The geometry of the basic instanton moduli space over the multi-Taub–NUT space (joint work with G.Etesi)

Szilárd Szabó

Rényi Institute of Mathematics

Budapest University of Technology

Kobe
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Outline

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- Possible topological applications...

Construction of the space

Fix $p_1, \ldots, p_k \in \mathbf{R}^3$ distinct points (the "nuts"). Denote by l_{ij} the straight line segment connecting p_i to p_j. Consider the principal bundle

$$P \longrightarrow \mathbf{R}^3 \setminus \{p_1, \ldots, p_k\}$$

with, for all $j \in \{1, \ldots, k\}$ and ε sufficiently small

$$c_1(P|_{S^2_\varepsilon(p_j)}) = -1.$$

Let

$$M = \coprod_j B^4_\varepsilon(p_j) \coprod P / \sim,$$

where $\sim$ is the equivalence relation induced by the Hopf-fibration.

Topological properties

- M inherits a smooth manifold structure and projection
$$\pi : M \to \mathbf{R}^3.$$
Denote the preimages $\pi^{-1}(p_j)$ by p_j.
- There exists on M a smooth S^1-action with fixed points $\{p_1, \ldots, p_k\}$, free on $M \setminus \{p_1, \ldots, p_k\}$. Denote by τ the infinitesimal generator of this action; τ is a smooth vector-field on $M \setminus \{p_1, \ldots, p_k\}$.
- M is non-compact, complete, simply-connected, orientable and spin.
- $H_2(M, \mathbf{Z})$ is generated by $k-1$ spheres $S_j^2 = \pi^{-1}(I_{j,j+1})$, intersecting along A_{k-1}.

Construction of the metric

Consider the potential function

$$V(\mathbf{x}) = 1 + \frac{1}{2}\sum_{j=1}^{k} \frac{1}{|\mathbf{x} - p_j|}$$

where $\mathbf{x} \in \mathbf{R}^3$ and $|.|$ stands for the Euclidean norm.
Notice:

- $\Delta_{\mathbf{R}^3} V = \delta_{p_1} + \cdots + \delta_{p_k}$;
- the differential form $*_3 \mathrm{d}V$ represents $2\pi c_1(P) \in H^2(P, \mathbf{Z})$.

It follows that there exists a connection 1-form $\omega \in \Omega^1(\mathbf{R}^3 \setminus \{p_1, \ldots, p_k\}, \mathrm{ad}(P))$ such that

$$*_3 \mathrm{d}V = \mathrm{d}\omega.$$

Denoting by (x, y, z) standard orthonormal coordinates in $\mathbf{R}^3$, set

$$g_V = V(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{1}{V}(\mathrm{d}\tau + \omega)^2.$$

Metric properties

- g_V extends smoothly to M.
- Gibbons-Hawking ansatz $\Rightarrow g_V$ is hyperKähler $(W^+ = 0, s = 0)$.
- Asymptotically locally flat (ALF): near infinity, up to a finite cover,
$$g_V \asymp \mathrm{d}r^2 + r^2 g_{S^2} + \mathrm{d}\tau^2,$$
where $r = |\pi(\mathbf{x}) - \pi(\mathbf{x}_0)|$ for any fixed $\mathbf{x}_0 \in M$.

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(M, g_V): multi-Taub–NUT space with nuts in $\{p_1, \ldots, p_k\}$, aka. ALF A_{k-1} gravitational instanton.

Instantons on 4-manifolds

Let (X, g) be an arbitrary orientable Riemannian 4-manifold. Fix an SU(2)-vector bundle

$$E \to X.$$

Let ∇ denote a Hermitian connection on E, and F stand for its curvature. Denote by $*$ the 4-dimensional Hodge operator. The anti-self-duality (ASD) equation for ∇ is

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where $\mathrm{d}vol_g$ stands for the volume form of g.
An ASD connection ∇ of finite energy is called an (anti-)instanton on E.

Moduli spaces of instantons on ALF spaces

Assume X is ALF. Write $\nabla = \mathrm{d} + A$ for $A \in \Omega^1(X \setminus B_R(\mathbf{x}_0), \mathrm{ad}(E))$ (or $A = \nabla - \Gamma$ for some flat SU(2)-connection Γ on $X \setminus B_R(\mathbf{x}_0)$).
We say ∇ satisfies the

1. weak holonomy condition for Γ if for some $C > 0$ and any $R >> 0$ we have

$$\|A\|_{L^2_1(M \setminus B_R(\mathbf{x}_0))} \leq C \|F\|_{L^2(M \setminus B_R(\mathbf{x}_0))};$$

2. rapid decay condition if

$$\lim_{r \to \infty} \sqrt{r} \|F\|_{L^2(M \setminus B_r(\mathbf{x}_0))} = 0.$$

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Theorem (G.Etesi – M.Jardim, 2008)

There exists a smooth moduli space $\mathcal{M}^{\mathrm{irr}}(X, E, \Gamma, e)$ of gauge equivalence classes of rapidly decaying instantons of energy e on E satisfying the weak holonomy condition for Γ.

The moduli space

Furthermore: for E the only smooth SU(2)-bundle on M and Γ the trivial connection over $M \setminus B_R(\mathbf{x}_0)$ a dimension count gives

$$\dim(\mathcal{M}^{\text{irr}}(M,1)) = 5.$$

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Theorem (G.Etesi – Sz.Szabó, 2008)

One connected component of the moduli space $\mathcal{M}(M,1)$ is

$$M\times]0,\infty]/\sim,$$

where for any $\mathbf{x} \in M$ and $e^{i\theta} \in S^1$ we have $(\mathbf{x},\infty) \sim (e^{i\theta}\mathbf{x},\infty)$. In particular, $\mathcal{M}(M,1)$ is a singular filling of M, with k singular points (p_j,∞) (corresponding to reducible solutions). A neighborhood of a reducible point is a cone over $\overline{\mathbf{CP}^2}$.

The conformal rescaling method

Suppose X is hyperKähler and spin, and let $f : X \to \mathbf{R}_+$ be a function with finitely many point-like singularities. Denote by Δ_g the Laplace-Beltrami operator.
Define $\tilde{g} = f^2 g$ to be the conformally rescaled Riemannian metric. Construct the Levi-Cività connection $\nabla^{LC}_{\tilde{g}} \rightsquigarrow \nabla^{+}_{\tilde{g}}$ the corresponding connection on the positive spinor bundle $S^+ \to X$.

Facts

- *$\nabla^{+}_{\tilde{g}}$ is independent of $f \mapsto cf$ for $c \in \mathbf{R}_+$;*
- *If $W^{+}_{g} = 0$ then $\nabla^{+}_{\tilde{g}}$ is ASD $\Leftrightarrow (\Delta_g + \frac{1}{6} s_g) f = 0$.*

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Furthermore, we have $e(\nabla^+_{\tilde{g}}) = \#\,\mathrm{Sing}(f)$ (supposing some conditions on f...)

Green function

Fix $\mathbf{x}_0 \in X$, and call r the distance function to x_0 in X. A function $G_{\mathbf{x}_0} : X \setminus \mathbf{x}_0 \to \mathbf{R}_+$ is called the minimal positive Green function centered at $\mathbf{x}_0$ if

1. $\Delta_g G_{\mathbf{x}_0} = \delta_{\mathbf{x}_0}$;
2. $G_{\mathbf{x}_0} = O(r^{-2})$, $\mathrm{d}G_{\mathbf{x}_0} = O(r^{-3})$ as $r \to 0$;
3. $G_{\mathbf{x}_0} \to 0$ as $r \to \infty$;

Theorem (Varopoulos, 1983)

Suppose $\mathrm{Ric}_g \geq 0$ *and for some* $\mathbf{x}_0$ *the following holds:*

$$\int_1^\infty \frac{r}{\mathrm{Vol}_g(B_r(\mathbf{x}_0))} < \infty.$$

Then at all $\mathbf{x} \in X$ *the minimal positive Green function* $G_{\mathbf{x}}$ *exists.*

A 5-parameter family of solutions

For the multi-Taub–NUT space M, we have $\mathrm{Ric}_{g_V} = 0$ and

$$\mathrm{Vol}_{g_V}(B_r(\mathbf{x}_0)) \asymp cr^3,$$

so for all $\mathbf{x} \in M$ we get $G_{\mathbf{x}}$. We obtain a family

$$f_{\mathbf{x},\lambda} = \frac{1}{\lambda} + G_{\mathbf{x}}$$

of harmonic functions and by conformal rescaling corresponding solutions $\nabla^+_{f_{\mathbf{x},\lambda}}$, parametrized by

$$(\mathbf{x},\lambda) \in M \times]0,\infty].$$

Near $\lambda = 0$: infinitely concentrated ("Dirac-type") solutions, near $\lambda = \infty$: "centerless" solutions.

The twistor space of a hyperKähler manifold

Let (X, g) be a simply connected hyperKähler 4-manifold: I, J, K Kähler structures satisfying the relations of the quaternion group

$$I^2 = J^2 = K^2 = -\mathrm{Id}, \quad IJ = -JI = K.$$

For all $(x, y, z) \in S^2$ the endomorphism

$$I_{(x,y,z)} = xI + yJ + zK$$

is also a Kähler structure.

Let i stand for the standard complex structure on $\mathbf{CP}^1$, and set

$$Z_X = X \times \mathbf{CP}^1,$$

endowed with the almost-complex structure

$$J_{(\mathbf{x},(x,y,z))} = I_{(x,y,z)}(\mathbf{x}) \times \mathrm{i}.$$

Z_X (also denoted Z) is the twistor space of X.

PROPERTIES OF THE TWISTOR SPACE

- Atiyah-Hitchin-Singer: J is integrable if and only if $W_g^+ = 0$.
- For all $\mathbf{x} \in X$, the line $\mathbf{CP}^1_{\mathbf{x}} = \pi_1^{-1}(\mathbf{x})$ is holomorphic with normal bundle

$$N_{\mathbf{CP}^1_{\mathbf{x}}} \cong \mathcal{O}_{\mathbf{CP}^1}(1) \oplus \mathcal{O}_{\mathbf{CP}^1}(1).$$

- The anti-podal map $\sigma : \mathbf{CP}^1 \to \mathbf{CP}^1$ induces an anti-holomorphic involution (real structure) $\sigma : Z_X \to Z_X$.
- The lines $\mathbf{CP}^1_{\mathbf{x}}$ are the real lines of a locally complete 4 complex dimensional family of lines called the twistor lines.
- $\pi_2 : Z_X \to \mathbf{CP}^1$ is a holomorphic fibration; denote by $\mathcal{O}_Z(k)$ the sheaf $\pi_2^* \mathcal{O}_{\mathbf{CP}^1}(k)$.

Penrose transform

For any $U \subset X$ open, we have an isomorphism

$$H^1(\pi_1^{-1}(U), \mathcal{O}_Z(-2)) \cong \ker \Delta_g|_U.$$

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Furthermore, denote by $\mathcal{I}_{\mathbf{x}}$ the ideal sheaf of $\mathbf{CP}^1_{\mathbf{x}}$ in Z_X. Then,

$$\mathcal{I}_{\mathbf{x}}|_{Z_X \setminus \mathbf{CP}^1_{\mathbf{x}}} \cong \mathcal{O}_Z,$$

so

$$\mathrm{Ext}^1(Z_X; \mathcal{I}_{\mathbf{x}}, \mathcal{O}_Z(-2)) \hookrightarrow H^1(Z_X \setminus \mathbf{CP}^1_{\mathbf{x}}, \mathcal{O}_Z(-2)).$$

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To have a finite-dimensional subspace of harmonic functions, we need a compactification of Z.

THE COMPLEX STRUCTURES OF M

Consider $S^2 \subset \mathbf{R}^3$, pick $e_1 \in S^2$ be arbitrary, and extend it to an oriented orthonormal basis (e_1, e_2, e_3) of $\mathbf{R}^3$. Consider the orthonormal basis of M:

$$\left(\xi_0 = \sqrt{V}\frac{\partial}{\partial\tau}, \xi_j = \frac{1}{\sqrt{V}}e_j\right) \quad j = 1, 2, 3.$$

Define the (almost-)complex structure J_{e_1} by

$$\xi_0 \mapsto \xi_1 \mapsto -\xi_0$$
$$\xi_2 \mapsto \xi_3 \mapsto -\xi_2.$$

M as a complex surface

Theorem (Kronheimer-Andersen-LeBrun)

1. *If e_1 is not parallel to any l_{ij}, then (M, J_{e_1}) is the smooth surface*

$$\left(xy - \prod_{j=1}^{k} (z - p_j) \right) \subset \mathbf{C}^3$$

for some mutually distinct $p_j \in \mathbf{C}$.

2. *If e_1 is parallel to some l_{ij}, then (M, J_{e_1}) is the resolution of singularities of*

$$\left(xy - \prod_{j=1}^{k} (z - p_j) \right) \subset \mathbf{C}^3$$

for some $p_j \in \mathbf{C}$ (where $p_i = p_j$ if e_1 is parallel to l_{ij}).

THE TWISTOR SPACE OF M

Consider the total space W of the fibration

$$\mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \to \mathbf{CP}^1.$$

Let x, y and z stand for the canonical sections of the components. Then there exist $p_j \in H^0(\mathbf{CP}^1, \mathcal{O}(2))$ such that Z_M is the hypersurface

$$xy - \prod_{j=1}^{k}(z - p_j).$$

A smooth compactification

Compactify W into

$$\mathbf{P}(\mathcal{O}(k)\oplus\mathcal{O})\oplus\mathbf{P}(\mathcal{O}(k)\oplus\mathcal{O})\oplus\mathbf{P}(\mathcal{O}(2)\oplus\mathcal{O})\to\mathbf{CP}^1,$$

and let

$$(x:u),(y:v),(z:w)$$

denote the canonical homogeneous coordinates on the components. Denote by Z^* the singular hypersurface

$$xyw^k-uv\prod_{j=1}^{k}(z-p_j).$$

Then Z^* arises from Z by adding 4 Hirzebruch-surfaces. Furthermore, Z^* admits an A_k-singularity at infinity; resolving it, we get a smooth compactification

$$\overline{Z}\to Z^*.$$

Claim

For all $\mathbf{x} \in M$ *we have*

$$\dim_{\mathbf{C}}(\mathrm{Ext}^1(\overline{Z}_M; \mathcal{I}_{\mathbf{x}}, \mathcal{O}_Z(-2))) = 2.$$

Proof.

Uses: Ext long exact sequence, Leray spectral sequence and rationality of the A_k singularity. □

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We can consider extension classes as

- some rank 2 sheaves on $\overline{Z}_M$, or
- harmonic functions via Penrose transform and

$$\mathrm{Ext}^1(\overline{Z}_M; \mathcal{I}_{\mathbf{x}}, \mathcal{O}_Z(-2)) \hookrightarrow H^1(Z_X \setminus \mathbf{CP}^1_{\mathbf{x}}, \mathcal{O}_Z(-2)).$$

Claim (Atiyah, 1981)

The harmonic functions $f : M \setminus \mathbf{x} \to \mathbf{C}$ *coming from* $\mathrm{Ext}^1(\overline{Z}_M; \mathcal{I}_{\mathbf{x}}, \mathcal{O}_Z(-2))$ *satisfy*

- $f \to const$ *as* $r \to \infty$;
- $f(\mathbf{y}) \asymp \frac{c}{|\mathbf{x}-\mathbf{y}|^2}$ *as* $\mathbf{y} \to \mathbf{x}$.

So, this 2-dimensional family corresponds to the functions

$$\lambda + \mu G_{\mathbf{x}}$$

with $\lambda, \mu \in \mathbf{C}$. Restricting to $\lambda, \mu \in \mathbf{R}_+$ and dividing by λ, we get the functions

$$1 + \frac{1}{\lambda} G_{\mathbf{x}}.$$

Idetification of centerless instantons

Let $F_{\mathbf{x}}$ denote the rank 2 vector bundle on $\overline{Z}$ corresponding to $G_{\mathbf{x}}$. One can cover $\overline{Z}$ by 2 open sets U, V, so that the gluing matrix of $F_{\mathbf{x}}$ is

$$\frac{1}{\nu}\begin{pmatrix} -x & \theta \\ -h & y \end{pmatrix}.$$

Observe that the gluing matrix of $F_{e^{i\tau}\mathbf{x}}$ only differs from this by a factor of $e^{i\tau}$.

So $F_{\mathbf{x}}$ is isomorphic to $F_{e^{i\tau}\mathbf{x}}$, hence so are the instantons $\nabla_{G_{\mathbf{x}}}$ and $\nabla_{e^{i\tau}G_{\mathbf{x}}}$.

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Further topics to study:

- Complete Riemannian metric on $\mathcal{M}$: hyperbolic on D^2, multi-Taub–NUT for fixed λ.

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Further topics to study:

- Complete Riemannian metric on $\mathcal{M}$: hyperbolic on D^2, multi-Taub–NUT for fixed λ.
- Determining the moduli space for Γ a non-trivial flat connection on E at infinity: $k-1$ possible choices, for each one the corresponding moduli space is smooth.
- Describe explicitly the framed moduli space: a hyperKähler 8-manifold, a singular SU(2)-fibration over $\mathcal{M}$.