| $V$ | $D_5^{(1)}$ | $q^{-1}+3q^{-2}t^2+q^{-3}t^2$ |
| $V_{\text{deg}}$ | $D_6^{(1)}$ | $q^{-1}+2q^{-2}t^2+q^{-3}t^2$ |
| $III(D6)$ | $D_6^{(1)}$ | $q^{-1}+2q^{-2}t^2+q^{-3}t^2$ |
| $III(D7)$ | $D_7^{(1)}$ | $q^{-1}+q^{-2}t^2+q^{-3}t^2$ |
| $III(D8)$ | $D_8^{(1)}$ | $q^{-1}+q^{-3}t^2$ |
| $IV$ | $E_6^{(1)}$ | $q^{-1}+2q^{-2}t^2+q^{-3}t^2$ |
| $II$ | $E_7^{(1)}$ | $q^{-1}+q^{-2}t^2+q^{-3}t^2$ |
| $I$ | $E_8^{(1)}$ | $q^{-1}+q^{-3}t^2$ |

# Idea of proof of Proposition

Analysis of Leray spectral sequence ${}_L E_2^{k,l}$ of $h$:

| | | | |
|---|---|---|---|
| $k=2$ | $0$ | $0$ | $0$ |
| $k=1$ | $0$ | $H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}})$ | $0$ |
| $k=0$ | $\mathbb{C}$ | $\mathbb{C}^{b_1(\mathcal{M})}$ | $\mathbb{C}$ |
| | $l=0$ | $l=1$ | $l=2$ |

Standard algebraic topology shows that

- $b_1(\mathcal{M}) = 0$,
- $\dim_{\mathbb{C}} H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}}) = 10 - \chi(F_\infty^{PX})$,
- the following map is surjective

$$H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) \to H^2(h^{-1}(Y_{-1}), \mathbb{C}) = \mathbb{C}.$$

# End of proof of the Proposition

We get

$$
\begin{aligned}
\mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C}) &\cong \mathrm{Im}(\mathbf{H}^2(Y, \mathbb{R}h_* \underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})) \\
&\cong \mathbb{C}, \\
\mathrm{Gr}^P_{-2} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C}) &= \mathrm{Ker}(\mathbf{H}^2(Y, \mathbb{R}h_* \underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})) \\
&\cong \mathbb{C}^{10-\chi(F^{PX}_\infty)}.
\end{aligned}
$$

# Betti spaces and affine cubic surfaces

Fricke–Klein 1926, van der Put–Saito '09: for each $X$ there exists a quadric

$$Q^{PX} \in \mathbb{C}[x_1, x_2, x_3]$$

such that

$$\mathcal{M}_{\mathrm{B}}^{PX} = (f^{PX}) \subset \mathbb{C}^3$$

where

$$f^{PX}(x_1, x_2, x_3) = x_1 x_2 x_3 + Q^{PX}(x_1, x_2, x_3).$$

# Compactifications of Betti spaces

Let

$$F^{PX} \in \mathbb{C}[x_0, x_1, x_2, x_3]_3$$

be the homogenization of $f^{PX}$ and set

$$\overline{\mathcal{M}}_{\mathrm{B}}^{PX} = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]/(F^{PX})).$$

Projective cubic surface (possibly) with singularities at

$$P_1 = [0:1:0:0], \quad P_2 = [0:0:1:0], \quad P_3 = [0:0:0:1].$$

Let

$$\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX} \to \overline{\mathcal{M}}_{\mathrm{B}}^{PX}$$

denote the minimal resolution of singularities.

## Total Milnor number and weight polynomial

Define the total Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ as

$$N^{PX} = \sum_{j=1}^{3} \mu(P_j)$$

where $\mu(P_j)$ is the Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ at $P_j$.

### Proposition

*We have*

$$WH^{PX}(q,t) = 1 + (4 - N^{PX}) q^{-1} t^2 + q^{-2} t^2.$$

# Table of weight polynomials

| $X$ | Singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ | $WH^{PX}(q,t)$ |
|---|---|---|
| $VI$ | $\emptyset$ | $1+4q^{-1}t^2+q^{-2}t^2$ |
| $V$ | $A_1$ | $1+3q^{-1}t^2+q^{-2}t^2$ |
| $V_{\mathrm{deg}}$ | $A_2$ | $1+2q^{-1}t^2+q^{-2}t^2$ |
| $III(D6)$ | $A_2$ | $1+2q^{-1}t^2+q^{-2}t^2$ |
| $III(D7)$ | $A_3$ | $1+q^{-1}t^2+q^{-2}t^2$ |
| $III(D8)$ | $A_4$ | $1+q^{-2}t^2$ |
| $IV$ | $A_1+A_1$ | $1+2q^{-1}t^2+q^{-2}t^2$ |
| $II$ | $A_1+A_1+A_1$ | $1+q^{-1}t^2+q^{-2}t^2$ |
| $I$ | $A_2+A_1+A_1$ | $1+q^{-2}t^2$ |

# Compactifying divisors

The divisor at infinity of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$D = L_1 \cup L_2 \cup L_3$$

where $L_i$ are lines pairwise intersecting each other in $P_1, P_2, P_3$. The nerve complex of the divisor at infinity of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$\mathcal{N}^{PX} = A^{(1)}_{N^{PX}+2} = I_{N^{PX}+3}.$$

# The first page of the weight spectral sequence

Deligne: spectral sequence ${}_W E_r$ abutting to $H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C})$ with ${}_W E_1^{-n,k+n}$ given by

| | | | |
|---|---|---|---|
| $k+n=4$ | $\bigoplus_{p \in \mathcal{N}_1^{PX}} H^0(p, \mathbb{C})$ | $\bigoplus_{L \in \mathcal{N}_0^{PX}} H^2(L, \mathbb{C})$ | $H^4\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$ |
| $k+n=3$ | $0$ | $0$ | $0$ |
| $k+n=2$ | $0$ | $\bigoplus_{L \in \mathcal{N}_0^{PX}} H^0(L, \mathbb{C})$ | $H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$ |
| $k+n=1$ | $0$ | $0$ | $0$ |
| $k+n=0$ | $0$ | $0$ | $H^0\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$ |
| | $-n=-2$ | $-n=-1$ | $-n=0$ |

# The first differentials of the weight spectral sequence

The only non-trivial differentials $d_1$ on ${}_WE_1$ are:

$$\oplus_{p\in\mathcal{N}_1^{PX}} H^0(p,\mathbb{C}) \xrightarrow{\delta} \oplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \xrightarrow{\delta_4} H^4\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right)$$

$$\oplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \xrightarrow{\delta_2} H^2\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right).$$

Algebraic topology of cubic surfaces shows that

$$\delta_4 : \oplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \twoheadrightarrow H^4\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right)$$

$$\delta_2 : \oplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \hookrightarrow H^2\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right) \cong \mathbb{C}^7.$$

# Dimensions of graded pieces for the weight filtration

We derive

$$\mathrm{Gr}_0^W H^0(\mathcal{M}_{\mathrm{B}}^{PX}) \cong \mathbb{C}$$
$$\mathrm{Gr}_{-2}^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong \mathbb{C}^{4-N^{PX}}$$
$$\mathrm{Gr}_{-4}^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong \mathbb{C}.$$

# Singularities of the Higgs field

Let $X = \mathbb{C}P^1$, $r = 2$, $n = 2$, two irregular singularities:

- Katz-invariant $\frac{1}{2}$ at $z = 0$, i.e. of the form

$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}z}{z^2} + \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}z}{z} + O(1)\mathrm{d}z$$

  with $b_1 \neq 0$ fixed;
- Katz-invariant 1 at $z = \infty$, i.e. of the form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} dz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \frac{dz}{z} + \text{lower order terms}$$

  with $a \neq 0, b \in \mathbb{C}$ fixed.

# Spectral data of irregular Higgs bundles

Refined version of Beauville–Narasimhan–Ramanan.

**Theorem (Sz '15)**

*There exists a birational morphism*

$$\tilde{\sigma} : \widetilde{Z} \to H_2 = P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2))$$

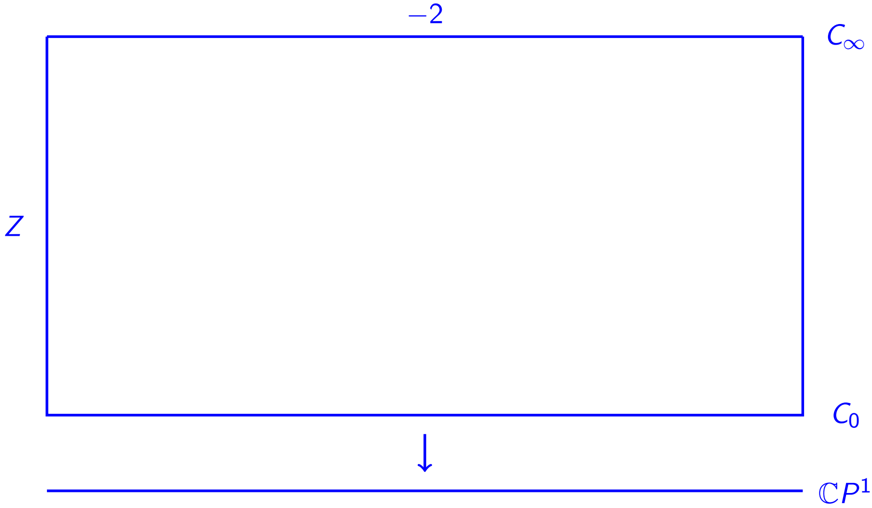
*to the Hirzebruch surface $H_2$ such that there exists an equivalence of categories between the groupoids*

1. *parabolic Higgs bundles $(\mathcal{E}, \theta)$ of rank 2 on $\mathbb{C}P^1$ with irregular singularities as above, and a compatible parabolic structure,*
2. *$\mathbb{R}$-parabolic pure sheaves $S_\bullet$ of dimension 1 and rank 1 with parabolic divisor*

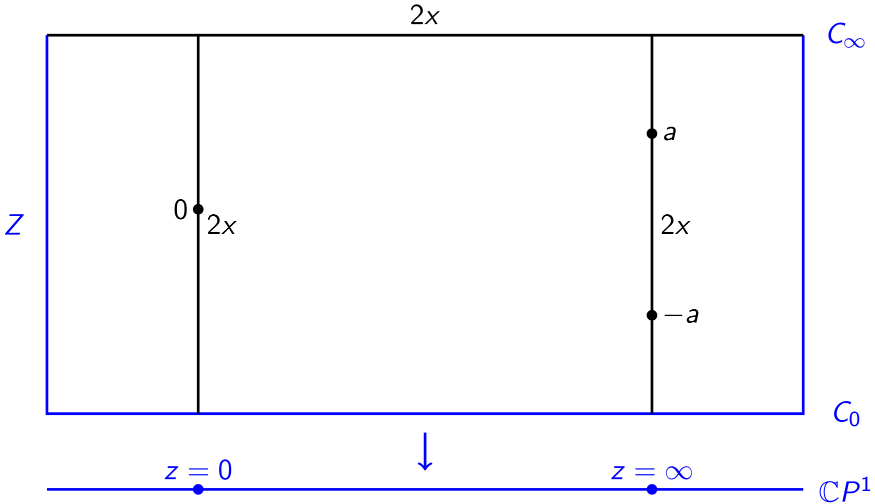
$$(p \circ \tilde{\sigma})^{-1}(\{0, \infty\})$$

*on $\widetilde{Z}$, with support $\widetilde{\Sigma}$ satisfying a list of properties.*

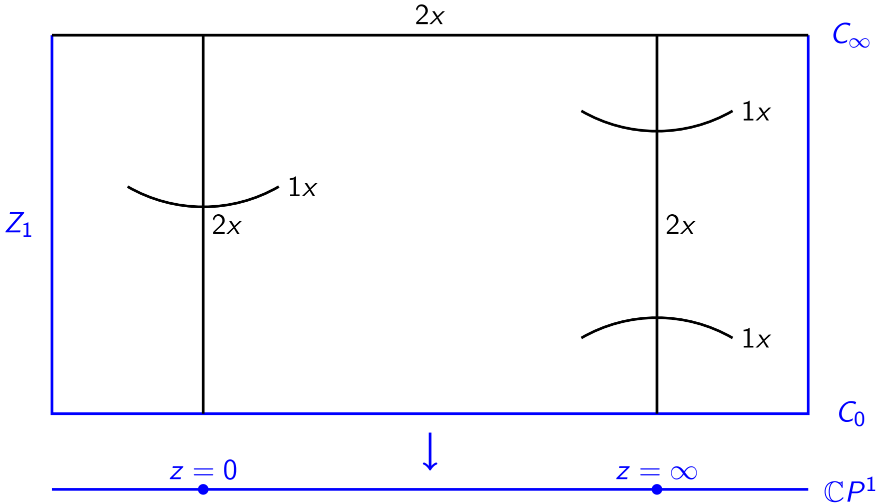
# Hirzebruch surface $H_2 = P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2))$



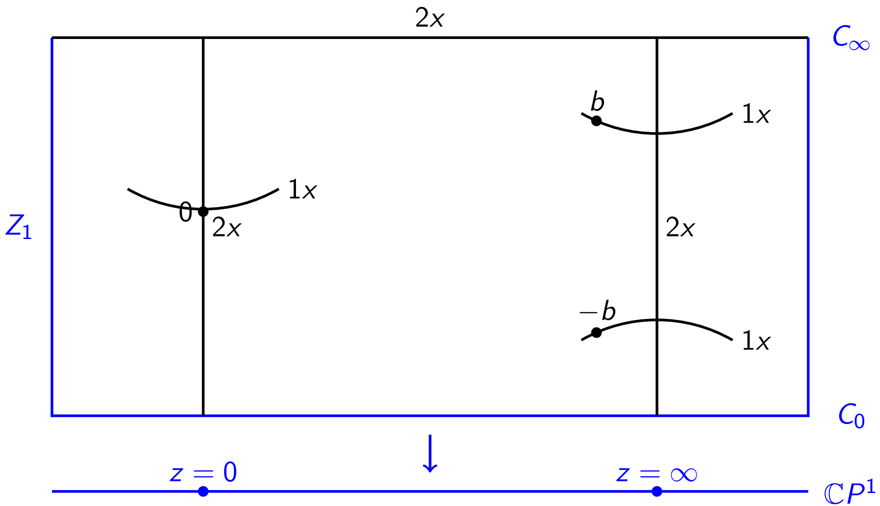
# Base points on $H_2$



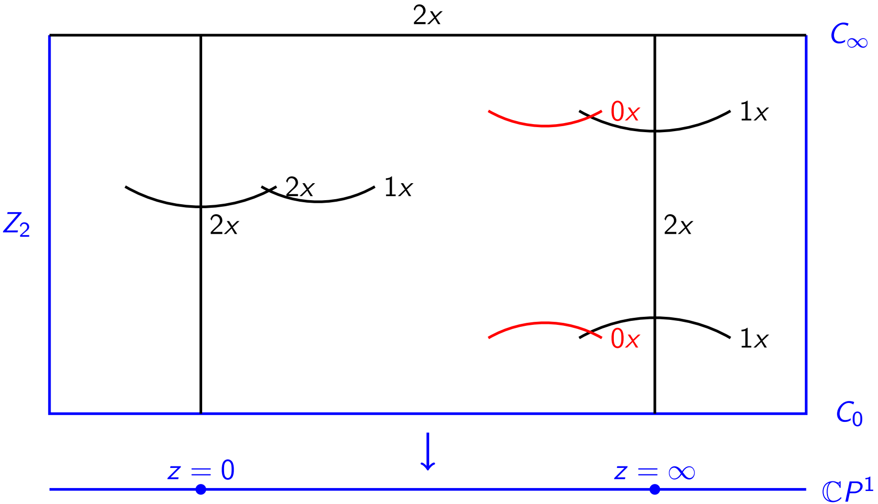
# The first blow-up



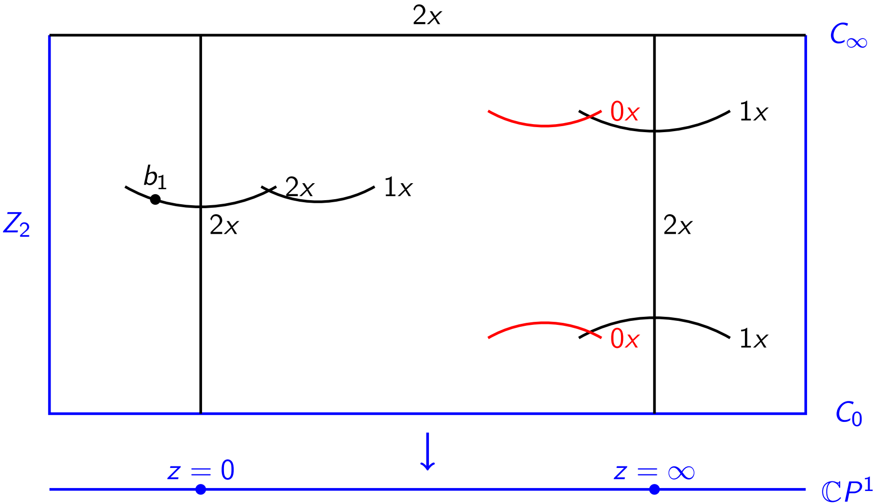
# Base points on the first blow-up



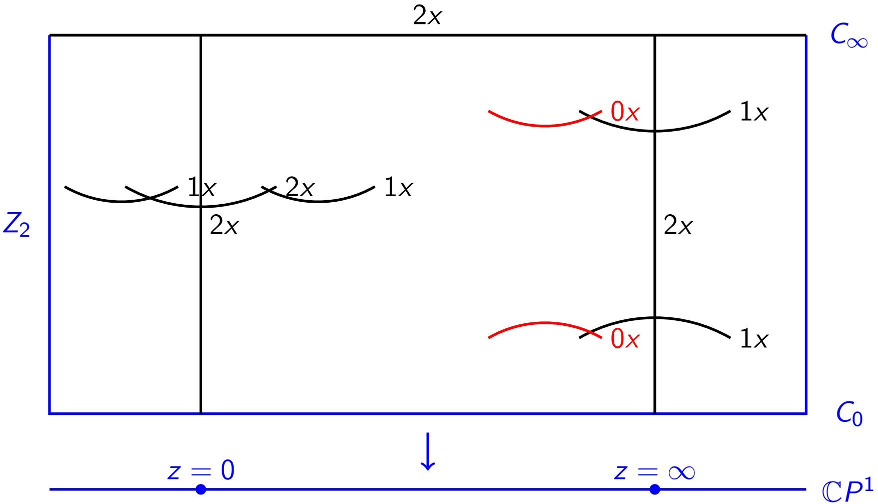
# The second blow-up



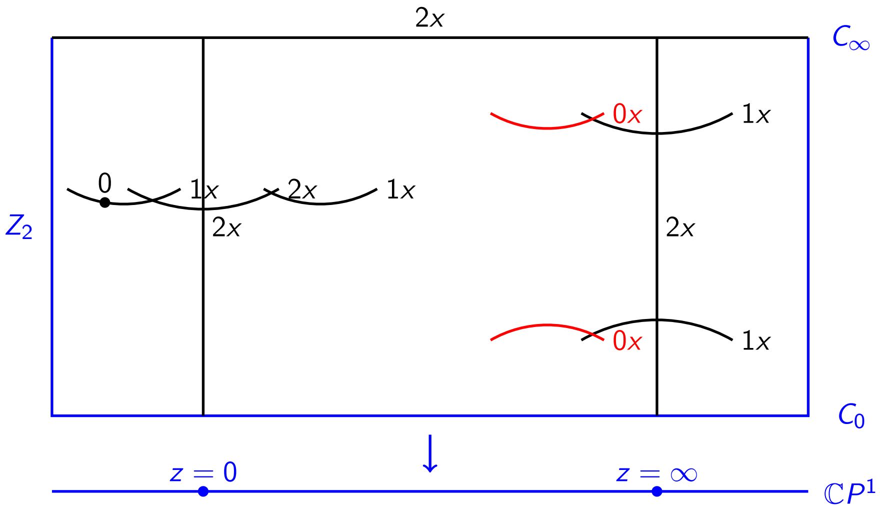
# Base point on the second blow-up



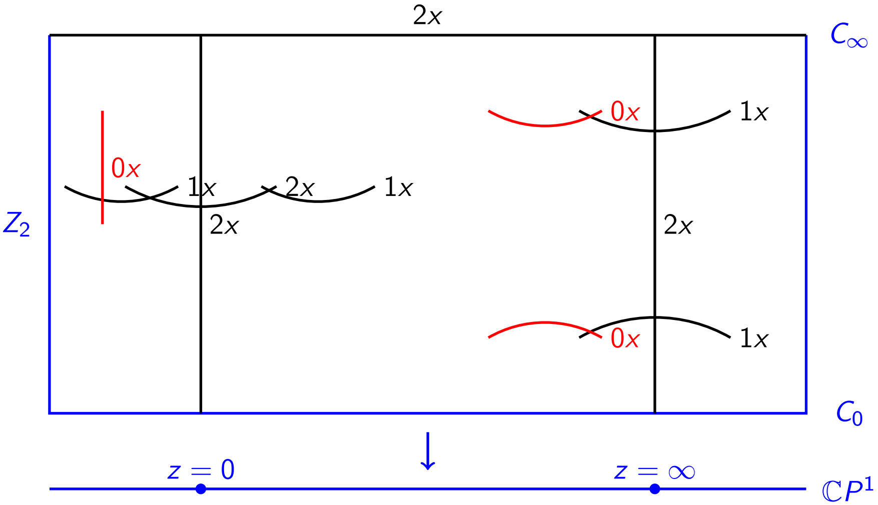
# The third blow-up



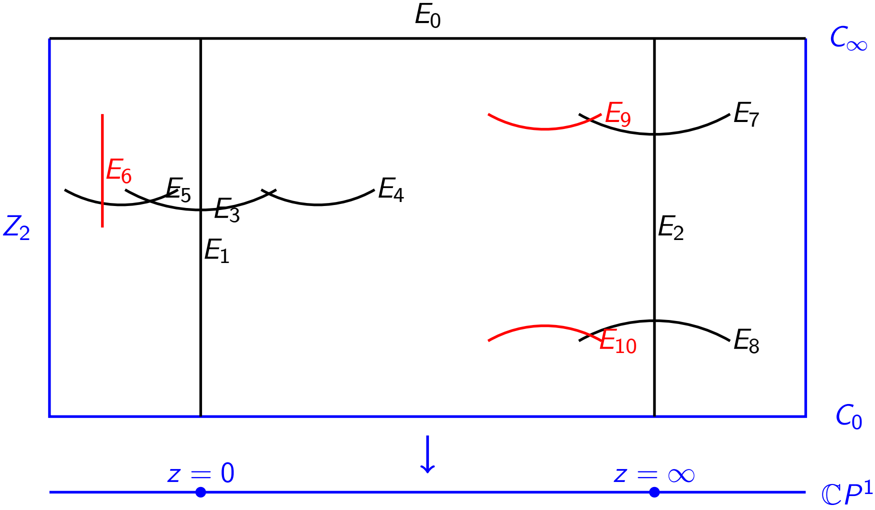
# Base point on the third blow-up



# The fourth blow-up



# Divisors on $\widetilde{Z}$



# The elliptic fibration

We consider the linear system $X$ of elliptic curves $\widetilde{\Sigma}$ on $\widetilde{Z}$ such that

$$\begin{aligned}
[\widetilde{\Sigma}] \cdot [E_0] = [\widetilde{\Sigma}] \cdot [E_1] = [\widetilde{\Sigma}] \cdot [E_2] = [\widetilde{\Sigma}] \cdot [E_3] = [\widetilde{\Sigma}] \cdot [E_4] \\
= [\widetilde{\Sigma}] \cdot [E_5] = [\widetilde{\Sigma}] \cdot [E_7] = [\widetilde{\Sigma}] \cdot [E_8] = 0, \\
[\widetilde{\Sigma}] \cdot [E_9] = [\widetilde{\Sigma}] \cdot [E_{10}] = 1 \\
[\widetilde{\Sigma}] \cdot [E_6] = 2.
\end{aligned}$$

Properties of this family:

- no base points;
- 1-dimensional, parameterized by $\mathbb{C}P^1$;
- the fiber over $\infty \in \mathbb{C}P^1$ is a degenerate elliptic curve of type $D_7^{(1)} = I_3^*$:

$$F_\infty^{PIII(D7)} = 2 \cdot E_0 + 2 \cdot E_1 + 2 \cdot E_2 + 2 \cdot E_3 + E_4 + E_5 + E_7 + E_8$$

# Singular fibers of the fibration

Introduce

$$B = \mathbb{C}P^1 \setminus \{\infty\} \subset \mathbb{C}P^1.$$

and let

$$X_{|B} \to B$$

be the restriction of $X$ to $B$. Set

$$\Delta = a^3 b_1 \left(16 b^3 - 54 a b_1\right).$$

Then, the singular fibers of $X_{|B}$ are

1. if $\Delta = 0$, then a type $II$ and an $I_1$ fibers;
2. if $\Delta \neq 0$, then three $I_1$ fibers.

# Relative Picard

The fibration has sections

$$\sigma : B \to X.$$

Abel–Jacobi: for any smooth $\widetilde{\Sigma} = X_b$, get

$$\begin{aligned} \widetilde{\Sigma} &\cong \mathrm{Pic}^0(\widetilde{\Sigma}) \\ x &\mapsto (x - \sigma(b)). \end{aligned}$$

For $b \in B$ such that $X_b$ is of type $I_1$ or $II$, D'Souza '79 and Altman–Kleiman '90 show that the compactified Jacobian exists and is biregular to $X_b$.

# Stokes matrices and monodromy near $\infty$

van der Put–Saito: let $\nabla$ be an irregular connection in the corresponding de Rham space. Near $\infty$, with respect to some trivialization $f_1, f_2$ the formal monodromy and the Stokes matrices read as

$$\begin{pmatrix} e^{4i\pi a} & 0 \\ 0 & e^{-4i\pi a} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}$$

for some $c_1, c_2 \in \mathbb{C}$. The topological monodromy is then

$$M_\infty = \begin{pmatrix} e^{4i\pi a} & 0 \\ 0 & e^{-4i\pi a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}$$

# Stokes matrices and monodromy near 0

Near 0, with respect to some trivialization $e_1, e_2$ the formal monodromy and the Stokes matrix are

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$$

for some $e \in \mathbb{C}$. The topological monodromy is then

$$M_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} = \begin{pmatrix} -e & -1 \\ 1 & 0 \end{pmatrix}$$

# Link and affine cubic

The trivializations $e_1, e_2$ and $f_1, f_2$ are related by a matrix

$$L = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}.$$

We then have the relation

$$M_0 L^{-1} M_\infty L = \mathrm{I}_2.$$

After some eliminations and changes of variables, this gives the relation

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0.$$

for some $\alpha \in \mathbb{C}^\times$.

# Determining the singularities of $\overline{\mathcal{M}}_B^{PIII(D7)}$

The only singular point of $\overline{\mathcal{M}}_B^{PIII(D7)}$ is $[0:0:0:1]$, of local equation

$$x_1x_2 + x_0x_1^2 + x_0x_2^2 + \alpha x_0^2x_1 + x_0^2x_2.$$

We extract its homogeneous terms of degree 3:

$$f_3 = x_0x_1^2 + x_0x_2^2 + \alpha x_0^2x_1 + x_0^2x_2.$$

We plug $x_1 = 0, x_0 = 1$ in $f_3$:

$$f_3(x_0, 0, x_2) = x_2^2 + x_2.$$

This has non-vanishing linear term in $x_2$. Similarly, plugging $x_2 = 0$ and $x_0 = 1$ in $f_3$ we get a non-vanishing linear term $\alpha x_1$.
Bruce–Wall '79: the singularity is then of type $A_3$.