THE EXTENSION OF A FUCHSIAN EQUATION ONTO
THE PROJECTIVE LINE

by

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Abstract. — In this paper, we show how to extend the holomorphic connection induced by a scalar Fuchsian equation on an open subset of the projective line as a logarithmic connection on the projective line in such a way that the eigenvalues of the residue of the integrable connection in a logarithmic point agree with the exponents of the equation.

1. Introduction

Denote by $\mathbb{P}^1$ the complex projective line and fix $p_1, \ldots, p_n$ in its affine part $\mathbb{A}^1$. We denote by $p_0$ the point at infinity $\infty$, by $P$ the set $\{p_0, \ldots, p_n\}$ and by $P_0$ the set $\{p_1, \ldots, p_n\}$. We set

$$\psi(z) = (z - p_1) \cdots (z - p_n).$$

For $w = w(z)$ a holomorphic function of the complex variable $z$, let $w^{(k)}$ its $k$-th order differential with respect to $z$. Let $G_1, \ldots, G_m$ be polynomials in $z$ such that the degree of $G_k$ is at most $k(n - 1)$. Then, the scalar homogeneous differential equation of order $m$ with meromorphic coefficients

$$w^{(m)} - \frac{G_1(z)}{\psi(z)} w^{(m-1)} - \cdots - \frac{G_m(z)}{\psi(z)^{m}} w = 0,$$

is called a Fuchsian equation. We shall denote the left-hand side by $\mathcal{L}(w)$. It has the property that all of its solutions near any point of the singular set $P$ have moderate growth, i.e. in any fixed sector with vertex the singular point $p_j$, the absolute value $|w(z)|$ of any given solution at the point $z$ can be bounded from above by a polynomial of $|z - p_j|^{-1}$. More precisely (see 15.23. [Inc26]), there exist complex numbers $\varepsilon_k^j$ for $k \in \{1, \ldots, m\}$ called
the exponents of the equation, such that a linearly independent set of solutions in such a sector can be given by functions with asymptotic behaviour

\[(z - p_j)^{\varepsilon_k}\]

as \(z \to p_j\). One has a characterisation of the exponents in terms of the coefficients of \(\mathcal{L}\). Namely, rewrite (2) as

\[
w^{(m)} - \frac{G^j_1(z)}{z - p_j} w^{(m-1)} - \cdots - \frac{G^j_m(z)}{(z - p_j)^m} w = 0,
\]

where the coefficients

\[
G^j_k(z) = \frac{G_k(z)(z - p_j)^k}{\psi(z)^k}
\]

are holomorphic functions in a neighborhood of \(p_j\). We introduce the notation

\[\varepsilon_k = \varepsilon(\varepsilon - 1) \cdots (\varepsilon - k + 1).\]

Then, the \(\varepsilon_k^j\) are the solutions of the indicial equation

\[
[\varepsilon]_m - G^j_1(p_j)[\varepsilon]_{m-1} - \cdots - G^j_{m-1}(p_j)\varepsilon - G^j_m(p_j) = 0.
\]

It is also well-known that the first \(m\) derivatives of the solutions of (2) give rise upon analytic continuation to a representation (called the monodromy representation) of the fundamental group of \(\mathbb{P}^1 \setminus P\) into \(GL(m, \mathbb{C})\). In this paper, we will assume the following:

**Condition 1.1.** — For any fixed \(j \in \{0, \ldots, n\}\), all the values \(\varepsilon_k^j\) for \(k \in \{1, \ldots, m\}\) are distinct modulo \(\mathbb{Z}\). Furthermore, the monodromy representation of the equation is irreducible.

**Remark 1.2.** — The first of these conditions is clearly generic. The second one holds for example if no non-trivial sub-sum of the set of exponents is an integer, hence it is also generic.

On the other hand, let \(E\) be a holomorphic vector bundle of rank \(m\) on \(\mathbb{P}^1\). Denote by \(\mathcal{O}\) the sheaf of holomorphic functions on \(\mathbb{P}^1\), by \(\Omega^1\) the sheaf of holomorphic 1-forms, and by \(\Omega^1(P)\) the sheaf of meromorphic 1-forms with at most simple poles in the points of \(P\) and no other poles. An integrable connection logarithmic in \(P\) over \(E\) is a sheaf map

\[
D : E \to E \otimes \Omega^1(P)
\]
satisfying the Leibniz-rule
\[ D(fe) = \frac{df}{dz}edz + fD(e) \]
for any \( f \in \Gamma(U, \mathcal{O}) \) and \( e \in \Gamma(U, E) \) on any open set \( U \). In dimension 1, the word \textit{integrable} is superfluous for the lack of higher holomorphic differential forms, hence we shall usually omit it. For any \( p_j \in P \), the \textit{residue} of \( D \) in \( p_j \) is a well-defined endomorphism of the fiber \( E_{p_j} \) of \( E \) at \( p_j \), which we shall denote by \( \text{res}(p_j, D) \).

There are various ways of associating a logarithmic connection \( (E, D) \) to a Fuchsian equation \( \mathcal{L} \) in such a way, that the solutions of the equation on any open set \( U \subset \mathbb{P}^1 \setminus P \) be in one-to-one correspondence with the sections of \( E \) on \( U \) annihilated by \( D \). For example, Deligne’s Riemann-Hilbert correspondence (see e.g. Theorem 4.4. [Mal87]) ensures the existence of a logarithmic lattice for the local system associated to the equation, such that all the eigenvalues of the residues lie inside the strip \( 0 \leq \Re(z) < 1 \) in \( \mathbb{C} \). In any case, it is known that the eigenvalues of the residue of such an extension \( (E, D) \) at \( p_j \) agree up to integers with the exponents of \( \mathcal{L} \). We show that an extension can be given where this is true not only up to integers:

\textit{Theorem 1.3.} — For any Fuchsian equation \( \mathcal{L} \) there exists an associated logarithmic connection \( D \) on the holomorphic bundle
\[ (7) \quad \mathcal{O} \oplus \mathcal{O}(1-n) \oplus \ldots \oplus \mathcal{O}((m-1)(1-n)) \]
such that for any \( p_j \in P \) the set of eigenvalues of \( \text{res}(p_j, D) \) is equal to the set of exponents of \( \mathcal{L} \) at \( p_j \). Conversely, for any logarithmic connection \( D \) on \( (7) \), there exists a Fuchsian equation \( \mathcal{L} \) such that \( D \) is associated to \( \mathcal{L} \).

\textit{Remark 1.4.} — (i). The same result, without equality of the eigenvalues of the residue and the exponents, was already obtained by A. Bolibruh (see Propositions 6.13 and 6.14, [vdPS03]). However, our proof is different from his.

(ii). By the residue formula, the sum of the eigenvalues of the residue of a logarithmic connection in all logarithmic points is equal to the negative of the degree of the underlying vector bundle. The degree of \( (7) \) is equal to \((1-n)m(m-1)/2\). Therefore, our theorem gives a new proof of the classical Fuchs’ relation stating that the sum of all the exponents of an equation in all singular points is equal to \((n-1)m(m-1)/2\).
2. Proof of Theorem 1.3

We start by explicitly constructing an extension of a Fuchsian equation with the desired property. On the affine part $\mathbb{A}^1$ of $\mathbb{P}^1$, let us introduce the expressions

\[ w_1 = w, \]
\[ w_2 = \psi \frac{dw}{dz}, \]
\[ \vdots \]
\[ w_m = \psi^{m-1} \frac{d^{m-1}w}{dz^{m-1}}. \]

Here, $w$ is taken to be a local section of the structure sheaf $\mathcal{O}$ of $\mathbb{P}^1$. Denote by $\psi' = d\psi/dz$, and consider the algebraic integrable connection on $\mathbb{A}^1$ with logarithmic poles at $P_0$ which can be written in the trivialisation (8) as

\[ D_A = d^{1,0} - \frac{A(z)}{\psi(z)} dz, \]

where $A(z)$ is the modified companion matrix

\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \psi' & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2\psi' & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (m-2)\psi' & 1 \\ G_m & G_{m-1} & G_{m-2} & \cdots & G_2 & G_1 + (m-1)\psi' \end{pmatrix} \]

of equation (2). One readily checks that a meromorphic function $w$ on some open set $U \subset \mathbb{P}^1 \setminus P$ is annihilated by $\mathcal{L}$ if and only if the vector $(w_1, w_2, \ldots, w_m)$ is a parallel section of $D_A$ over $U$ for some (hence, only one) vector $(w_2, \ldots, w_m)$. This gives the extension of $\mathcal{L}$ over $P_0$ as a logarithmic connection. We now need to define the extension over $\infty$. For this purpose, let $\zeta = z^{-1}$ be the coordinate at infinity. Then, because of the degree conditions on the $G_k$, Fuchs’ theorem implies that on the open set $\mathbb{P}^1 \setminus (P \cup \{0\})$ the equation $\mathcal{L}(w) = 0$ is equivalent to

\[ \frac{d^m w}{d\zeta^m} (\zeta^{-1}) = \frac{G_0(\zeta)}{\zeta} \frac{d^{m-1}w}{d\zeta^{m-1}} (\zeta^{-1}) + \cdots + \frac{G_m(\zeta)}{\zeta^m} w(\zeta^{-1}), \]
where $G^0_k$ are some holomorphic functions of $\zeta$. Define the lattice at infinity by the trivialisation

\begin{align*}
    w^0_1 &= w \\
    w^0_2 &= \zeta \frac{dw}{d\zeta} \\
    &\vdots \\
    w^0_m &= \zeta^{m-1} \frac{d^{m-1}w}{d\zeta^{m-1}}.
\end{align*}

By the above, the solutions on $\mathbb{P}^1 \setminus P$ of $\mathcal{L}$ are in one-to-one correspondence with the first components of parallel sections of the logarithmic connection

\begin{equation}
    D_A = d^{1,0} - \frac{A^0(\zeta)}{\zeta} d\zeta,
\end{equation}

where

\begin{equation}
    A^0 = \begin{pmatrix}
        0 & 1 & 0 & 0 & \cdots & 0 \\
        0 & 1 & 1 & 0 & \cdots & 0 \\
        \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
        0 & 0 & 0 & \cdots & (m-2) & 1 \\
        G^0_m & G^0_{m-1} & G^0_{m-2} & \cdots & G^0_2 & G^0_1 + (m-1)
    \end{pmatrix}.
\end{equation}

The extensions (8) and (12) define a holomorphic bundle $E$ on $\mathbb{P}^1$, and the formulae (9)-(10) and (13)-(14) induce a logarithmic connection $D$ on $E$.

Let us show that these extensions over the singular points satisfy the properties claimed. First, we identify the type of the underlying holomorphic bundle. As

\[
    \zeta \frac{dw}{d\zeta} = -z \frac{dw}{dz}
\]

and

\[
    \begin{bmatrix} z, \frac{d}{dz} \end{bmatrix} = \begin{bmatrix} \zeta, \frac{d}{d\zeta} \end{bmatrix} = -1,
\]

we deduce by induction expressions of the form

\[
    \zeta^k \frac{d^k w}{d\zeta^k} = (-z)^k \frac{d^k w}{dz^k} + a_{k,k-1} (-z)^{k-1} \frac{d^{k-1} w}{dz^{k-1}} + \cdots + a_{k,1} (-z) \frac{dw}{dz}
\]
for constants $a_{k,l} \in \mathbb{Z}$. It follows that on the intersection $\mathbb{P}^1 \setminus P$ of the domains of the two trivialisations, these latter are linked by the matrix

\[
\text{diag} \left( 1, \frac{\psi(z)}{z}, \ldots, \frac{\psi(z)^{m-1}}{z^{m-1}} \right)
\begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & \ldots \\
0 & a_{2,1} & 1 & 0 & \ldots \\
0 & a_{3,1} & a_{3,2} & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

This matrix is clearly conjugate under $Gl(m, \mathbb{Q})$ to

\[
\text{diag} \left( 1, \frac{\psi(z)}{z}, \ldots, \frac{\psi(z)^{m-1}}{z^{m-1}} \right).
\]

Now, for large $|z|$ one has the asymptotic

\[
\frac{\psi(z)}{z} \asymp z^{n-1},
\]

so the leading order term of the above diagonal matrix is

\[
\text{diag} \left( 1, z^{n-1}, \ldots, z^{(m-1)(n-1)} \right).
\]

As this latter is the gluing matrix for the holomorphic bundle (7), we deduce that this is the type of the underlying holomorphic vector bundle $E$ of the constructed logarithmic connection.

Next, we find the eigenvalues of $\text{res}(p_j, D)$ with respect to these trivialisations. First, observe that the gauge transformation which has matrix with respect to the trivialisation (8)

\[
\text{diag} \left( 1, \frac{z - p_j}{\psi(z)}, \ldots, \frac{(z - p_j)^{m-1}}{\psi(z)^{m-1}} \right)
\]

is holomorphic at $p_j$. It obviously maps the trivialisation (8) into

\[
w_1 = w
\]

\[
w_2 = (z - p_j) \frac{dw}{dz}
\]

\[
\vdots
\]

\[
w_m = (z - p_j)^{m-1} \frac{d^{m-1}w}{dz^{m-1}}.
\]

With respect to this trivialisation, the form of the connection (9) is written

\[
d^{1,0} \frac{A^j(z)}{z - p_j} dz,
\]
with

\[
A_j = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 1 & 0 & \ldots & 0 \\
0 & 0 & 2 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (m - 2) & 1 \\
G^j_m & G^j_{m-1} & G^j_{m-2} & \ldots & G^j_2 & G^j_1 + (m - 1)
\end{pmatrix},
\]

where the holomorphic functions \(G^j_k\) were introduced in (4). Clearly, the characteristic polynomial of this matrix is precisely the indicial equation (5) of equation (2). The computation is analogous for \(p_0 = \infty\).

We now come to the proof of the second statement. It is a classically known fact that any finite-rank system of first-order differential equations on \(\mathbb{P}^1 \setminus P\) is associated to a scalar linear differential equation \(\mathcal{L}\) with singularities in the points \(P\) plus some additional points \(S\) called apparent singularities. At the points of \(S\) the coefficients of \(\mathcal{L}\) may have poles, but all of its solutions extend holomorphically. In Section 4 of [Oht82], the zero and pole sets of the Wronskian corresponding to a global section of the dual holomorphic bundle are analyzed. The zero set of the Wronskian is precisely the set of apparent singularities of the associated scalar equation. Notice that this set depends on the choice of global section of the dual bundle. In particular, the following result is implicit:

**Lemma 2.1 (Ohtsuki).** — Suppose that \((E, D)\) is a connection on \(\mathbb{P}^1\) logarithmic at \(P\), such that the dual holomorphic vector bundle \(E^*\) admits a global section. Then, there exists a scalar Fuchsian equation \(\mathcal{L}\) such that \((E, D)\) is associated to \(\mathcal{L}\), and the number of apparent singularities of \(\mathcal{L}\) is not greater than

\[
\deg(E^*) + (n - 1)\frac{m(m - 1)}{2}.
\]

Let us show how this lemma implies our second statement. Suppose we are given a logarithmic connection \(D\) on the bundle

\[
E = \mathcal{O} \oplus \mathcal{O}(1 - n) \oplus \ldots \oplus \mathcal{O}((m - 1)(1 - n)).
\]

Then, the connection induced by \(D\) on the bundle

\[
E((m - 1)(n - 1)) = \mathcal{O}((m - 1)(n - 1)) \oplus \mathcal{O}((m - 2)(n - 1)) \oplus \ldots \oplus \mathcal{O}
\]

is also logarithmic. The dual of this bundle is

\[
(E((m - 1)(n - 1)))^* = \mathcal{O}((m - 1)(1 - n)) \oplus \mathcal{O}((m - 2)(1 - n)) \oplus \ldots \oplus \mathcal{O},
\]
which clearly admits the global section \((0, \ldots, 0, 1)\). On the other hand, we have
\[
\deg \left((E((m - 1)(n - 1)))^*\right) = (1 - n) \frac{m(m-1)}{2}.
\]
The lemma implies that a scalar equation \(\mathcal{L}\) without apparent singularities can be found such that \((E((m - 1)(n - 1)), D)\) is a logarithmic connection associated to \(\mathcal{L}\). Then, since \((E, D)\) and \((E((m - 1)(n - 1)), D)\) only differ in their extension over \(P\), we infer that the logarithmic connection \((E, D)\) is also associated to the genuine Fuchsian equation \(\mathcal{L}\), and the proof is finished.

References


