On mathematical results of Stephen Hawking
Összintézeti szeminárium, Rényi Alfréd Matematikai Kutatóintézet

Szabó Szilárd

2018. október 15.

Differentiable manifolds

Let M be a connected separable Hausdorff-space and $n \in \mathbb{N}$. We say that M has the structure of an n-dimensional differentiable manifold if there exists a family of open subsets $\{U_i\}_{i \in I}$ of M such that

$$M = \cup_{i \in I} U_i,$$

and for each $i \in I$ there exists a homeomorphism

$$\varphi_i : U_i \to V_i$$

where $V_i \subseteq \mathbb{R}^n$ is an open subset, such that for each $i, j \in I$ the map

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$$

is differentiable (equivalently, a diffeomorphism) on its domain of definition.

Tangent spaces, vector fields

Let M be an n-dimensional differentiable manifold and $x \in M$. The tangent space of M at x is the space of all curves $\gamma : (-\varepsilon, \varepsilon) \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = x$ and for some (equivalently, any) $i \in I$ the map $\varphi_i \circ \gamma_1$ is differentiable, up to the following equivalece relation: two such curves γ_1, γ_2 are equivalent iff for some (equivalently, any) $i \in I$ such that $x \in U_i$ we have

$$(\varphi_i \circ \gamma_1)'(0) = (\varphi_i \circ \gamma_2)'(0).$$

The tangent space of M at x is a vector space of dimension n, denoted by $T_x M$. A vector field on M is a section of the tangent bundle, i.e. a "smooth map"

$$X : x \mapsto X(x) \in T_x M.$$

Riemannian manifolds

A Riemannian metric on an n-dimensional differentiable manifold M is a collection g of symmetric positive definite bilinear forms

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

smoothly varying with x. A differentiable manifold M equipped with a Riemannian metric is called a Riemannian manifold.
Any Riemannian manifold admits a canonical metric

$$d(x, y) = \inf_{\gamma}(l(\gamma))$$

where the infimum ranges over all C^1-paths $\gamma : [0, 1] \to M$ connecting x to y and

$$l(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} \mathrm{d}t.$$

All Riemannian manifolds we consider will be complete for the topology induced by d.

Connections

A connection on M is a procedure that associates a vector field $\nabla_Y X$ to any two given vector fields X, Y, satisfying the following properties:

- additivity in Y: for any vector fields Y_1, Y_2 we have $\nabla_{Y_1+Y_2} X = \nabla_{Y_1} X + \nabla_{Y_2} X$;
- linearity in Y over scalar fields: for any function f on M we have $\nabla_{fY} X = f \nabla_Y X$;
- additivity in X: for any vector fields X_1, X_2 we have $\nabla_Y (X_1 + X_2) = \nabla_Y X_1 + \nabla_Y X_2$;
- Leibniz' rule: for any function f on M we have $\nabla_Y (fX) = (Y.f)X + f \nabla_Y X$, where $(Y.f)(x)$ is the derivative at x of f on any curve representing $Y(x)$.

Lie bracket, torsion

Let X, Y be two vector fields on a Riemannian manifold (M, g). There exists a vector field $[X, Y]$ such that for any function f on M we have

$$X.(Y.f) - Y.(X.f) = [X, Y].f$$

The vector field $[X, Y]$ is called the Lie bracket of X and Y.
The torsion of a connection ∇ on M with respect to the directions X, Y is defined as

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Compatible connections, Levi-Cività connection

A connection ∇ on a Riemannian manifold (M, g) is compatible if for any vector fields X, Y, Z we have

$$Z.(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

There exists a unique compatible connection ∇ on (M, g) such that for any vector fields X, Y we have

$$T_\nabla(X, Y) = 0.$$

This connection is called the Levi-Cività connection of (M, g), and denoted by ∇^{LC}.

The Riemannian curvature of the Levi-Cività connection

Given a Riemannian manifold (M, g), one may associate to it the vector field

$$R_x(X, Y, Z) = \nabla_X(\nabla_Y Z)(x) - \nabla_Y(\nabla_X Z)(x) - \nabla_{[X,Y]}Z(x)$$

for any vector fields X, Y, Z. Allowing X, Y, Z to vary, this provides us with the Riemannian curvature tensor R_x of (M, g) at x.

Let us denote by Ω_x^2 the space of skew-symmetric bilinear forms on T_xM: it is a real vector space endowed with the metric induced by g_x. There exist certain symmetries of R_x which allow us to view it as an orthogonal transformation:

$$R_x \in \mathrm{O}(\Omega_x^2, g_x).$$

The Ricci curvature

One may define a tensor field

$$Ric_x \in \mathrm{O}(T_x M, g_x)$$

called the Ricci curvature of (M, g) by the following formula:

$$Ric_x(X, Y) = \sum_{i=1}^{n} g_x(R_x(E_i, X, Y), E_i(x)),$$

where $E_1(x), \ldots, E_n(x)$ denote an arbitrary orthonormal basis of $(T_x M, g_x)$.

Einstein's equation

Fix a differentiable manifold and let g be any Riemannian metric on M. The vacuum Einstein equation of (Riemannian) general relativity reads as:

$$Ric_x(g) = \Lambda g_x.$$

for some $\Lambda \in \mathbb{R}$ (called the cosmological constant). Here, we emphasized the dependence of Ric_x on g by writing $Ric_x(g)$. Nowadays, it is accepted that $\Lambda > 0$. Nevertheless, from now on we take $\Lambda = 0$.

The Hodge operator

Let (M, g) be an oriented Riemannian manifold of dimension 4.
The volume form of g is the alternating quadrilinear form

$$\mathrm{dV}_x : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{R}$$

defined by

$$\mathrm{dV}_x(E_1(x), \dots, E_4(x)) = 1$$

for any direct orthonormal basis $E_1(x), \dots, E_4(x)$ of (T_xM, g_x).
Given any $\alpha, \beta \in \Omega^2_x$ we may associate another alternating quadrilinear form $\alpha \wedge \beta$ called their wedge product by the formula

$$(\alpha \wedge \beta)(X_1, X_2, X_3, X_4) = \frac{1}{4} \sum_{\sigma \in S_4} \mathrm{sgn}(\sigma) \alpha(X_{\sigma(1)}, X_{\sigma(2)}) \beta(X_{\sigma(3)}, X_{\sigma(4)}).$$

The Hodge operator of (M, g) is the unique linear map $* : \Omega^2_x \to \Omega^2_x$ such that for all $\alpha, \beta \in \Omega^2_x$ we have

$$\alpha \wedge *\beta = g_x(\alpha, \beta)\mathrm{dV}_x.$$

Selfdual and anti-selfdual forms

A basic property of the Hodge operator:

$$* \circ * = \mathrm{Id}_{\Omega^2_x}.$$

In particular, setting

$$\Omega^{\pm}_x \subset \Omega^2_x$$

for the ± 1-eigenspace of $*$, there exists a decomposition

$$\Omega^2_x = \Omega^+_x \oplus \Omega^-_x.$$

Elements of Ω^+_x and of Ω^-_x are called selfdual, respectively anti-selfdual 2-forms. It turns out that

$$\dim(\Omega^+_x) = 3 = \dim(\Omega^-_x),$$

and Ω^+_x and Ω^-_x are swapped under change of orientation of M.

The decomposition of the curvature tensor

The Riemannian curvature tensor decomposes with respect to the decomposition $\Omega_x^+ \oplus \Omega_x^-$ as follows:

$$\begin{pmatrix} \frac{s}{12}\mathrm{Id}_{\Omega_x^+} + W^+ & Ric_0^t \\ Ric_0 & \frac{s}{12}\mathrm{Id}_{\Omega_x^-} + W^- \end{pmatrix},$$

where

- $s = \mathrm{tr}(Ric)$ is the scalar curvature of (M, g),
- $Ric_0 = Ric - \frac{s}{4}g$ is the trace-free Ricci tensor,
- W^+, W^- are the positive, respectively negative Weyl tensors of (M, g).

Gravitational anti-instantons

A non-compact Riemannian 4-manifold (M, g) is said to be a gravitational anti-instanton if

$$Ric = 0, \quad W^+ = 0,$$

and the Riemannian curvature $|R_x|_{g_x} \to 0$ as the distance $d(x, x_0) \to \infty$ for some fixed reference point x_0.
In view of the previous decomposition, this is equivalent to requiring

$$R|_{\Omega^+} = 0.$$

If moreover M is simply connected, then flatness of Ω^+ is equivalent to the existence of a global orthonormal basis $\omega_I, \omega_J, \omega_K$ of parallel selfdual 2-forms.
4-manifolds with an ONB of parallel selfdual 2-forms are called hyper-Kähler.

Complex structures

To sum up: a gravitational anti-instanton with simply connected underlying manifold M is a hyper-Kähler 4-manifold $(M, g, \omega_I, \omega_J, \omega_K)$.
Given a symmetric $(0,2)$-tensor field g and an anti-symmetric $(0,2)$-tensor field ω_I one may construct a skew-adjoint $(1,1)$-tensor field (i.e., an endomorphism-field) I by the prescription

$$\omega_I(X, IY) = g(X, Y).$$

In our case, it follows from $\omega_I \in \Omega^+$ that this endomorphism field satisfies

$$I^2 = -\mathrm{Id}_{TM}.$$

It turns out that for a hyper-Kähler manifold $(M, g, \omega_I, \omega_J, \omega_K)$ each of the associated endomorphism fields I, J, K endow M with the structure of a complex manifold (in a way compatible with g).

Quiver varieties

General construction of hyper-Kähler manifolds by H. Nakajima (1990) based on an idea of M. Atiyah, V. Drinfeld, N. Hitchin and Y. Manin (1978): to any finite graph $\Gamma = (V, E)$, a map

$$\vec{d} : V \to \mathbb{N}$$

(called dimension vector) and a sufficiently general vector

$$\vec{\zeta} : V \to \mathbb{C} \oplus \mathbb{R}$$

satisfying

$$\sum_{v \in V} d(v)\zeta(v) = 0$$

one can associate a non-compact hyper-Kähler manifold $(M(\Gamma, \vec{d}, \vec{\zeta}), g)$, called quiver variety.

A_k ALE-spaces as quiver varieties

Particular case of quiver varieties: multi-Eguchi–Hanson spaces. Let Γ be a cycle of length $k+1$, and $\vec{d} \equiv \vec{1}$. Then, g is a gravitational instanton on the smoothing $M(\Gamma, \vec{1}, \vec{\zeta})$ of the cyclic Kleinian surface singularity

$$(xy + z^{k+1} = 0) \subset \mathbb{C}^3.$$

Their geometry at infinity is Asymptotically Locally Euclidean (ALE).

Asymptotically Locally Euclidean manifolds

Observe: in radial co-ordinates, the Euclidean metric on $\mathbb{R}^4$ is given by

$$g_{R^4} = r^2 g_{S^3} + \mathrm{d}r^2$$

where r is the distance to 0 and g_{S^3} is the round metric on the three-sphere S^3.

More generally, let (M, g_M) be a non-compact Riemannian 4-manifold. We say that (M, g_M) is Asymptotically Locally Euclidean (ALE) if there exists a compact set $K \subset M$ and a compact Riemannian 3-manifold (S, g_S) such that there is a diffeomorphism

$$M \setminus K = S \times \mathbb{R},$$

with respect to which we have the following asymptotic equality:

$$g_M \approx r^2 g_S + \mathrm{d}r^2 \quad \text{as} \quad r \to \infty.$$

The A_k quiver varieties satisfy this definition with $S = S^3/\mathbb{Z}_{k+1}$.

Asymptotically Locally Flat manifolds

A non-compact Riemannian 4-manifold (M, g_M) is Asymptotically Locally Flat (ALF) if there exists

- a compact subset $K \subset M$
- a compact 3-manifold $(S, g|_S)$
- a fibering
$$S^1 \to S \to \Sigma$$
over some compact Riemannian surface (Σ, g_Σ)
- a scalar $\beta > 0$ (physically interpreted as the inverse of temperature)
- a diffeomorphism
$$M \setminus K = S \times \mathbb{R}$$

such that we have the following asymptotic equality:

$$g_M \approx r^2 g_\Sigma + \beta \mathrm{d}\theta^2 + \mathrm{d}r^2 \quad \text{as} \quad r \to \infty.$$

ALE versus ALF

Advantages of ALF as compared to ALE 4-manifolds:

- mathematically, Hausel–Hunsicker–Mazzeo (2004) showed that any ALF-space M has a smooth compactification $\overline{M}$ by Σ at infinity such that

$$H^2(\overline{M}) \cong H^2_{L^2}(M);$$

- physically, one may consider the limit of these spaces as $\beta \to \infty$, and Witten (2009) used this to explain electric-magnetic duality between two 5-dimensional theories.

The Gibbons–Hawking ansatz

Let $p_1, \ldots, p_{k+1} \in \mathbb{R}^3$ be distinct points and set

$$U = \mathbb{R}^3 \setminus \{p_1, \ldots, p_{k+1}\}.$$

Let

$$V : U \to \mathbb{R}$$

be a harmonic function such that the 3-dimensional Hodge dual

$$\left[\frac{*\mathrm{d}V}{2\pi}\right] \in H^2(U, \mathbb{Z})$$

is an integer homology element (integrality property). By Chern–Weil theory, there exists then a principal S^1-bundle over U endowed with a connection 1-form η satisfying

$$*\mathrm{d}V = \mathrm{d}\eta.$$

Let M^0 be the total space of this S^1-bundle endowed with the Riemannian metric

$$g_{M^0} = \frac{1}{V}\eta^2 + V(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2).$$

Gravitational anti-instantons with an S^1-symmetry

The Riemannian manifold (M^0, g_{M^0}) given by the Gibbons–Hawking ansatz is a gravitational anti-instanton admitting an S^1-action preserving g and the 2-forms $\omega_I, \omega_J, \omega_K$. Conversely, we have the following.

Theorem (G. Gibbons, S. Hawking (1979))

Let $(M, g, \omega_I, \omega_J, \omega_K)$ be a gravitational anti-instanton, and assume there exists an S^1-action preserving $g, \omega_I, \omega_J, \omega_K$. Then there exists a harmonic function V on some open subset of $\mathbb{R}^3$ satisfying the integrality property such that $(M, g, \omega_I, \omega_J, \omega_K)$ is of the form given by the ansatz.

Sketch of the proof.

Let T denote the vector field of infinitesimal generators of the action, and θ its dual 1-form, $\theta_I = I(\theta)$, etc. Set

$$V = \frac{1}{|T|^2}, \quad \eta = V\theta, \quad \theta_I = -\mathrm{d}x_1, etc.$$

The A_k ALE and ALF gravitational anti-instantons

The gravitational anti-instanton (M^0, g_{M^0}) given by the Gibbons–Hawking ansatz is not complete. However, in the following important examples M^0 has a natural completion M by the points $p_1, \ldots, p_{k+1}$ so that the neighbourhood of any p_j is homeomorphic to the Hopf-fibration. Furthermore, g_{M^0} extends to M as a complete metric g_M.

- multi-Eguchi–Hanson spaces: for the choice

$$V(x) = \sum_{i=1}^{k+1} \frac{1}{2|x - p_i|},$$

  we get the A_k quiver variety with dimension vector $\vec{d} \equiv \vec{1}$;
- multi-Taub–NUT spaces: for the choice

$$V(x) = \beta^{-1} + \sum_{i=1}^{k+1} \frac{1}{2|x - p_i|},$$

  we get an ALF metric with the same underlying smooth manifold as the multi-Eguchi–Hanson spaces.

Classification of ALE gravitational anti-instantons

Theorem (P. Kronheimer (1989))

Let (M, g_M) be a complete gravitational anti-instanton of type ALE. Then, there exists a finite subgroup $\Gamma \subset SU(2)$ such that M is diffeomorphic to the smoothing (equivalently, minimal resolution) of $\mathbb{C}^2/\Gamma$, and g_M belongs to a family of metrics parameterized by $\vec{\zeta} \in H^{1,1}(M, \mathbb{C})$.

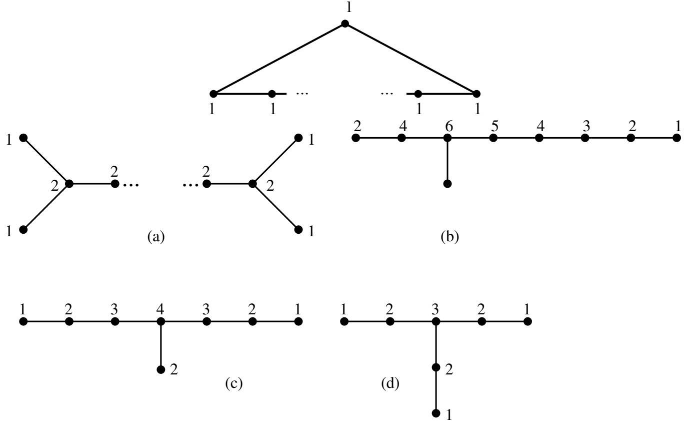
Finite subgroups $\Gamma \subset \mathrm{SU}(2)$ are classified as the lifts under

$$\mathrm{SU}(2) \to \mathrm{SO}(3)$$

of isometry groups of polygons and Platonic solids: A_k (cyclic), D_k (dihedral), E_8 (icosahedral), E_7 (octahedral), E_6 (tetrahedral).

ALE gravitational anti-instantons and quiver varieties

Kronheimer's result shows that all ALE gravitational anti-instantons arise from the quiver variety construction, for the affine Dynkin diagram associated to Γ by the McKay-correspondence, and $\vec{d}$ equal to the positive integer generator of the radical of the associated quadratic form:



Classification of ALF gravitational anti-instantons

Let (M, g_M) be a complete gravitational anti-instanton of type ALF. Then, there exists a compact $K \subset M$ such that $M \setminus K$ is homeomorphic to either

1. cyclic type: $\mathbb{R}_+ \times S^2 \times S^1$ or $\mathbb{R}_+ \times (S^3/(\mathbb{Z}/k+1))$;
2. dihedral type: $\mathbb{R}_+ \times (S^3/D_{k+1})$ where D_{k+1} is the binary dihedral group of order $4k+4$.

Theorem (V. Minerbe (2009))

Any ALF gravitational anti-instanton with topological behaviour at infinity of cyclic type is diffeomorphic to $\mathbb{R}^3 \times S^1$ *or a multi-Taub–NUT space.*

Conjecture (S. Cherkis, A. Kapustin (2007))

Any ALF gravitational anti-instanton of dihedral type is diffeomorphic to a Cherkis–Hitchin space.

Further constructions of gravitational anti-instantons

A rich source of gravitational anti-instantons: moduli spaces of instantons on hyper-Kähler 4-manifolds, or of various dimensionally reduced equations thereof. These spaces often turn out to be ALG or ALH-spaces (defined by applying alphabetical induction to ALE and ALF).

Classification of ALG/ALH metrics: to a large extent an unknown area.

Gaiotto–Moore–Neitzke conjecture (2013) on the precise behaviour of the ALG metric of moduli spaces of Hitchin's equations on a Riemann surface, partly proved by Mazzeo–Swoboda–Weiss–Witt.