Perversity equals weight for Painlevé spaces

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Conformal field theory, isomonodromy tau-functions and Painlevé equations
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Outline

Hodge theory, Riemann–Hilbert

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Filtrations

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Wild non-abelian Hodge theory (NAHT)

Simpson '90, Biquard–Boalch '04: fix

- C: smooth projective curve over $\mathbb{C}$
- $r \geqslant 2$ rank
- $p_1, \ldots, p_n \in C$ irregular singularities (with local charts z_j)
- a flag type and parabolic weights at each p_j
- an irregular type $Q_j \in \mathfrak{t} \otimes \mathbb{C}((z_j))/\mathbb{C}[[z_j]]$ and an adjoint orbit $\mathcal{O}_j$ of the residue at each p_j.

The space of solutions of Hitchin's equations

$$D^{0,1}\theta = 0$$

$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection D on a rank r smooth Hermitian vector bundle (V, h) and a field $\theta : V \to V \otimes \Omega^{1,0}_C$ having prescribed singular behaviour near $p_j \rightsquigarrow$ hyper-Kähler moduli space $\mathcal{M}_{\text{Hod}}$.

de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\mathsf{Hod}}$ with a geometric meaning:

- de Rham: $\mathcal{M}_{\mathsf{dR}}$ parameterising poly-stable parabolic connections with irregular singularities
- Dolbeault: $\mathcal{M}_{\mathsf{Dol}}$ parameterising poly-stable parabolic Higgs bundles with higher-order poles.

By non-abelian Hodge theory, $\mathcal{M}_{\mathsf{dR}}$ and $\mathcal{M}_{\mathsf{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\mathsf{Hod}}$).

Irregular Riemann–Hilbert correspondence

- Birkhoff, Mebkhout, Kashiwara, Jurkat, Deligne, Malgrange: equivalence between the categories of irregular connections and Stokes-filtered local systems.
- Boalch '07: algebraic construction of wild character varieties $\mathcal{M}_{\mathrm{B}}$ parameterising Stokes data.
- Irregular Riemann–Hilbert correspondence (RH): bi-analytic map

$$\mathrm{RH} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}.$$

Conclusion: $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are all diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}$), in particular they have the same cohomology spaces.

PAINLEVÉ SPACES

From now on, we set $C = \mathbb{C}P^1$ and we assume $r = 2$ and $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Hod}} = 4$. There exists a finite list

$$PI, PII, PIII(D6), PIII(D7), PIII(D8), PIV, PV_{\mathrm{deg}}, PV, PVI$$

of irregular types with this property, called Painlevé cases. From now on, we let

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

and we write PX to refer to one of the above Painlevé cases. We therefore have smooth non-compact Kähler surfaces

$$\mathcal{M}_{\mathrm{dR}}^{PX}, \quad \mathcal{M}_{\mathrm{Dol}}^{PX}, \quad \mathcal{M}_{\mathrm{B}}^{PX}$$

diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}^{PX}$) for any fixed X.

Middle perversity t-structure

Given an algebraic variety Y, consider the derived category

$$D^b(Y, \mathbb{Q})$$

of bounded complexes of $\mathbb{Q}$-vector spaces K on Y with constructible cohomology sheaves of finite rank.
Beilinson–Bernstein–Deligne '82: truncation functors

$${}^{\mathfrak{p}}\tau_{\leqslant i} : D^b(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leqslant i}(Y, \mathbb{Q})$$

encoding the support condition for the middle perversity function, giving rise to a system of truncations

$$0 \to \cdots \to {}^{\mathfrak{p}}\tau_{\leqslant -p} K \to {}^{\mathfrak{p}}\tau_{\leqslant -p+1} K \to \cdots \to K$$

Perverse filtration on Dolbeault spaces

Hitchin '87: for $\mathcal{M}_{\mathrm{Dol}}$ a Dolbeault moduli space there exists a surjective map

$$h : \mathcal{M}_{\mathrm{Dol}} \to Y = \mathbb{C}^N.$$

Consider

$$K = \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}} \in D^b(Y, \mathbb{Q}).$$

The perverse filtration P on

$$\mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

is defined as

$$P^p\mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = \mathrm{Im}(\mathbf{H}^*(Y, {}^{\mathfrak{p}}\tau_{\leqslant -p}\mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}})).$$

We define the perverse Hodge polynomial of $\mathcal{M}_{\mathrm{Dol}}$ by

$$PH(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) q^i t^k.$$

Flag filtration on Dolbeault spaces

For an affine variety Y of dimension n consider a generic flag

$$Y_{-n} \subset \cdots \subset Y_{-1} \subset Y_0 = Y,$$

where Y_p are the intersections of Y with a fixed generic linear flag under a fixed projective embedding. Given any K one may consider the sequence of complexes

$$0 \subseteq K_{Y \setminus Y_{-1}} \subseteq \cdots \subseteq K_{Y \setminus Y_{-n}} \subseteq K.$$

It gives rise to the flag filtration F defined by

$$F^i H^l(Y,K) = \mathrm{Ker}(H^l(Y,K) \to H^l(Y_{i-1}, K|_{Y_{i-1}})).$$

Theorem (de Cataldo–Migliorini '10)

For Y affine we have

$$P^p H^l(Y,K) = F^{p+l} H^l(Y,K).$$

Weight filtration on Betti spaces

As $\mathcal{M}_{\mathrm{B}}$ is an affine algebraic variety, Deligne's Hodge II. ('71) shows that $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ carries a weight filtration W. We derive a polynomial

$$WH(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}^W_{2i} H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{C}) q^i t^k.$$

Hausel–Rodriguez-Villegas '08: WH is indeed a polynomial.

$P = W$ conjecture

Theorem (de Cataldo–Hausel–Migliorini '12)

If C is compact and $r = 2$, then for the Dolbeault and Betti spaces corresponding to each other under non-abelian Hodge theory and the Riemann–Hilbert correspondence, we have

$$PH(q,t) = WH(q,t).$$

Conjecture (de Cataldo–Hausel–Migliorini '12)

The same assertion holds for any rank r.

$P = W$ in the Painlevé cases

Let us set

$$PH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) q^i t^k,$$

$$WH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}_{2i}^W H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) q^i t^k.$$

Theorem (Sz '18)

For each

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

we have

$$PH^{PX}(q,t) = q^{-1} WH^{PX}(q,t).$$

Simpson's conjecture

Theorem (Sz '18)

There exists a smooth compactification $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ of $\mathcal{M}_{\mathrm{B}}^{PX}$ by a simple normal crossing divisor D with nerve complex $\mathcal{N}^{PX}$ and for some sufficiently large compact set $K \subset \mathcal{M}_{\mathrm{B}}^{PX}$ a homotopy commutative square

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}^{PX} \setminus K & \longrightarrow & \mathcal{M}_{\mathrm{B}}^{PX} \setminus K \\ \downarrow h & & \downarrow \phi \\ D^{\times} & \longrightarrow & |\mathcal{N}^{PX}|. \end{array}$$

Here, h denotes the Hitchin map, $D^{\times} \subset Y$ is a neighbourhood of ∞ in the Hitchin base, and the top row is the diffeomorphism coming from NAHT and the irregular RH correspondence.

Hitchin fibration

Irregular Hitchin map

$$h : \mathcal{M}_{\mathrm{Dol}}^{PX} \to Y = \mathbb{C}.$$

Theorem (Ivanics–Stipsicz–Szabó '17)

There exists an embedding

$$\mathcal{M}_{\mathrm{Dol}}^{PX} \hookrightarrow E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$$

and an elliptic fibration

$$\tilde{h} : E(1) \to \mathbb{C}P^1$$

extending h.

Denote by F_{∞}^{PX} the non-reduced curve $E(1) \setminus \mathcal{M}_{\mathrm{Dol}}^{PX} = \tilde{h}^{-1}(\infty)$.

Euler characteristic and perverse polynomial

Proposition

We have

$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-3} H^0(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 1$$
$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 1$$
$$\dim_{\mathbb{Q}} \mathrm{Gr}^P_{-2} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{Q}) = 10 - \chi(F^{PX}_{\infty}).$$

In particular, we have

$$PH^{PX}(q,t) = q^{-1} + (10 - \chi(F^{PX}_{\infty}))q^{-2}t^2 + q^{-3}t^2.$$

Table of perverse polynomials

X	F_∞^{PX}	$PH^{PX}(q,t)$
VI	$D_4^{(1)}$	$q^{-1}+4q^{-2}t^2+q^{-3}t^2$
V	$D_5^{(1)}$	$q^{-1}+3q^{-2}t^2+q^{-3}t^2$
V_{deg}	$D_6^{(1)}$	$q^{-1}+2q^{-2}t^2+q^{-3}t^2$
$III(D6)$	$D_6^{(1)}$	$q^{-1}+2q^{-2}t^2+q^{-3}t^2$
$III(D7)$	$D_7^{(1)}$	$q^{-1}+q^{-2}t^2+q^{-3}t^2$
$III(D8)$	$D_8^{(1)}$	$q^{-1}+q^{-3}t^2$
IV	$E_6^{(1)}$	$q^{-1}+2q^{-2}t^2+q^{-3}t^2$
II	$E_7^{(1)}$	$q^{-1}+q^{-2}t^2+q^{-3}t^2$
I	$E_8^{(1)}$	$q^{-1}+q^{-3}t^2$

Idea of proof of Proposition

Analysis of Leray spectral sequence ${}_LE_2^{k,l}$ of h:

$k=2$	0	0	0
$k=1$	0	$H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}})$	0
$k=0$	$\mathbb{C}$	$\mathbb{C}^{b_1(\mathcal{M})}$	$\mathbb{C}$
	$l=0$	$l=1$	$l=2$

Standard algebraic topology shows that

- $b_1(\mathcal{M}) = 0$,
- $\dim_{\mathbb{C}} H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}}) = 10 - \chi(F_\infty^{PX})$,
- the following map is surjective

$$H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) \to H^2(h^{-1}(Y_{-1}), \mathbb{C}) = \mathbb{C}.$$

End of proof of the Proposition

We get

$$\begin{aligned} \mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C}) &\cong \mathrm{Im}(\mathbf{H}^2(Y, \mathbb{R}h_* \underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})) \\ &\cong \mathbb{C}, \\ \mathrm{Gr}^P_{-2} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C}) &= \mathrm{Ker}(\mathbf{H}^2(Y, \mathbb{R}h_* \underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})) \\ &\cong \mathbb{C}^{10-\chi(F^{PX}_\infty)}. \end{aligned}$$

Moment map on monodromy and Stokes data

P. Boalch (2007): Let G be a reductive group over $\mathbb{C}$, $U_\pm$ unipotent radicals of a pair of opposite Borels, $T = U_+ \cap U_-$, $\mathfrak{t} = L(T)$, $k \geqslant 1$ and set

$$\tilde{\mathcal{C}} = G \times (U_+ \times U_-)^{k-1} \times \mathfrak{t}.$$

Then $\tilde{\mathcal{C}}$ is a quasi-Hamiltonian $G \times T$-space with multiplicative moment map

$$\begin{aligned}(\mu, \exp)\colon \tilde{\mathcal{C}} &\to G \times T \\ (C, S_1, \ldots, S_{2k-2}, \Lambda) &\mapsto (C^{-1} S_{2k-2} \cdots S_1 e^{2i\pi\Lambda} C, e^{-2i\pi\Lambda})\end{aligned}$$

Quasi-Hamiltonian construction of $\mathcal{M}_B$

Internally fused double: quasi-Hamiltonian G-space $\mathbf{D} = G \times G$ with action of G by diagonal conjugation.
Fusion operation on quasi-Hamiltonian $G \times T$-spaces M_1, M_2: a quasi-Hamiltonian $G \times T \times T$-space

$$M_1 \circledast M_2.$$

Let $g = g(C)$. The wild character variety is the fusion

$$\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \tilde{\mathcal{C}}_1 \circledast \cdots \circledast \tilde{\mathcal{C}}_n$$

with $\mathbb{D}$ appearing g times. In concrete terms it can be given by equations

$$[A_1, B_1] \cdots [A_g, B_g] \mu_1 \cdots \mu_n = \mathbb{I},$$

up to G-action. It inherits a holomorphic symplectic structure.

Betti spaces and affine cubic surfaces

Fricke–Klein 1926, van der Put–Saito '09: for each X there exists a quadric

$$Q^{PX} \in \mathbb{C}[x_1, x_2, x_3]$$

such that

$$\mathcal{M}_{\mathrm{B}}^{PX} = (f^{PX}) \subset \mathbb{C}^3$$

where

$$f^{PX}(x_1, x_2, x_3) = x_1 x_2 x_3 + Q^{PX}(x_1, x_2, x_3).$$

Compactifications of Betti spaces

Let

$$F^{PX} \in \mathbb{C}[x_0, x_1, x_2, x_3]_3$$

be the homogenization of f^{PX} and set

$$\overline{\mathcal{M}}_{\mathrm{B}}^{PX} = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]/(F^{PX})).$$

Projective cubic surface (possibly) with singularities at

$$P_1 = [0:1:0:0], \quad P_2 = [0:0:1:0], \quad P_3 = [0:0:0:1].$$

Let

$$\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX} \to \overline{\mathcal{M}}_{\mathrm{B}}^{PX}$$

denote the minimal resolution of singularities.

Total Milnor number and weight polynomial

Define the total Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ as

$$N^{PX} = \sum_{j=1}^{3} \mu(P_j)$$

where $\mu(P_j)$ is the Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ at P_j.

Proposition

We have

$$WH^{PX}(q,t) = 1 + (4 - N^{PX})q^{-1}t^2 + q^{-2}t^2.$$

Table of weight polynomials

X	Singularities of $\overline{\mathcal{M}}_B^{PX}$	$WH^{PX}(q,t)$
VI	$\emptyset$	$1+4q^{-1}t^2+q^{-2}t^2$
V	A_1	$1+3q^{-1}t^2+q^{-2}t^2$
V_{deg}	A_2	$1+2q^{-1}t^2+q^{-2}t^2$
$III(D6)$	A_2	$1+2q^{-1}t^2+q^{-2}t^2$
$III(D7)$	A_3	$1+q^{-1}t^2+q^{-2}t^2$
$III(D8)$	A_4	$1+q^{-2}t^2$
IV	A_1+A_1	$1+2q^{-1}t^2+q^{-2}t^2$
II	$A_1+A_1+A_1$	$1+q^{-1}t^2+q^{-2}t^2$
I	$A_2+A_1+A_1$	$1+q^{-2}t^2$

Compactifying divisors

The divisor at infinity of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$D = L_1 \cup L_2 \cup L_3$$

where L_i are lines pairwise intersecting each other in P_1, P_2, P_3.
The nerve complex of the divisor at infinity of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$\mathcal{N}^{PX} = A^{(1)}_{N^{PX}+2} = I_{N^{PX}+3}.$$

The first page of the weight spectral sequence

Deligne: spectral sequence ${}_W E_r$ abutting to $H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C})$ with ${}_W E_1^{-n,k+n}$ given by

$$
\begin{array}{lrrr}
k+n=4 & \bigoplus_{p \in \mathcal{N}_1^{PX}} H^0(p,\mathbb{C}) & \bigoplus_{L \in \mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) & H^4\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right) \\
k+n=3 & 0 & 0 & 0 \\
k+n=2 & 0 & \bigoplus_{L \in \mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) & H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right) \\
k+n=1 & 0 & 0 & 0 \\
k+n=0 & 0 & 0 & H^0\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right) \\
 & -n=-2 & -n=-1 & -n=0
\end{array}
$$

The first differentials of the weight spectral sequence

The only non-trivial differentials d_1 on ${}_W E_1$ are:

$$\oplus_{p\in\mathcal{N}_1^{PX}} H^0(p,\mathbb{C}) \xrightarrow{\delta} \oplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \xrightarrow{\delta_4} H^4\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right)$$

$$\oplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \xrightarrow{\delta_2} H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right).$$

Algebraic topology of cubic surfaces shows that

$$\delta_4 : \oplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \twoheadrightarrow H^4\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right)$$

$$\delta_2 : \oplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \hookrightarrow H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX},\mathbb{C}\right) \cong \mathbb{C}^7.$$

Dimensions of graded pieces for the weight filtration

We derive

$$\mathrm{Gr}_0^W H^0(\mathcal{M}_{\mathrm{B}}^{PX}) \cong \mathbb{C}$$
$$\mathrm{Gr}_{-2}^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong \mathbb{C}^{4-N^{PX}}$$
$$\mathrm{Gr}_{-4}^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong \mathbb{C}.$$

$P = W$ — identification of the filtrations

The two tables (Dolbeault/Betti) show the numerical version of the theorem.

We saw:

- $\mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C}) \cong \mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})$
- $\mathrm{Gr}^W_{-4} H^2(\mathcal{M}^{PX}_{\mathrm{B}}, \mathbb{C}) \cong \ker(\delta)$.

Need to show that NAHT maps $\mathbf{H}^2(Y_{-1}, \mathbb{R}h_* \underline{\mathbb{C}}|_{Y_{-1}})$ isomorphically onto $\ker(\delta)$.

Key observation:

$$H^2(\mathcal{M}^{PX}_{\mathrm{Hod}} \setminus K, \mathbb{C}) \cong \mathbb{C},$$

and the generators of both $\mathrm{Gr}^P_{-3} H^2(\mathcal{M}^{PX}_{\mathrm{Dol}}, \mathbb{C})$ and $\mathrm{Gr}^W_{-4} H^2(\mathcal{M}^{PX}_{\mathrm{B}}, \mathbb{C})$ represent non-trivial classes in this cohomology space.

Simpson's conjecture — partitions of unity

Components of the compactifying divisor D^{PX} of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$:

$$L_1, \ldots, L_{N^{PX}+3}$$

in circular order. Consider open tubular neighbourhoods

$$T_1, \ldots, T_{N^{PX}+3}$$

of these components, such that $T_i \cap T_j \neq \emptyset$ if and only if $L_i \cap L_j \neq \emptyset$. Let

$$\rho_1, \ldots, \rho_{N^{PX}+3}$$

be a partition of unity subordinate to the open sets T_i, and define the map

$$\begin{aligned} \phi : T_{\mathrm{B}}^{PX} = T_1 \cup \cdots \cup T_{N^{PX}+3} &\to \mathbb{R}^{N^{PX}+3} \\ x &\mapsto (\rho_1(x), \ldots, \rho_{N^{PX}+3}(x)). \end{aligned}$$

Simpson's conjecture — homotopy commutativity

It is easy to see that $\mathrm{im}(\phi)$ is homotopy equivalent to the nerve complex $\mathcal{N}^{PX}$ of D^{PX}.
Need to show: the loop α in $T_{\mathrm{B}}^{PX} \cap \mathcal{M}_{\mathrm{B}}^{PX}$ coming from the fundamental group of D^{PX} maps to a generator of the fundamental group of S^1 under ϕ.
This statement is the Poincaré dual in $\partial T_{\mathrm{B}}^{PX}$ of $P = W$.

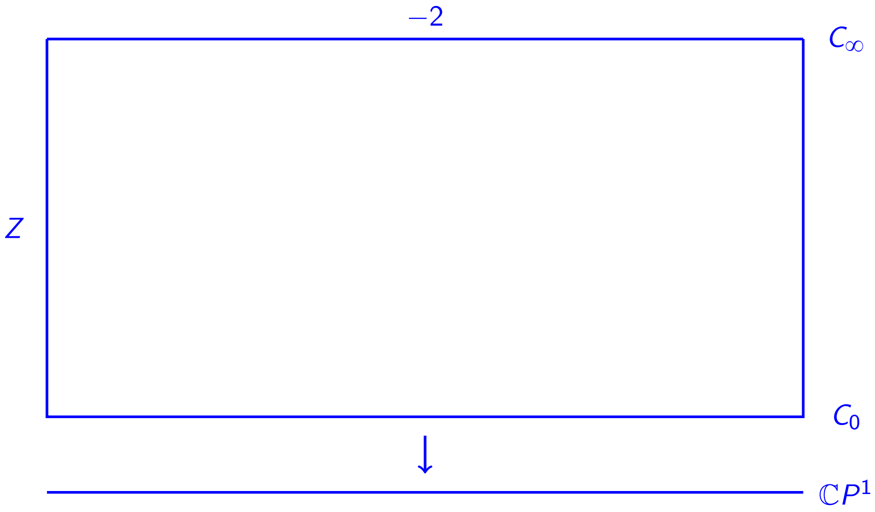
Fixing the singularity types

Joint work with P. Ivanics and A. Stipsicz: description of $\mathcal{M}^s_{\mathrm{Dol}}$ in $\dim_{\mathbb{C}} = 2$.
Let $X = \mathbb{C}P^1$, $r = 2$, $n = 2$, singularities:

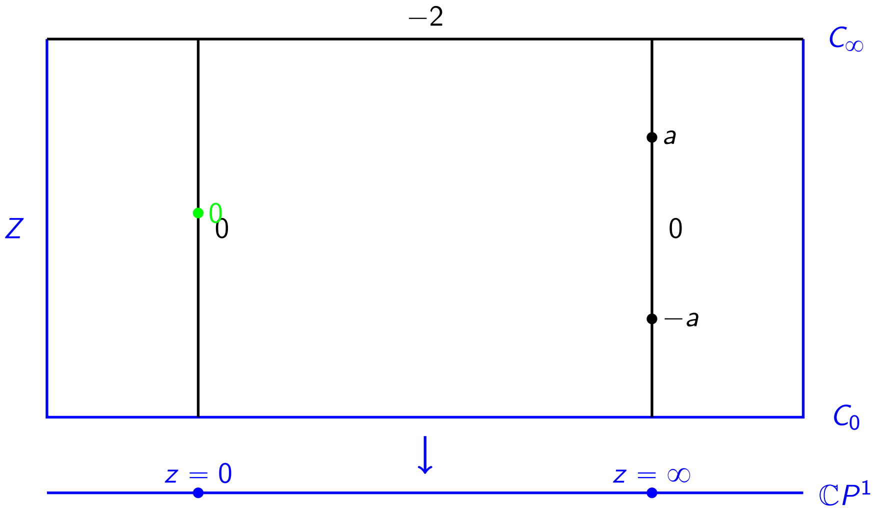
- p_1 at $z = 0$: logarithmic, with full flag parabolic filtration and nilpotent residue compatible with the filtration
- p_2 at $z = \infty$: Poincaré rank 2, with trivial parabolic filtration and local form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} z dz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} dz + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{dz}{z} + \text{lower order terms}$$

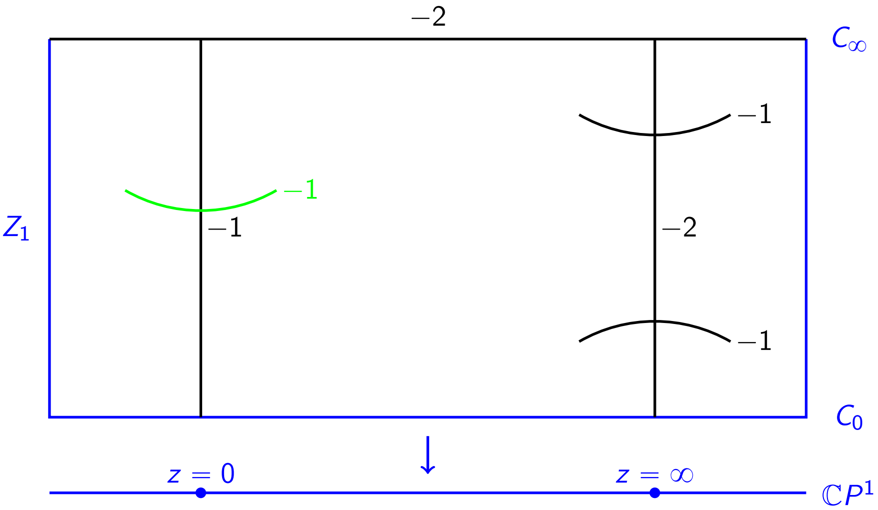
Hirzebruch surface $H_2 = \mathbf{P}(\mathcal{O}_{¶} \oplus \mathcal{O}_{¶}(2))$



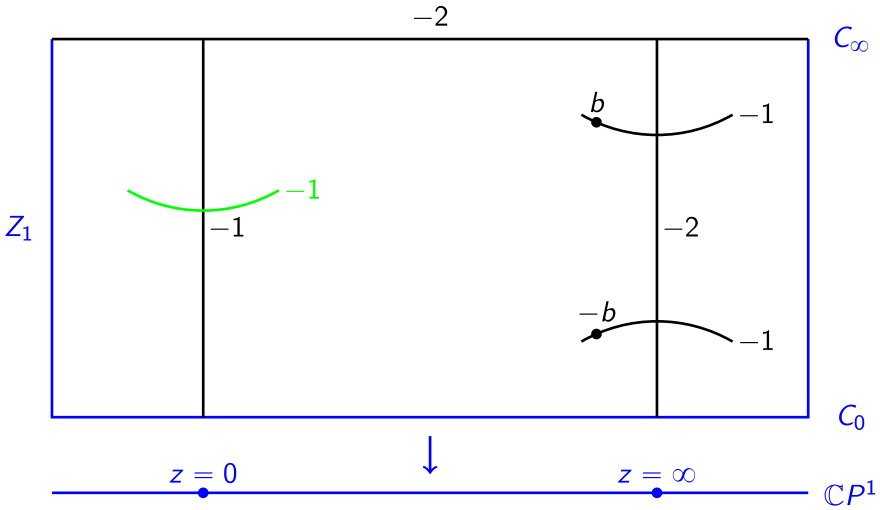
Base points on H_2



The first blow-up



Base points on the first blow-up



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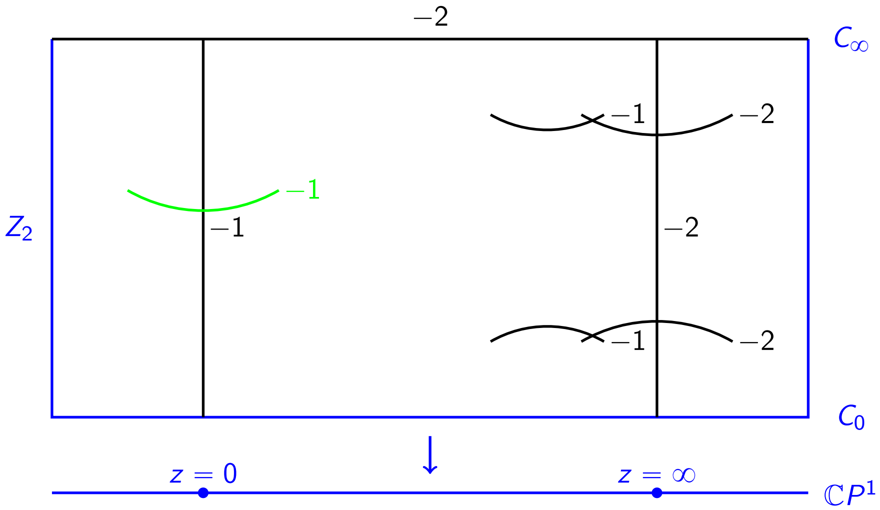
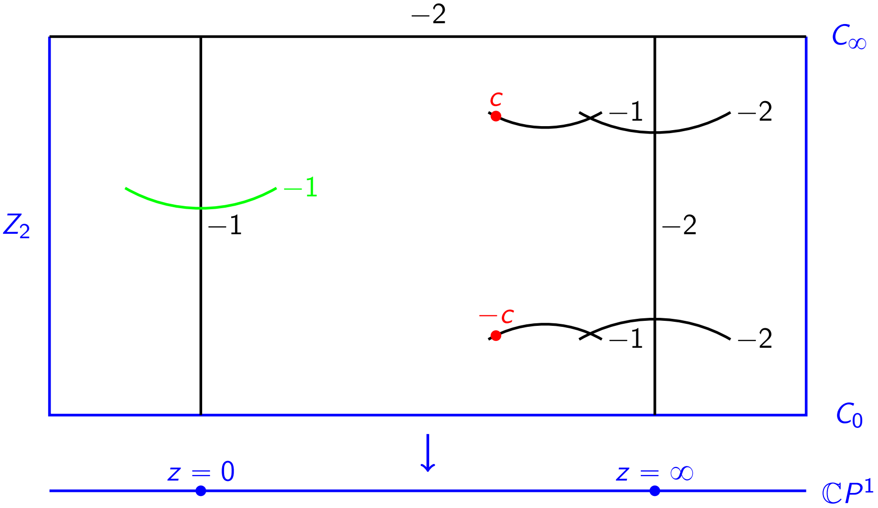
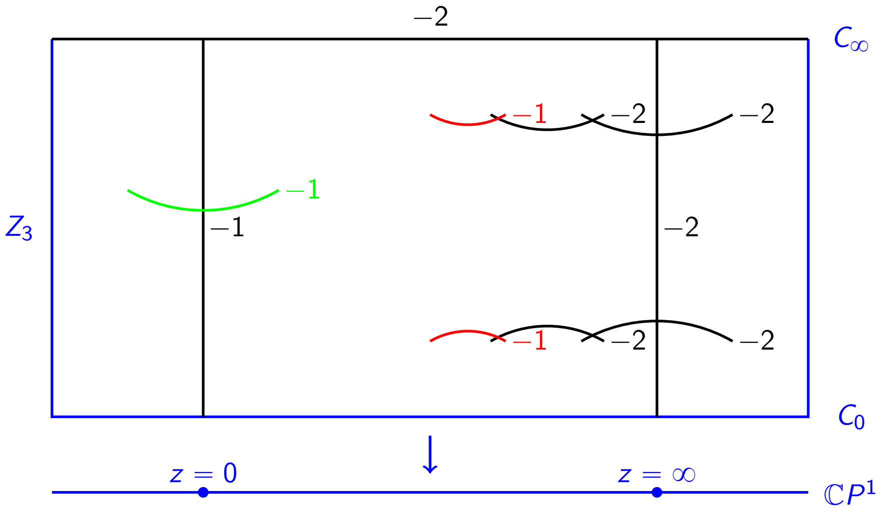


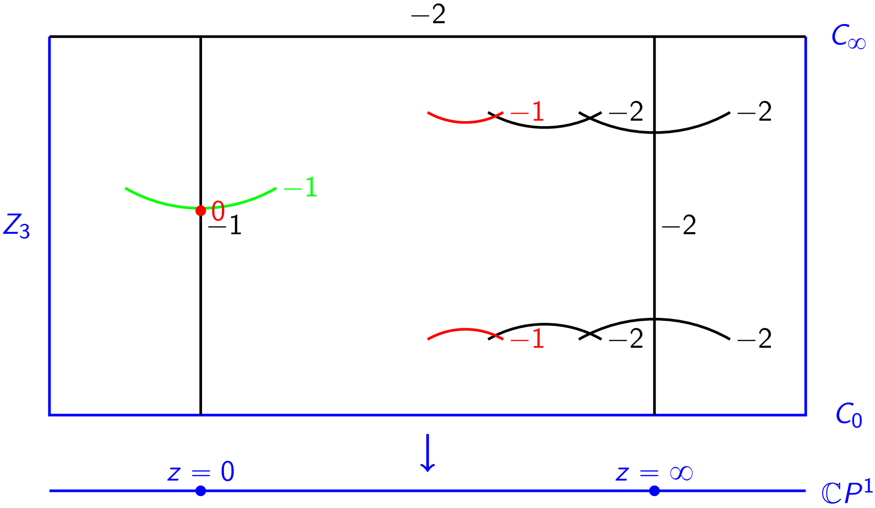
Base points on the second blow-up



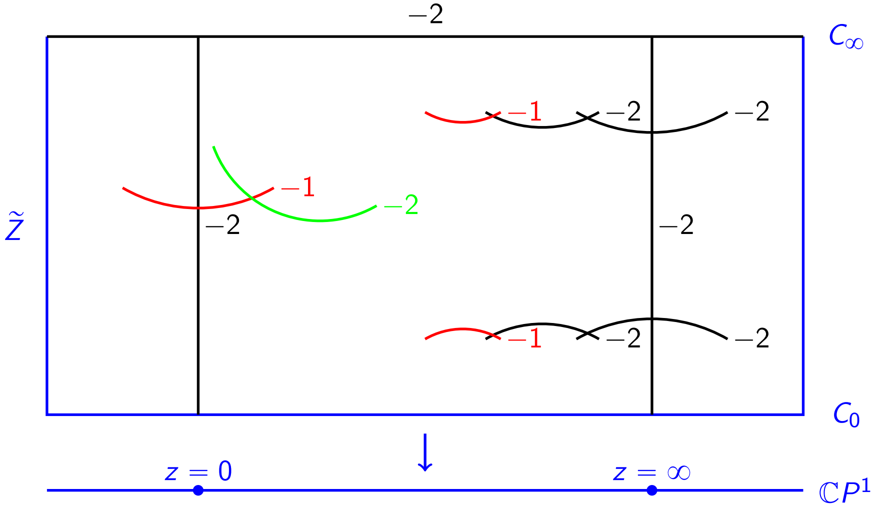
The third blow-up



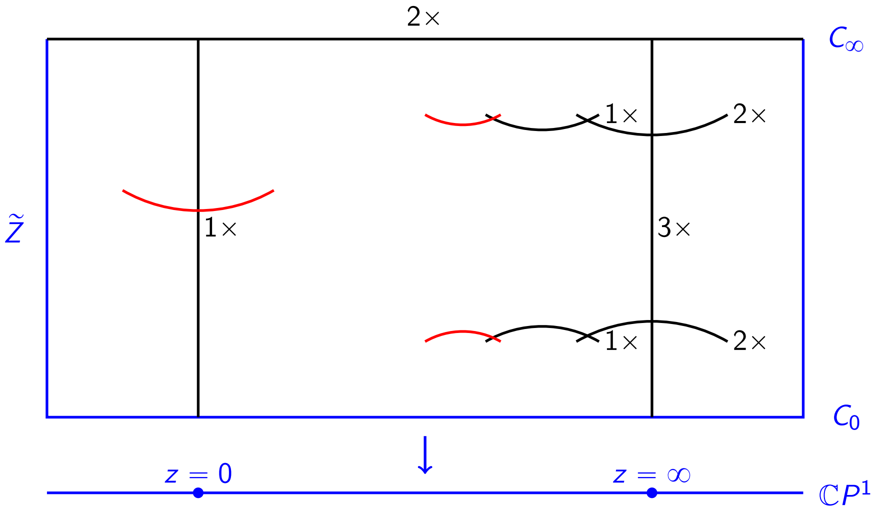
A further base point



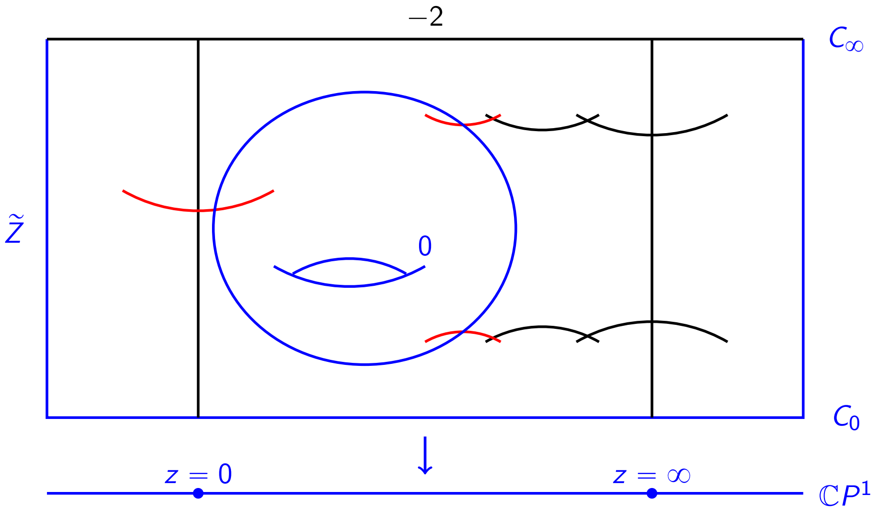
Blowing up the last base point



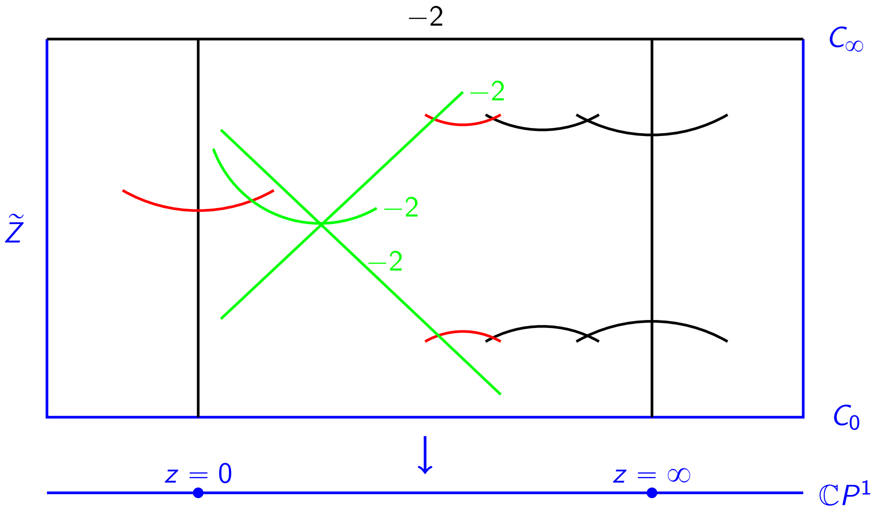
An anticanonical $\widetilde{E}_6$-fiber in the pencil



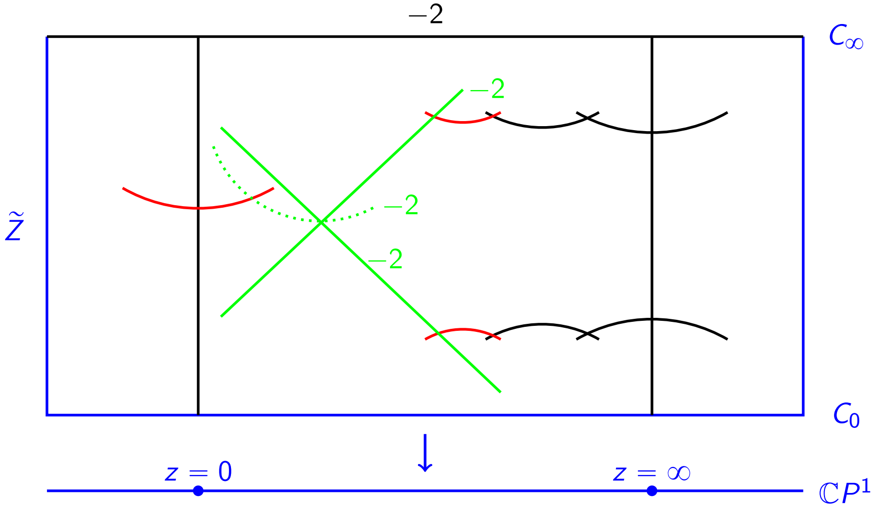
The generic curve in the pencil



A type IV singular curve in the fibration



The corresponding spectral curve



Relative Picard over the smooth locus

The fibration has sections

$$\sigma : B \to X.$$

Abel–Jacobi: for any smooth $\tilde{\tilde{\Sigma}} = X_b$, get

$$\begin{aligned} \tilde{\tilde{\Sigma}} &\cong \mathrm{Pic}^0(\tilde{\tilde{\Sigma}}) \\ x &\mapsto (x - \sigma(b)). \end{aligned}$$

Spectral sheaf and residue

Lemma

For any curve Z in the corresponding pencil (smooth or singular), the endomorphism $\mathrm{Res}_{p_1}\theta$ has non-trivial nilpotent part if and only if the spectral sheaf $\mathcal{S}$ is a locally free sheaf on Z_t near P.

Proof.

Tensoring the defining exact sequence

$$0 \to p^*\mathcal{E} \otimes K^\vee(-3\{p_2\} - \{p_1\}) \xrightarrow{\zeta - p^*\theta} p^*\mathcal{E} \to \mathcal{S} \to 0$$

by $\mathcal{O}_{\mathbb{F}_2}/p^*\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{p_1}$, we get:

$$\mathcal{S}|_{p_1} \cong \mathrm{coker}(\mathrm{Res}_{p_1}\theta).$$

□

Strata of the relative Picard over the singular locus

For suitable values of the parameters, there is only one singular fiber of the fibration, of type IV. The corresponding spectral curve Z_t consists of two sections of the Hirzebruch surface, simply tangent to each other on the fiber over p_1.
Stable Higgs bundles with spectral curve Z_t and $\mathrm{Res}_{p_1}(\theta)$ having non-trivial nilpotent part are parameterized by

$$\mathbb{C}_{\delta_+,\delta_-} \coprod \mathbb{C}_{\delta_+-1,\delta_-+1}.$$

Stable Higgs bundles with spectral curve Z_t and $\mathrm{Res}_{p_1}(\theta) = 0$ are parameterized by a point. The choice of compatible quasi-parabolic lines are parameterized by $\mathbb{C}P^1$.

Relative Picard over the singular locus and wall-crossing

In all, the corresponding Hitchin fiber is parameterized by

$$\mathbb{C}_{\delta_+,\delta_-} \coprod \mathbb{C}_{\delta_+ -1,\delta_- +1} \coprod \mathbb{C}P^1,$$

therefore is of type IV too.
The actual value of $(\delta_+, \delta_-) \in \mathbb{Z}^2$ depends on the choice of parabolic structure. Effect of crossing a wall in the space of parabolic structures:

$$(\delta_+, \delta_-) \rightsquigarrow (\delta_+ - 1, \delta_- + 1).$$

Singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PIV}$

van der Put–Saito: the Betti moduli space is the affine cubic

$$x_1x_2x_3 + x_1^2 - (s_2^2 + s_1s_2)x_1 - s_2^2x_2 - s_2^2x_3 + s_2^2 + s_1s_2^3 = 0$$

with $s_1 \in \mathbb{C}, s_2 \in \mathbb{C}^\times$.
Singular points of $\overline{\mathcal{M}}_{\mathrm{B}}^{PIV}$ over $x_0 = 0$: $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$, both of type A_1. So, $N^{PIV} = 2$ and

$$WH^{PIV}(q, t) = 1 + 2q^{-1}t^2 + q^{-2}t^2.$$