Hyper-Kähler isometry of Fourier–Laplace–Nahm transform

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Outline

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Notations

X: smooth projective curve over $\mathbf{C}$
$G = \mathrm{Gl}_r(\mathbf{C})$
$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$P = \{p_0, p_1, \dots, p_n\}$: a finite set of distinct points in X
$\mathcal{O}$: sheaf of holomorphic functions
Ω^k: sheaf of smooth k-forms
$\mathbf{\Omega}^1$: sheaf of holomorphic 1-forms
$\mathbf{\Omega}^1(*P)$: sheaf of meromorphic 1-forms on X with poles of arbitrarily high order at P

Irregular Higgs bundles on curves

Let $\mathcal{E}$ be a holomorphic vector bundle of rank r on X and

$$\theta : \mathcal{E} \longrightarrow \mathbf{\Omega}^1(*P) \otimes_{\mathcal{O}} \mathcal{E}$$

a meromorphic Higgs field.
We write

$$\theta = T_n \frac{dz}{z^n} + \cdots + T_2 \frac{dz}{z^2} + O(z^{-1})dz$$

with respect to some local analytic coordinate z centered at $p \in P$ and some holomorphic trivialisation of $\mathcal{E}$.
We assume that $T_2, \dots, T_n$ belong to some torus $\mathfrak{t} \subset \mathfrak{gl}_r(\mathbf{C})$.

Parabolic structure at singular points

A compatible parabolic structure for θ at p is the choice of

$$\alpha \in \mathfrak{t}_{\mathbf{R}}.$$

Up to conjugation we may assume $\mathfrak{t}$ consists of diagonal matrices, so we have

$$\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_r).$$

To α we associate the parabolic subgroup

$$P_\alpha = \{g \in \mathrm{Gl}_r(\mathbf{C}) | \quad z^\alpha g z^{-\alpha} \text{ exists as } z \to 0\}$$

of G with Lie-algebra denoted by $\mathfrak{p}_\alpha$.

Residues

Let

$$H \subset \mathsf{Gl}_r(\mathbf{C})$$

stand for the common centraliser of $T_2, \ldots, T_n$ and $\mathfrak{h}$ for its Lie-algebra.
We assume that

$$T_1 \in \mathcal{O} \subset \mathfrak{h} \cap \mathfrak{p}_\alpha$$

lies in a fixed semi-simple adjoint orbit, defined by eigenvalues

$$\lambda_1, \ldots, \lambda_r.$$

Stability of irregular parabolic Higgs bundles

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_k^j$$

and

$$\text{par-slope}(\mathcal{E}) = \frac{\text{par-deg}(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

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$(\mathcal{E}, \theta)$ is said to be parabolically stable if for all non-trivial proper subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\text{Im}(\theta|_{\mathcal{F}}) \subset \mathbf{\Omega}^1(*P) \otimes \mathcal{F}$, one has

$$\text{par-slope}(\mathcal{F}) < \text{par-slope}(\mathcal{E}).$$

Moduli spaces of irregular Higgs bundles

Theorem (O. Biquard – P. Boalch 2004)

For generic values of the parameters the moduli space $\mathcal{M}^s_{\mathrm{Dol}}$ of stable irregular parabolic Higgs bundles with prescribed singularity data up to gauge transformations is a smooth complete hyper-Kähler manifold.

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Remark

The proof relies on a "wild" non-abelian Hodge theory connecting the moduli problem of irregular parabolic Higgs bundles to a de Rham moduli problem of irregular parabolic connections (E, D).

Deformation theory and Dolbeault complex structure

From now on, the parameters are assumed to be generic so that $\mathcal{M}_{\mathrm{Dol}}^s$ is smooth and complete, and $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}^s$.
The tangent space of $\mathcal{M}_{\mathrm{Dol}}^s$ at $(\mathcal{E}, \theta)$ can be naturally identified as a subspace

$$T_{(\mathcal{E},\theta)}\mathcal{M}_{\mathrm{Dol}}^s \subset \mathbf{H}^1(X; \mathcal{E}nd(\mathcal{E}) \xrightarrow{\mathrm{ad}_\theta} \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}(*P))$$

of the first hypercohomology space of the Dolbeault complex for endomorphisms. With this identification, the Dolbeault complex structure I on $\mathcal{M}_{\mathrm{Dol}}^s$ is induced by the standard complex structure of $\mathbf{H}^1$.

Dolbeault holomorphic symplectic structure

Given a hyper-Kähler manifold (M, g, I, J, K), let

$$\omega_J(.,.) = g(., J.), \quad \omega_K(.,.) = g(., K.)$$

be the Kähler forms and

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K.$$

Then (I, Ω_I) defines a holomorphic symplectic structure on M. For $\mathcal{M}^s_{\mathrm{Dol}}$ with g, I, J, K defined by wild non-abelian Hodge theory the Dolbeault holomorphic symplectic form is given by

$$\Omega_I((\dot{A}, \dot{\Phi}), (\dot{B}, \dot{\Psi})) = \langle (\dot{A}, \dot{\Phi}) \cup (\dot{B}, \dot{\Psi}), [X] \rangle.$$

Isometries between moduli spaces

Question

Are there isometries between irregular Dolbeault moduli spaces ?

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Yes, some are given by Nahm transformation.

Assumption on points at finite distance

From now on we let $X = \mathbf{P}^1$, $p_1, \dots, p_n \in \mathbf{C}$, $p_0 = \infty$.
θ is supposed to have a logarithmic singularity (i.e., $n = 0$) at p_j for $j \in \{1, \dots, n\}$: in a local trivialisation of E near p_j, one has

$$\theta = \frac{T^j(z)}{z - p_j} dz,$$

where T^j is a holomorphic $\mathfrak{gl}$-valued function defined near p_j.
Furthermore, the residue

$$T^j(p_j) = \mathrm{diag}(0, \dots, 0, \lambda^j_{r_j+1}, \dots, \lambda^j_r),$$

is diagonal, with λ^j_k non-zero and generic.

ASSUMPTION AT INFINITY

θ is supposed to have an irregular singularity with $n - 1 = 1$ at infinity: in a local trivialisation of E near ∞, one has

$$\theta = Adz + B\frac{dz}{z} + \text{lower order terms},$$

where

$$A = \operatorname{diag}(\xi_1, \dots, \xi_1, \dots\dots\dots, \xi_{n'}, \dots.., \xi_{n'})$$
$$B = \operatorname{diag}(\lambda_1^0, \dots, \lambda_{a_2}^0, \dots\dots, \lambda_{1+a_{n'}}^0, \dots, \lambda_r^0)$$

(the leading order term and residue, respectively). Here the ξ_k are pairwise distinct constants, and the λ_l^0 are generic non-zero. (Notation:$a_1 = 0, a_{n'+1} = r$.)

Dolbeault complex

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.

Define a sheaf $\mathcal{F}$ by

$$0 \to \mathcal{E} \otimes \mathbf{\Omega}^1 \to \mathcal{F} \to \oplus_{j=1}^{n} \mathrm{Im}(\mathrm{res}_{p_j} \theta) \otimes \mathbf{\Omega}^1_p(p) \to 0.$$

Define the Dolbeault complex $\mathcal{D}ol$ by

$$\mathcal{E} \xrightarrow{\theta} \mathcal{F}$$

with the terms in degrees 0 and 1.

Fourier–Laplace–Nahm Transform

Consider the product

$$\mathbf{C} \times \widehat{\mathbf{C}}$$

with projection maps $\pi, \hat{\pi}$. We define a Poincaré Higgs-bundle:

$$\mathcal{P} = \left(\mathcal{O}_{\mathbf{C} \times \widehat{\mathbf{C}}}, \frac{1}{2}(\xi \mathrm{d}z + z \mathrm{d}\xi) \right).$$

with $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ the standard coordinate.
The transformed Higgs bundle is defined by

$$\mathcal{N}(\mathcal{E}, \theta) = \mathbf{R}_1 \hat{\pi}_* (\pi^* \mathcal{D}ol \otimes \mathcal{P})$$

on $\widehat{\mathbf{C}} \setminus \widehat{P}$. It admits an extension to $\widehat{\mathbf{P}}^1$.

Properties of the transform

We set

$$\mathcal{N}(\mathcal{E}, \theta) = (\widehat{\mathcal{E}}, \widehat{\theta}).$$

Then (Sz 2007):

- $\widehat{\theta}$ has the same kind of singularities as θ (with different local parameters).
- $\mathcal{N}$ preserves (poly-)stability.
- $\mathcal{N}$ is involutive (up to a sign).

Isometry

Theorem (Sz 2014)

$\mathcal{N}$ is a hyper-Kähler isometry.

Strategy of proof: show

$$I \mapsto \widehat{I}$$

$$J \mapsto \widehat{J}$$

$$\Omega_I \mapsto \Omega_{\widehat{I}}$$

Complex structures

The transformation of the complex structure I is immediate from the definition (c.f. Aker–Sz 2014).
The transformation of the complex structure J follows from identification with minimal extension followed by Fourier–Laplace transform of the underlying holonomic $\mathcal{D}$-module (Sz 2012).
From now on we will focus on the transformation of Ω_I.

Beauville–Narasimhan–Ramanan correspondence

Set

$$L = \mathbf{\Omega}^1_{\mathbf{P}^1}(P) = \mathbf{\Omega}^1_{\mathbf{P}^1}(p_1 + \cdots + p_n + 2 \cdot \infty)$$

and consider the ruled surface

$$Z = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus L) \xrightarrow{\pi} \mathbf{P}^1$$

with relatively ample line bundle $\mathcal{O}(1)$ and global sections

$$x \in H^0(Z, \mathcal{O}(1) \otimes \pi^* L), \quad y \in H^0(Z, \pi^* L).$$

Consider the cokernel sheaf $M_{(\mathcal{E},\theta)}$ defined by

$$0 \to \pi^*(\mathcal{E} \otimes L^\vee) \xrightarrow{\pi^*\theta \otimes y - \pi^* Id_{\mathcal{E}} \otimes x} \pi^*\mathcal{E} \otimes \mathcal{O}(1) \to M_{(\mathcal{E},\theta)} \to 0.$$

Beauville–Narasimhan–Ramanan correspondence, cont'd

It is possible to recover $(\mathcal{E}, \theta)$ from $M_{(\mathcal{E},\theta)}$:

$$\mathcal{E} = \pi_* M_{(\mathcal{E},\theta)}, \quad \theta = \pi_*(x : M \to M \otimes \pi_* L \otimes \mathcal{O}(1)).$$

The support $S_{(\mathcal{E},\theta)}$ of $M_{(\mathcal{E},\theta)}$ is called spectral curve.
The above associations induce an equivalence between the categories of

Higgs bundles with integral spectral curve S

and

torsion sheaves of pure dimension 1 and of rank 1 on Z supported away from (y)

Irregular BNR correspondence

If we fix the (semi-simple) irregular type of θ as above, then there exists a refinement of BNR (Sz arXiv:1502.02003, motivated by Kontsevich and Soibelman 2013).
Namely, there exists a birational morphism

$$\widetilde{Z} \cdots \to Z$$

such that sheaves on $\widetilde{Z}$ satisfying some properties correspond to irregular Higgs bundles on X with fixed irregular type and integral spectral curve.

Hilbert scheme of curves

$\widetilde{Z}$ is a holomorphic Poisson surface with Liouville symplectic form ω degenerating along some effective Weil divisor D_∞.
One can define a Hilbert scheme

$$\mathrm{Hilb}(r)$$

of curves $S \subset \widetilde{Z}$ having the same Hilbert polynomial as a generic curve satisfying the properties of the irregular BNR correspondence,

$$\mathrm{Hilb}^0(r) \subset \mathrm{Hilb}(r)$$

the connected component of a given S_0, and

$$B \subset \mathrm{Hilb}^0(r)$$

the Zariski open subset parameterising smooth irreducible curves S not contained in D_∞.

Moduli spaces of sheaves on Poisson surfaces

Consider moreover the relative Picard bundle

$$\mathrm{Pic}^d(\widetilde{Z}) \to B$$

whose fiber over $b \in B$ is the set of isomorphism classes of degree d line bundles over S_b.

Theorem (Donagi, Markman 1996)

B is smooth and $\mathrm{Pic}^d(\widetilde{Z})$ has a canonical Poisson structure Ω_{Mukai} whose symplectic leaves are obtained by prescribing the intersection of the curves S with D_∞.

Matching the symplectic structures

Then we have the

Proposition

The irregular BNR correspondence identifies Ω_I on (a Zariski open subset of) $\mathcal{M}^s_{\mathrm{Dol}}$ with Ω_{Mukai} on $\mathrm{Pic}^d(\widetilde{Z})$.

Proven in particular cases by Markman (1994), Hurtubise (1996) and Hurtubise–Harnad (2008).

Proof (sketch) of hyper-Kähler isometry

Known:

$$(\mathcal{E}, \theta) \quad (\widehat{\mathcal{E}}, \widehat{\theta})$$

have isomorphic spectral sheaves

$$M_{(\mathcal{E}(t), \Phi(t))} \cong M_{(\widehat{\mathcal{E}}(t), \widehat{\Phi}(t))}$$

on the open surface

$$T^*(\mathbf{C} \setminus P).$$

Therefore, the Proposition applied to the vectors

$$\widehat{T} = T_{(\mathcal{E},\Phi)}\mathcal{N}(T), \quad \widehat{X} = T_{(\mathcal{E},\Phi)}\mathcal{N}(X)$$

shows that

$$\Omega_{\widehat{I}}(\widehat{T}, \widehat{X}) = \Omega_{\mathsf{Mukai}}(\tilde{T}, \tilde{X})$$

too.

Fixing the singularity types

Ongoing joint work with A. Stipsicz: description of $\mathcal{M}^s_{\mathrm{Dol}}$ in $\dim_{\mathbf{C}} = 2$.
Let $X = \mathbf{P}^1$, $r = 2$, $n = 2$, singularities:

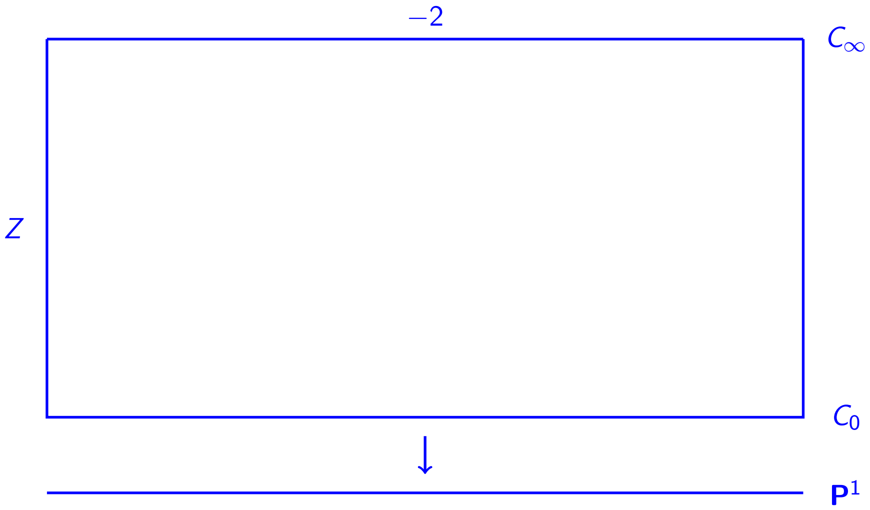
- $z = 0$: logarithmic, with trivial parabolic filtration and residue

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dz$$

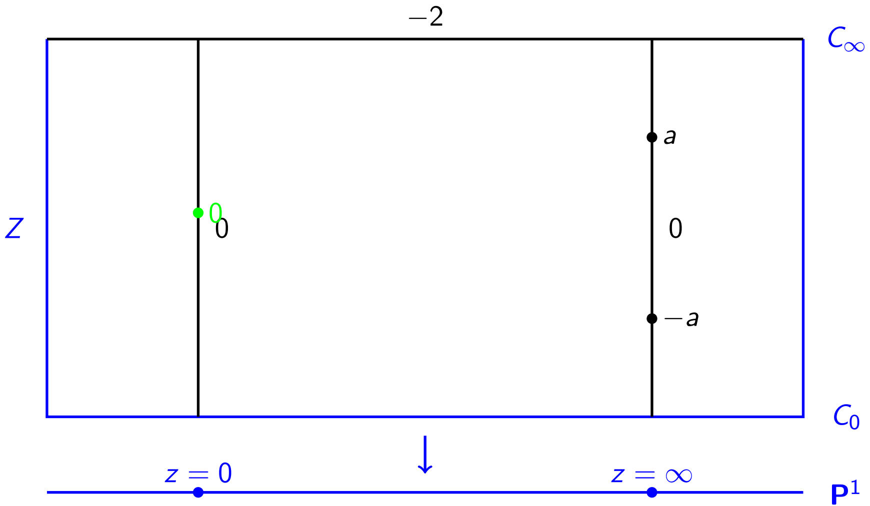
- $z = \infty$: Poincaré rank 2, with trivial parabolic filtration and local form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} zdz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} dz + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{dz}{z} + \text{lower order terms}$$

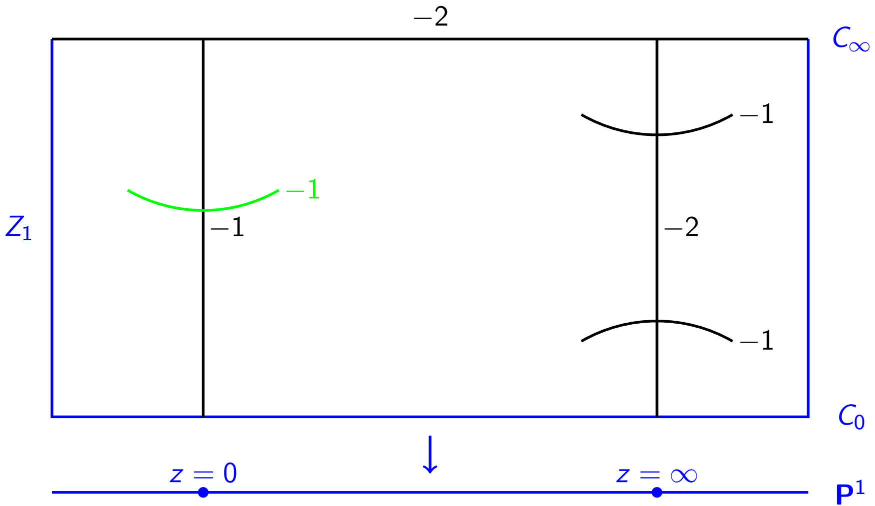
Hirzebruch surface $H_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2))$



Base points on H_2



The first blow-up



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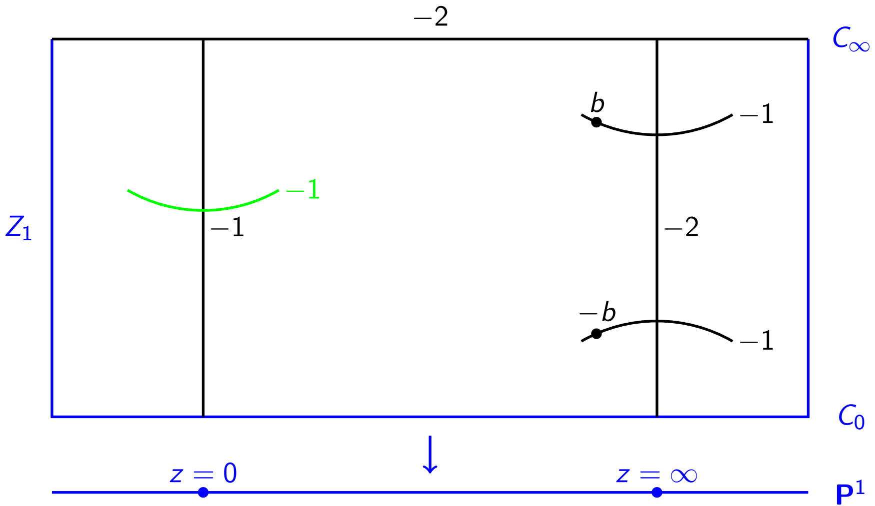
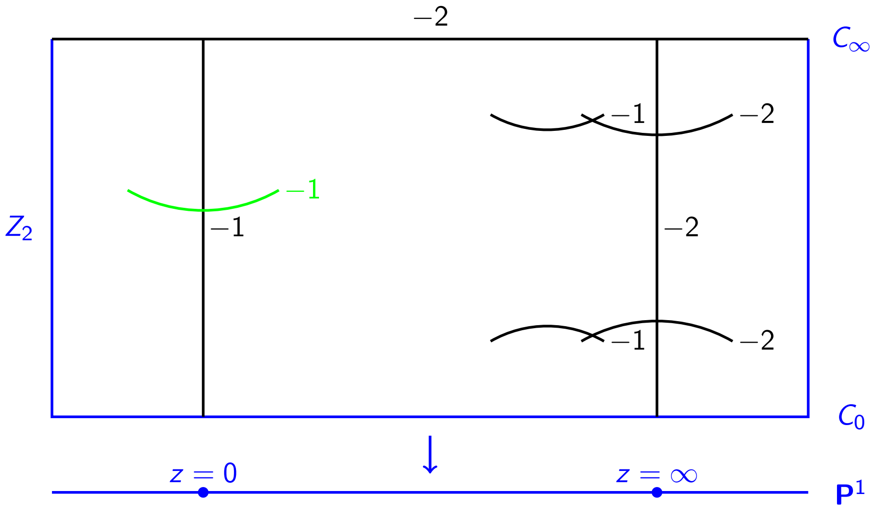
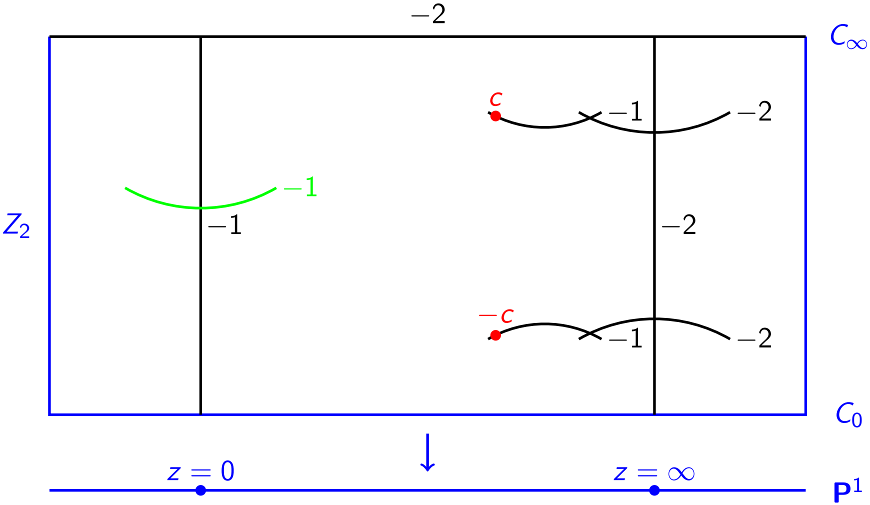


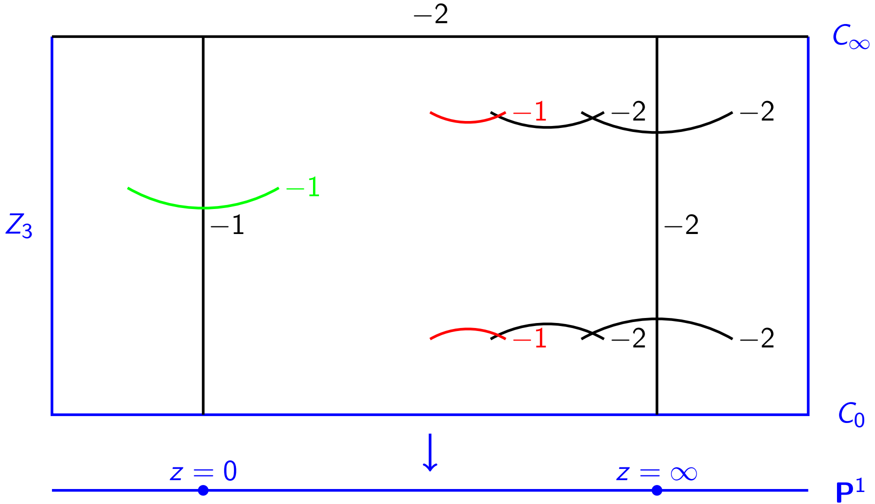
The second blow-up



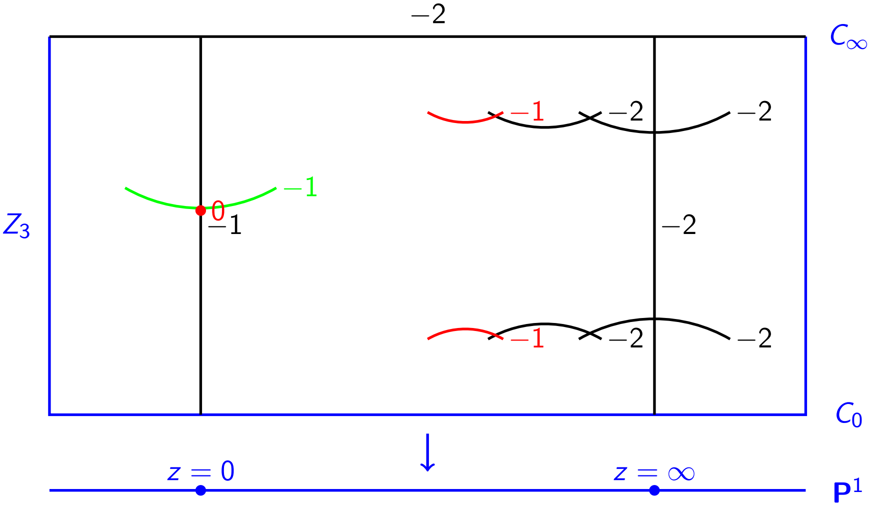
Base points on the second blow-up



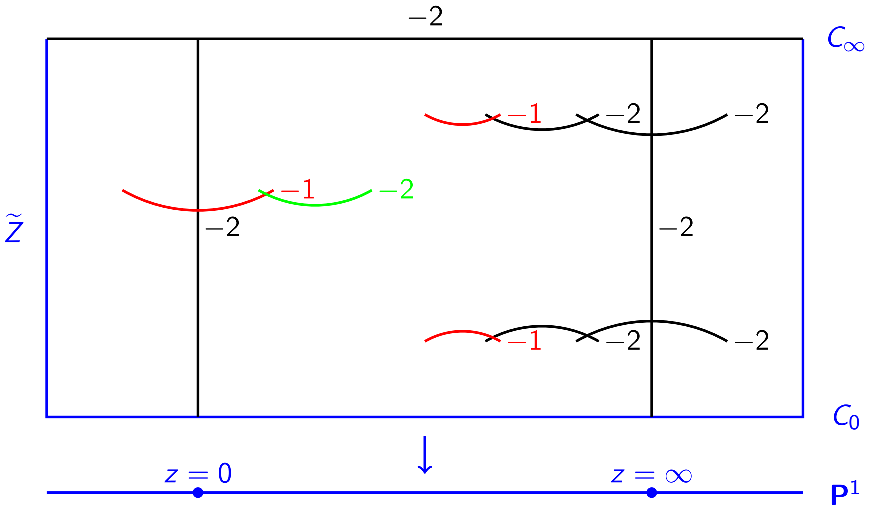
The third blow-up



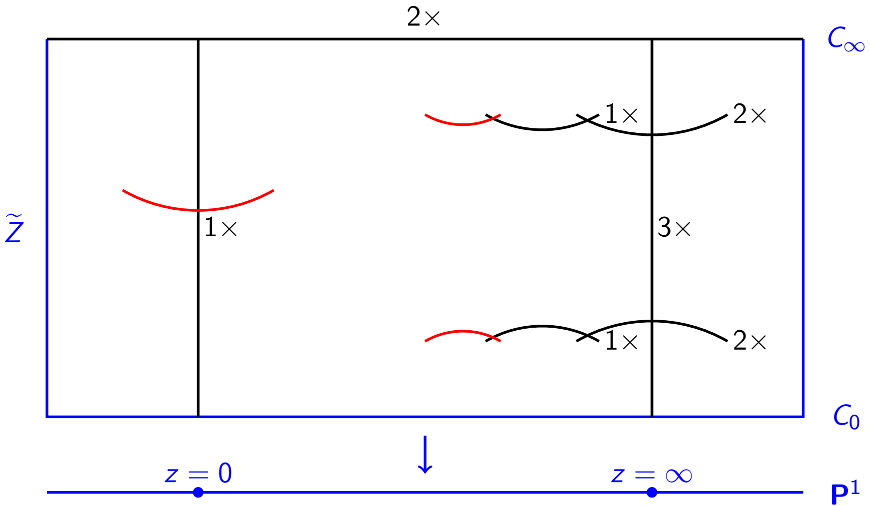
A further base point



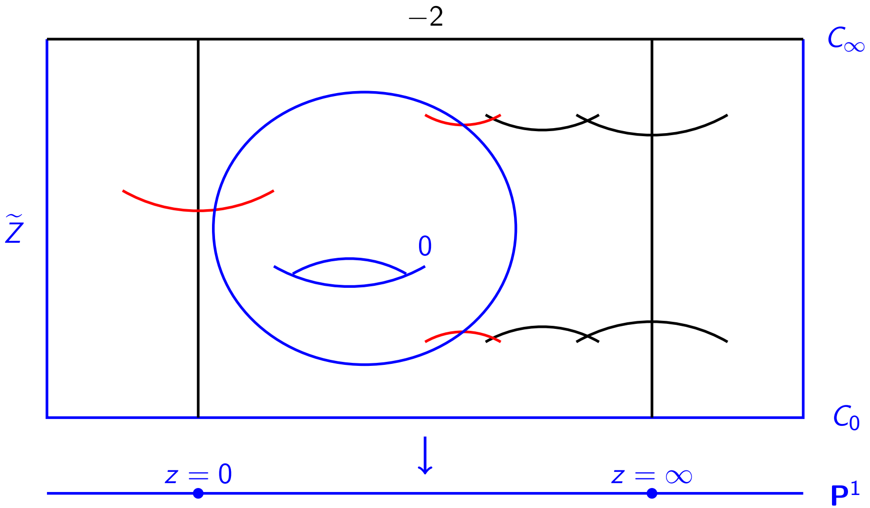
Blowing up the last base point



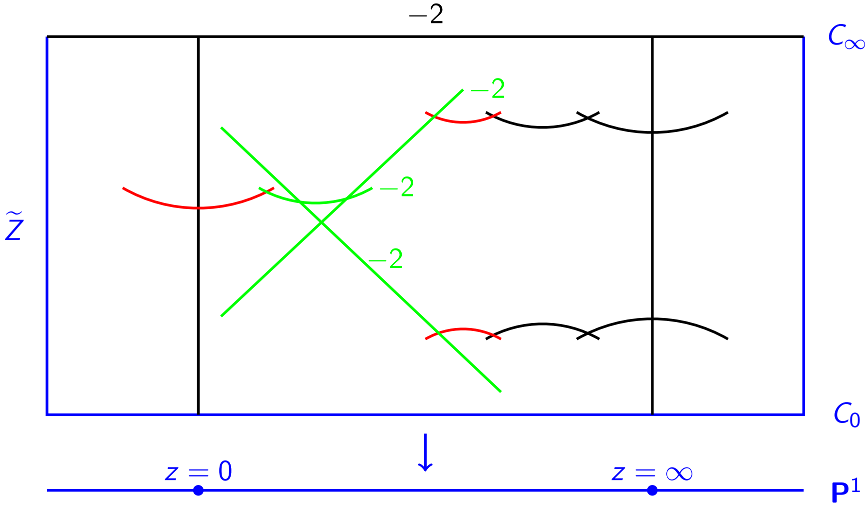
An anticanonical $\widetilde{E}_6$-fiber in the pencil



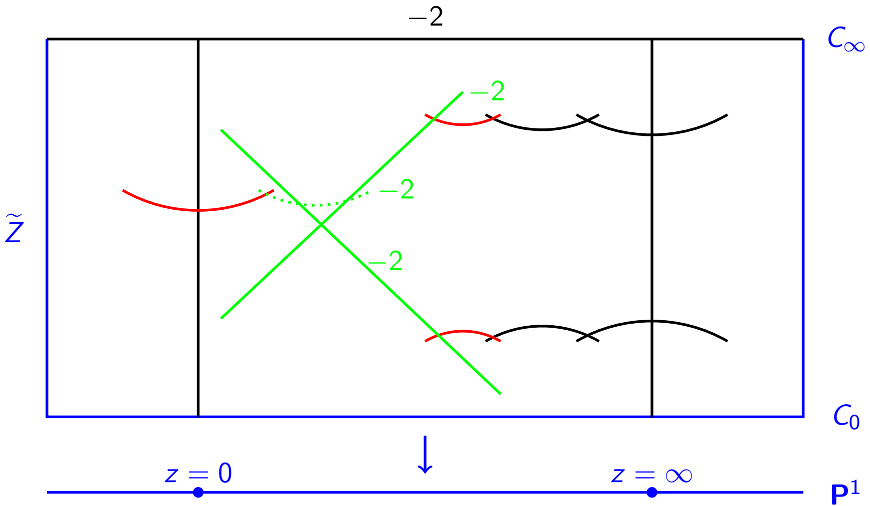
The generic curve in the pencil



An I_3 singular curve in the pencil



The corresponding spectral curve



Conclusion

In addition to the above singular fibers there exists a simple nodal $\mathbf{P}^1$ too in this elliptic pencil.
So we have

$$\mathrm{Hilb}(2) = \mathbf{P}^1_t \supset B = \mathbf{C}_t \setminus \{0,1\}.$$

Over B, the relative Jacobian gives a Zariski open subset of $\mathcal{M}^s_{\mathrm{Dol}}$.
On the other hand, over Hilb(2) the relative compactified Jacobian is not biregular to $\mathcal{M}^s_{\mathrm{Dol}}$.
The compactification of the moduli space is thus a relative dual of an Okamoto–Painlevé pair, as expected.
However, this compactification is of Uhlenbeck type.