SOLUTION SHEET OF GEOMETRY 1 MID-TERM, OCTOBER 17TH 2022

(1) Show that the orthocenter of any triangle is the incenter of its orthic triangle.

Solution: In an arbitrary triangle $ABC\Delta$, we denote by D, E, F the feet of the altitudes through A, B, C, respectively. We will show that then $FDB\angle = CDE\angle$. For that purpose, let F', F'' be the images of F under reflection about the sides BC and CA, respectively. It follows from the properties of reflection that $FDB\angle = BDF'\angle$. It follows from the solution of Fagnano's problem that F', D, E, F'' are aligned. In particular, this shows that $BDF'\angle = CDE\angle$. Combining these equalities, we get the desired equality. As a consequence, we see that

$$FDA\angle = \frac{\pi}{2} - FDB\angle = \frac{\pi}{2} - CDE\angle = ADE\angle,$$

Said differently, the height AD of $ABC\Delta$ is the bisector of the angle at D of the orthic triangle $DEF\Delta$. By symmetry of the argument, the other two heights of $ABC\Delta$ agree with the other two angle bisectors of $DEF\Delta$. In particular, the intersection H of the heights of $ABC\Delta$ agrees with the intersection of the angle bisectors of $DEF\Delta$, which is just its incenter.

(2) Using the notations introduced in our study of the 9-point circle (but without using the theorem itself), show directly that $B''A'C''\angle = FHE\angle$. Use this to give an alternative proof of the statement that the points $A', B', C', A'', B'', C''$ lie on the same circle.

Solution: Looking at the triangle $BCH\Delta$, we see that $A'B''\|CH$ because A', B'' are mid-points of its sides. Since C, H, F are aligned, this also shows that $A'B''\|HF$. Similarly, we find $A'C''\|HE$. Combining these shows that $B''A'C''\angle = FHE\angle$.

Now, $AFHE$ is a cyclic quadrangle because it admits two opposite right angles at E, F. Therefore, we have $FHE\angle = \pi - \alpha$. From the first statement of the exercise, we then deduce that $B''A'C''\angle = \pi - \alpha$. On the other hand, using the triangles $ABH\Delta, CAH\Delta$ we see that $A''B''\|AB$ and $A''C''\|AC$. This gives that $B''A''C''\angle = \alpha$. Since

$$B''A'C''\angle + B''A''C''\angle = (\pi - \alpha) + \alpha = \pi,$$

it follows from the cyclic quadrangle theorem that A' lies on the circumcircle of $A''B''C''\Delta$. By symmetry of the argument, the same holds for B', C' too.

(3) Given a regular 17-gon $P_0, \ldots, P_{16}$, construct a regular 85-gon.

Solution: Construct first a regular pentagon $Q_0, \ldots, Q_4$ with $Q_0 = P_0$ inscribed in the circumcircle of $P_0, \ldots, P_{16}$. We claim that $P_7 Q_2$ are two vertices of a regular 85-gon inscribed in the same circle. Indeed, denoting by O the center of the circle we have

$$\begin{aligned} Q_0 O Q_2 \angle &= 2 \cdot \frac{2\pi}{5} \\ P_0 O P_7 \angle &= 7 \cdot \frac{2\pi}{17}. \end{aligned}$$

It follows from this that

$$\begin{aligned} Q_2 O P_7 \angle &= 7 \cdot \frac{2\pi}{17} - 2 \cdot \frac{2\pi}{5} \\ &= 35 \cdot \frac{2\pi}{85} - 34 \cdot \frac{2\pi}{85} \\ &= \frac{2\pi}{85}. \end{aligned}$$

(4) For any triangle $ABC\Delta$, denote by D, E, G the third vertices of regular triangles placed externally on the sides BC, CA, AB respectively. Given only the points D, E, G, write down steps of a (ruler and compass) construction of A, B, C.

Solution: We can construct the point F such that $DEF\Delta$ is isosceles with angle $\frac{2\pi}{3}$ at F, lying on the same side of the line DE as G. We have seen in class that then $ABF\Delta$ is similar to $DEF\Delta$. It follows that $AFG\Delta$ is a right triangle with angle $\frac{\pi}{3}$ at F and a right angle at A. Therefore, we can construct A by constructing an angle $\frac{\pi}{3}$ at F on one side of the line FG, and taking its intersection with the circle having FG as diameter. Then, we can construct B by rotating A by angle $\pm\frac{2\pi}{3}$ about F. Finally, rotating A by angle $\pm\frac{\pi}{3}$ about E, we get C.