$P = W$ conjecture in lowest degree in Garnier case with 5 parabolic points

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Based on of Sz. *Perversity equals weight for Painlevé spaces* Adv. Math. (2021) and its continuation `arXiv:2103.00932`.

Using Mathematics that I learned from Prof. Masa-Hiko Saito, to whom I am grateful for continuous support and stimulating discussions.

Happy retirement, Masa-Hiko!

Basic notation

Fix the data

- $\Sigma = \mathbb{C}P^1$: complex projective line (more generally, a compact Riemann surface),
- $r = 2$ rank (i.e., $G = \mathrm{GL}_2(\mathbb{C})$),
- $t_0, \dots, t_n \in \mathbb{C}P^1$ logarithmic singularities (with local charts z_j),
- $D = t_0 + \cdots + t_n$ parabolic divisor,
- $\alpha_j^- = \frac{1}{4} < \alpha_j^+ = \frac{3}{4}$ Dolbeault parabolic weights

Hitchin's equations

Let (V, h) be a fixed rank 2 smooth Hermitian vector bundle on a compact Riemann surface Σ. We consider Hitchin's equations (1987)

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection D and a field $\theta : V \to V \otimes \Omega^{1,0}_C$.

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$$D^{0,1}\theta = 0 \in \Omega^2(\Sigma, \mathfrak{gl}(V)) \quad \text{complex equation}$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0 \in \Omega^2(\Sigma, \mathfrak{u}(V, h)) \quad \text{real equation}$$

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for a unitary connection D and a field $\theta : V \to V \otimes \Omega^{1,0}_C$.
The space of solutions of Hitchin's equations is preserved by unitary gauge transformations. Taking solutions up to unitary gauge equivalence $\rightsquigarrow$ $\mathcal{M}_{\text{Hod}}$ non-abelian Hodge moduli space, a hyper-Kähler manifold.

Tame harmonic bundles

Let Σ be a noncompact curve of the form

$$\Sigma = \overline{\Sigma} \setminus \{p_1, \ldots, p_n\},$$

with $\overline{\Sigma}$ compact. Tame harmonic bundles (C. Simpson, 1990): at the parabolic divisor we require

- θ has at most first-order poles at p_j,
- the eigenvalues of $\mathrm{res}_{p_j}(\theta)$ are equal to 0 (strongly parabolic),
- and with respect to a compatible trivialization,

$$h \approx \mathrm{diag}(|z_j|^{2\alpha_j^-}, |z_j|^{2\alpha_j^+}) = \mathrm{diag}(|z_j|^{\frac{1}{2}}, |z_j|^{\frac{3}{2}})$$

Konno, 1993: solutions up to unitary gauge equivalence form $\mathcal{M}_{\mathrm{Hod}}$ non-abelian Hodge moduli space, a hyper-Kähler manifold.

de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\text{Hod}}$ have a geometric meaning:

- de Rham: $\mathcal{M}_{\text{dR}}$ parametrizing S-equivalence classes of $\vec{\alpha}$-semistable parabolic integrable connections

$$\nabla = D + \theta + \theta^{\dagger_h}$$

  with logarithmic singularities with eigenvalues $\mu_j^{\pm} = \alpha_j^{\pm}$;
- Dolbeault: $\mathcal{M}_{\text{Dol}}$ parametrizing S-equivalence classes of $\vec{\alpha}$-semistable parabolic Higgs bundles $(\mathcal{E}, \theta)$ with first-order poles and nilpotent residue, where $\mathcal{E}$ is the holomorphic vector bundle structure on V defined by $D^{0,1}$.

By non-abelian Hodge theory (Hitchin–Kobayashi correspondence), $\mathcal{M}_{\text{dR}}$ and $\mathcal{M}_{\text{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\text{Hod}}$):

$$\text{NAHT}\colon \mathcal{M}_{\text{Dol}} \xrightarrow{\sim} \mathcal{M}_{\text{dR}}.$$

Hitchin map

Hitchin, 1987: for $\mathcal{M}_{\mathrm{Dol}}$ a Dolbeault moduli space there exists a surjective proper algebraic map of quasi-projective varieties

$$H\colon \mathcal{M}_{\mathrm{Dol}} \to Y = \mathbb{C}^N,$$

where $\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{Dol}} = 2N$. For $(\mathcal{E}, \theta)$ we set

$$\det(\zeta - \theta) = \zeta^r + s_1 \zeta^{r-1} + \cdots + s_r$$

for the characteristic polynomial of θ. Then,

$$H(\mathcal{E}, \theta) = (s_1, \ldots, s_r).$$

The generic fiber of H is an abelian variety, H is a completely integrable system.

Perverse filtration on Dolbeault spaces

The perverse filtration P on

$$\mathrm{H}^*(Y, \mathrm{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}) \cong H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

is defined as

$$P^p \mathrm{H}^*(Y, \mathrm{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}) = \mathrm{Im}(\mathrm{H}^*(Y, {}^{\mathfrak{p}}\tau_{\leq -p} \mathrm{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathrm{H}^*(Y, \mathrm{R}H_* \underline{\mathbb{Q}}_{\mathcal{M}}))$$

where

$${}^{\mathfrak{p}}\tau_{\leq i} : D^b(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leq i}(Y, \mathbb{Q})$$

are Beilinson–Bernstein–Deligne's truncation functors encoding the support condition for the middle perversity function

Flag filtration and perverse filtration

Let Y be an affine algebraic variety. Consider a generic flag

$$Y_{-n} \subset \cdots \subset Y_{-1} \subset Y_0 = Y,$$

where Y_p are the intersections of Y with a fixed generic linear flag under a fixed affine embedding. Given any $K \in D^b_{\text{constr}}(Y, \mathbb{Q})$ one may consider the sequence of complexes

$$0 \subseteq K_{Y \setminus Y_{-1}} \subseteq \cdots \subseteq K_{Y \setminus Y_{-n}} \subseteq K.$$

It gives rise to the flag filtration F defined by

$$F^i H^l(Y, K) = \text{Ker}(H^l(Y, K) \to H^l(Y_{i-1}, K|_{Y_{i-1}})).$$

Theorem (de Cataldo–Migliorini '10)

For Y affine we have $P^p H^l(Y, K) = F^{p+l} H^l(Y, K)$.

Character variety, weight filtration

Character variety $\mathcal{M}_{\mathrm{B}}$: moduli space parameterising (filtered) local systems ρ on $\mathbb{C}P^1 \setminus D$ up to global conjugacy, with eigenvalues on a simple positive loop around p_j given by

$$c_j^{\pm} = \exp(-2\pi\sqrt{-1}\alpha_j^{\pm}) = \pm\sqrt{-1}.$$

Known: $\mathcal{M}_{\mathrm{B}}$ is an affine algebraic variety.
Deligne, 1971: $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ carries a weight filtration W.

Topological representatives of W

ElZein–Némethi, 2002. Take a simple normal crossings compactification of $\mathcal{M}_{\mathrm{B}}$. Let

$$Q \in \tilde{D}^k \setminus \tilde{D}^{k+1}$$

be a generic point in the k-fold intersections of the irreducible components of the compactifying divisor. Let the corresponding divisor components be denoted by $Y_1, \ldots, Y_k$. The preimage $\Pi^{-1}(Q)$ of Q in the normal bundle of $Y_1 \cap \cdots \cap Y_k$ in $\mathcal{M}_{\mathrm{B}}$ deformation retracts onto a k-dimensional real torus

$$T^k \subset \partial N|_{Y_1 \cap \cdots \cap Y_k}$$

in the boundary of a tubular neighbourhood. The set of classes of these tori, for all k-tuple intersections, generate $\mathrm{Gr}^W_{-2k} H_k$, and the dual cohomology classes generate $\mathrm{Gr}^W_{2k} H^k$.

Riemann–Hilbert correspondence

Regular-singular Riemann–Hilbert correspondence: bi-analytic map

$$\mathrm{RH}\colon \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}},$$

obtained by analytical continuation of local parallel sections of ∇.

Conclusion:
$\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are diffeomorphic to each other, via

$$\psi = \mathrm{RH} \circ \mathrm{NAHT}\colon \mathcal{M}_{\mathrm{Dol}} \longrightarrow \mathcal{M}_{\mathrm{B}}.$$

Hitchin WKB-problem

Katzarkov–Noll–Pandit–Simpson, 2013: understand RH $\circ$ NAHT "close to infinity", i.e. for $(\mathcal{E}, \theta)$ with $|H(\mathcal{E}, \theta)| \gg 1$. Addressed by T. Mochizuki, 2016, for non-critical paths.

"Standard" WKB-method (...,Voros, 1983,..., Gaiotto–Moore–Neitzke, 2013,...), see K. Iwaki's talk: asymptotic analysis of solutions ψ of Schrödinger's equation

$$-\hbar^2 \frac{\mathrm{d}^2\psi(q)}{\mathrm{d}q^2} + V(q)\psi(q) = 0$$

as $\hbar \to 0$. This involves expressions of the form

$$\exp\left(\pm\hbar^{-1}\int_\gamma \sqrt{V(q)}\,\mathrm{d}q + O(\hbar)\right)$$

for paths $\gamma\colon [0,1] \to \Sigma$ avoiding the ramification points.

$P = W$ conjecture

Theorem (de Cataldo–Hausel–Migliorini '12)

If C is compact and $r = 2$, then for the Dolbeault and Betti spaces corresponding to each other under non-abelian Hodge theory and the Riemann–Hilbert correspondence, the filtrations P and W get mapped into each other.

Conjecture (de Cataldo–Hausel–Migliorini '12)

The same assertion holds for any rank r.

Geometric $P = W$ conjecture

C. Simpson, Geometric $P = W$ conjecture (2016): for a sufficiently large compact $K \subset \mathcal{M}_{\mathrm{Dol}}$ there exists a homotopy commutative square

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} \setminus K & \xrightarrow{\psi} & \mathcal{M}_{\mathrm{B}} \setminus \psi(K) \\ \downarrow H & & \downarrow \phi \\ Y \setminus H(K) & \longrightarrow & |\mathcal{N}|, \end{array}$$

where $\mathcal{N}$ is the nerve (or dual) complex of the boundary divisor and ϕ is a natural map.
Proposed to provide a geometric reason of $P = W$.

Geometric and cohomological $P = W$ in PVI case

Theorem (Sz. 2021)

In the Painlevé VI case, Geometric $P = W$ conjecture is true. Moreover, it implies the cohomological $P = W$ conjecture.

When comparing the dimensions of the graded pieces, we found an anomaly in the case PIII(D_8), mentioned in M-H. Saito's talk. Later, alternative approach in joint work with A. Némethi, for all Painlevé cases.

Idea of proof

We fix a generic element $q \in S^1 \subset Y$ and consider $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha})$ such that

$$H(\mathcal{E}, \theta) = q.$$

For $R > 0$ we have

$$H(\mathcal{E}, \sqrt{R}\theta) = Rq.$$

It is then possible to express the $R \to \infty$ asymptotic behaviour of $\mathrm{RH} \circ \mathrm{NAHT}$ in function of $\int_\gamma \sqrt{q}$ over various paths γ in $\mathbb{C}P^1$. Ingredients:

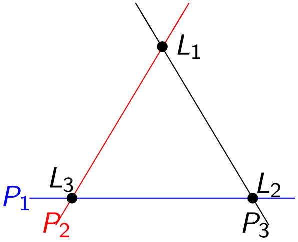
- results on asymptotic solutions of Hitchin's equations (T. Mochizuki),
- explicit description of $\mathcal{M}_{\mathrm{B}}$ (Fricke–Klein co-ordinates).

Compactifying divisors of Betti space

The divisor at infinity of $\overline{\mathcal{M}}_{\mathrm{B}}$ is

$$D = L_1 \cup L_2 \cup L_3 = (x_1 x_2 x_3) \subset \mathbb{C}P^2_\infty = \{[0 : x_1 : x_2 : x_3]\}$$

where $L_i = (x_i)$ are lines pairwise intersecting each other in P_1, P_2, P_3. Dual boundary divisor:



Arc decomposition of Hitchin base

Critical angles decompose S^1 into three closed arcs

$$S^1 = I_1 \cup I_2 \cup I_3$$

such that $\phi \circ \mathrm{RH} \circ \mathrm{NAHT}$ maps each I_j to P_j, and short intervals around points separating I_j's to L_k.
"Small $\leftrightarrow$ large" behaviour, c.f. C. Simpson's talk (and Kontsevich–Soibelman).

$P = W$ in lowest weight

Theorem (Sz. 2021, arXiv:2103.00932v2)

Let $n = 4$ and $\mathcal{M}_{\text{Dol}}$ and $\mathcal{M}_{\text{B}}$ denote the Dolbeault moduli space and character variety of $\mathbb{C}P^1$ with 5 logarithmic points in rank 2. Then, for every $0 \leq k \leq 4$ the regular singular Riemann–Hilbert correspondence and the non-abelian Hodge correspondence induce an isomorphism

$$\mathrm{Gr}_P^{-k-2} H^k(\mathcal{M}_{\text{Dol}}, \mathbb{Q}) \cong \mathrm{Gr}^W_{2k} H^k(\mathcal{M}_{\text{B}}, \mathbb{Q}).$$

Here, $-k-2$ is the lowest possibly non-trivial weight of P and $2k$ is the highest possibly non-trivial weight of W on H^k.

Idea of proof

Same approach as in the Painlevé case, up to

- complementing T. Mochizuki's results by those of R. Mazzeo and co-authors,
- replacing Fricke–Klein co-ordinates by C. Simpson's Fenchel–Nielsen type co-ordinates.

In this case, A. Komyo (2015) has described another compactification. It would be interesting to understand the map at infinity using his compactification too.

Hitchin base and map

Let $(\mathcal{E}, \theta)$ be a strongly parabolic Higgs bundle of rank 2 with 5 logarithmic points $t_0, \ldots, t_4$, and set $D = t_0 + \cdots + t_4$.
We then have $\mathrm{tr}(\theta) \equiv 0$. Hitchin base:

$$\mathcal{B} = \{q : \quad q(t_j) = 0 \text{ for all } 0 \leq j \leq 4\} \subset H^0(\mathbb{C}P^1, K^{\otimes 2}(2D)) \cong \mathbb{C}^7,$$

so $\dim_{\mathbb{C}}(\mathcal{B}) = 2$. Hitchin map:

$$\begin{aligned} H \colon \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha}) &\to \mathcal{B} \\ (\mathcal{E}, \theta) &\mapsto -\det(\theta). \end{aligned}$$

Spectral curve

For $q \in S^3_1 \subset \mathcal{B}$ we write $\zeta_\pm(Rq, z)$ for the roots of

$$\zeta^2 - Rq = 0,$$

specifically

$$\zeta_\pm(Rq, z) = \pm\sqrt{Rq(z,1)}.$$

Denote the corresponding meromorphic 1-forms by

$$Z_\pm(Rq, z) = \pm\sqrt{Rq(z,1)}\frac{\mathrm{d}z}{\prod_{j=0}^4 (z - t_j)}.$$

We denote by

$$X_{Rq} = \{([z:w], \pm\sqrt{Rq(z,w)})\} \to \mathbb{C}P^1$$

the Riemann surface of the bivalued function $\zeta_\pm(Rq, z)$.

Ramification divisor and Hopf fibration

We set

$$\Delta_q = \{z \in \mathbb{C} : \quad q(z) = 0\}$$

for the ramification divisor of X_{Rq} (independent of R). We then have

$$\Delta_q = \{t_0, t_1, t_2, t_3, t_4, t(q)\}$$

for some $t(q) \in \mathbb{C}P^1$. Namely,

$$q(z) = \frac{(az - b)\mathrm{d}z^{\otimes 2}}{\prod_{j=0}^{4}(z - t_j)}$$

for some $(a, b) \in \mathbb{C}^2$ with $at(q) - b = 0$. The map

$$t \colon S^3_1 \to \mathbb{C}P^1$$
$$q \mapsto t(q) = [b : a]$$

is the Hopf fibration.

Model connection

Let $h_{\mathcal{L}_{\mathcal{E}}}$ be the Hermitian–Einstein metric on the spectral sheaf $\mathcal{L}_{\mathcal{E}}$ corresponding to parabolic weights $\pm\frac{1}{2}$ at the points t_j and $t(q)$ respectively.

Let $h_{\mathcal{E},\infty}$ be the push-forward of $h_{\mathcal{L}_{\mathcal{E}}}$ on $\mathcal{E}$ over $\mathbb{C}P^1 \setminus \Delta_q$, and $\nabla_{h_{\mathcal{E},\infty}}$ the corresponding unitary Chern connection.

Introduce

$$\nabla^{\text{model}}_{\sqrt{R}} = \nabla_{h_{\mathcal{E},\infty}} + \begin{pmatrix} 2\Re Z_+(Rq, z) & 0 \\ 0 & 2\Re Z_-(Rq, z) \end{pmatrix}.$$

Connection forms

Over any simply connected subset of $\mathbb{C}P^1 \setminus \Delta_q$, let $p_{q,*}$ stand for the inverse of p_q^* on either branch of X_q. Let

$$B_{\det(\mathcal{E})} \in \Omega^1(\mathbb{C}P^1 \setminus D, \mathbb{C}), \qquad \frac{1}{2} p_q^* B_{\det(\mathcal{E})} + B_{\mathcal{L}_{\mathcal{E}}} \in \Omega^1(X_q \setminus \tilde{\Delta}_q, \mathbb{C})$$

stand for the connection forms of the flat abelian U(1)-connections $\nabla_{h_{\det(\mathcal{E})}}, \nabla_{h_{\mathcal{L}_{\mathcal{E}}}}$ with respect to some smooth unitary frames. The connection form of $\nabla_{h_{\mathcal{E},\infty}}$ then reads as

$$\begin{pmatrix} \frac{1}{2} B_{\det(\mathcal{E})} + p_{q,*} B_{\mathcal{L}_{\mathcal{E}}} & 0 \\ 0 & \frac{1}{2} B_{\det(\mathcal{E})} - p_{q,*} B_{\mathcal{L}_{\mathcal{E}}} \end{pmatrix}.$$

Asymptotic abelianization

Let $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$ denote the Hermite–Einstein metric and integrable connection associated to $(\mathcal{E}, \sqrt{R}\theta)$.

Theorem (T. Mochizuki '16)

Over any simply connected compact set $K \subset \mathbb{C} \setminus \Delta_q$ there exists a gauge transformation $g_{\sqrt{R}}$ such that

$$g_{\sqrt{R}} \cdot \nabla_{\sqrt{R}} - \nabla_{\sqrt{R}}^{\text{model}} \to 0$$

(measured with respect to $h_{\sqrt{R}}$) as $R \to \infty$, uniformly over K.

Fiducial solution, Painlevé 3

R. Mazzeo, J. Swoboda, H. Weiss, F. Witt '16 (near the ramification point $t(q)$), L. Fredrickson, R. Mazzeo, J. Swoboda, H. Weiss '20 (near parabolic points D): local models for the $R \gg 0$ behaviour of $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$, called fiducial solutions.
Near $t(q)$: let $\ell_{\sqrt{R}}$ be the solution of the Painlevé 3-type equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\right)\ell_{\sqrt{R}} = 8R\tilde{r}\sinh(2\ell_{\sqrt{R}})$$

satisfying the boundary behaviours

$$\ell_{\sqrt{R}}(\tilde{r}) \approx -\frac{1}{2}\log(\tilde{r}), \quad \tilde{r} \to 0+$$

$$\ell_{\sqrt{R}}(\tilde{r}) \approx \frac{1}{\pi}K_0\left(\frac{8}{3}\sqrt{R\tilde{r}^3}\right) \approx \frac{\sqrt{3}}{2\pi\sqrt{2}\sqrt[4]{R\tilde{r}^3}}e^{-\frac{8}{3}\sqrt{R\tilde{r}^3}}, \quad \tilde{r} \to \infty,$$

with K_0 the modified Bessel function of order 0.

Fiducial solution, approximate solution

Then, for a co-ordinate $\tilde{z}$ on the disc $|\tilde{z}| < 1$ introduce a unitary connection and Higgs field:

$$A^{\text{fid}}_{\sqrt{R}} = \left(\frac{1}{8} + \frac{1}{4}\tilde{r}\partial_{\tilde{r}}\ell_{\sqrt{R}}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} 2\sqrt{-1}\mathrm{d}\tilde{\varphi}$$

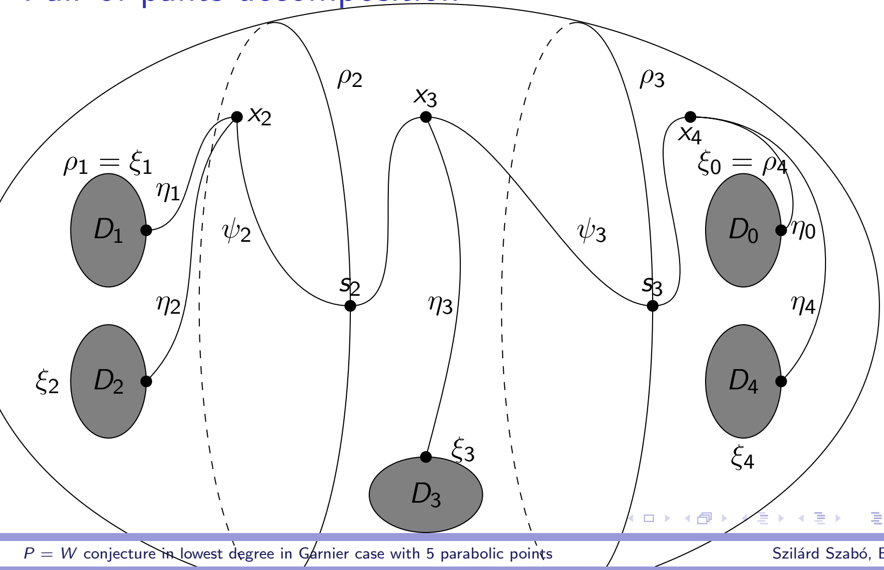
$$\theta^{\text{fid}}_{\sqrt{R}} = \begin{pmatrix} 0 & \tilde{r}^{1/2}e^{\ell_{\sqrt{R}}(\tilde{r})} \\ \tilde{z}\tilde{r}^{-1/2}e^{-\ell_{\sqrt{R}}(\tilde{r})} & 0 \end{pmatrix}\mathrm{d}\tilde{z}.$$

Gluing construction of the fiducial solution and Mochizuki's abelian form $\rightsquigarrow$ approximate solution $h^{\text{appr}}_{\sqrt{R}}$.

Theorem (MSWW '16, FMSW '20)

Assume that all the zeroes of q are simple. Then, there exists a small perturbation of the Hermitian metric $h^{\text{appr}}_{\sqrt{R}}$ that satisfies Hitchin's equation for $(\mathcal{E}, \sqrt{R}\theta)$.

Pair-of-pants decomposition



Fenchel–Nielsen type co-ordinates

Simpson '16: (an open subset of) $\mathcal{M}_B$ carries complex length co-ordinates

$$l_i = \operatorname{tr} \mathrm{RH}(\nabla)[\rho_i] \in \mathbb{C} \quad (i \in \{2, 3\}),$$

and complex twist co-ordinates

$$[p_i : q_i] \in \mathbb{C}P^1 \quad (i \in \{2, 3\}),$$

subject to the condition

$$p_i^2 + l_i p_i q_i + q_i^2 \neq 0.$$

Boundary divisor of character variety

Introduce

$\mathrm{Q} = \{(t, [p : q]) \in (\mathbb{C} \setminus \{\pm 2\}) \times \mathbb{C}P^1$ satisfying $p^2 + tpq + q^2 \neq 0\}$.

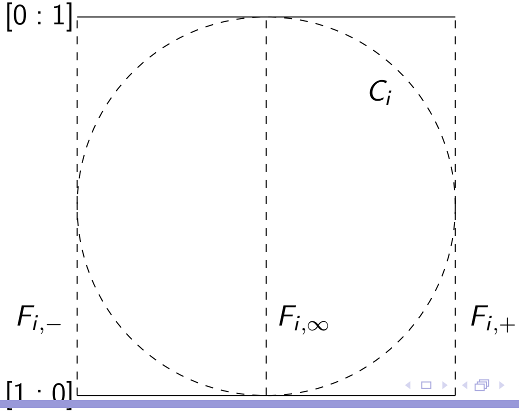
Simpson: homotopy type of the dual boundary complex of $\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})$ agrees with the one of Q^2

$$\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma}) \sim \mathbb{D}\partial\mathrm{Q}^2 \sim \mathbb{D}\partial\mathrm{Q} * \mathbb{D}\partial\mathrm{Q} \sim S^1 * S^1 \sim S^3.$$

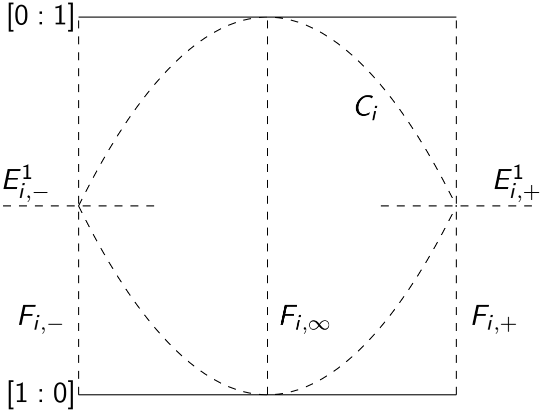
Boundary divisor of Q

Set

$$C_i = (p_i^2 + t_i p_i q_i + q_i^2), \quad F_{i,\pm} = \{t = \pm 2\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1.$$

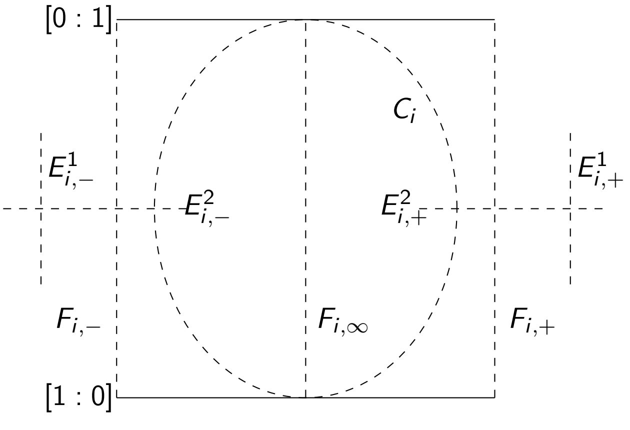


First blow-up



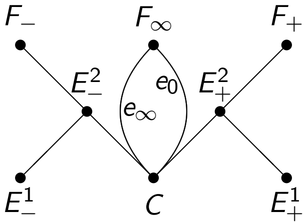
Still not SNC.

Second blow-up



This is SNC.

Dual complex of ∂Q



Parallel transport map

For any loop γ in $\mathbb{C}P^1 \setminus \Delta_q$ let us write

$$\mathrm{RH}(\nabla_{\sqrt{R}})[\gamma] = \begin{pmatrix} a(\gamma, Rq) & b(\gamma, Rq) \\ c(\gamma, Rq) & d(\gamma, Rq) \end{pmatrix}$$

with respect to unit diagonalizing frame of θ. For $0 \leq j \leq 2$ introduce

$$\pi_j(q) = \int_{x_2}^{t_j} Z_+(q, z) \in \mathbb{C},$$

$$\tau_j(q) = \frac{a t_j - b}{\prod_{0 \leq k \leq 4, k \neq j}(t_j - t_k)} \in \mathbb{C}.$$

Asymptotics of parallel transport around parabolic points

Proposition

Fix any $q \in S^3_1$ and consider the loop $\gamma = \xi_j$, a positively oriented circle around p_j, of radius $r_0 > 0$ in the Euclidean metric.

1. The behaviour of the diagonal entries of $\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]$ as $R \to \infty$ is given by the limits

$$a(\xi_j, q, R) \to 0$$
$$d(\xi_j, q, R) \to 0;$$

2. the behaviour of the off-diagonal entries is given by

$$b(\xi_j, q, R)e^{8\Re\sqrt{\tau_j}\sqrt{Rr_0}} \to \sqrt{-1}$$
$$c(\xi_j, q, R)e^{-8\Re\sqrt{\tau_j}\sqrt{Rr_0}} \to \sqrt{-1}.$$

Asymptotics of complex length co-ordinate l_2

Proposition

Fix $q \in S^3_1$ and consider the loop $\gamma = \rho_2$.

1. In case $\Re(\pi_1 - \pi_2) \neq 0$ we have the limit
$$l_2(\mathcal{E}, \sqrt{R}\theta)^{-1} 2\cosh\left(2\int_{\eta_2-\eta_1} B_{\mathcal{L}_{\mathcal{E}}} + 4\sqrt{R}\Re(\pi_2 - \pi_1)\right) \to -1$$
as $R \to \infty$.
2. In case $\Re(\pi_1 - \pi_2) = 0$ the limit of $l_2(\mathcal{E}, \sqrt{R}\theta)$ as $R \to \infty$ exists and is finite.

Asymptotics of complex length co-ordinate l_3

Proposition

Fix $q \in S_1^3$.

1. In case $\Re(\pi_4(q) - \pi_0(q)) \neq 0$ we have the limit

$$l_3(\mathcal{E}, \sqrt{R}\theta)^{-1} 2\cosh\left(2\int_{\eta_4-\eta_0} B_{\mathcal{L}_{\mathcal{E}}} + 4\sqrt{R}\Re(\pi_4 - \pi_0)\right) \to -1$$

   as $R \to \infty$.
2. In case $\Re(\pi_4(q) - \pi_0(q)) = 0$ the limit of $l_3(\mathcal{E}, \sqrt{R}\theta)$ as $R \to \infty$ exists and is finite.

Choice of radii

From now on, we assume that

- the radii of ξ_1, ξ_2 are equal to each other, denoted by $s_2 > 0$,
- the radius of ξ_3 is some independently chosen $s_3 > 0$, and
- the radii of ξ_0, ξ_4 are equal to each other and to another parameter $s_4 > 0$.

Limit of complex twist co-ordinate $[p_2 : q_2]$

Proposition

Fix $q \in S^3_1$ such that $\Re(\pi_2 - \pi_1) \neq 0$. Then, the complex twist co-ordinate $[p_2 : q_2]$ associated to Rq converges to $[0 : 1]$ as $R \to \infty$ if the conditions

$$\int_{\psi_2} \Re Z_+ < 2\Re(2\sqrt{s_2\tau_2} - \sqrt{s_2\tau_1} - \sqrt{s_3\tau_3})$$

$$|\Re(\pi_1 - \pi_2)| = 2\sqrt{s_2}\Re(\sqrt{\tau_1} - \sqrt{\tau_2})$$

$$\int_{\eta_2 - \eta_1} B_{\mathcal{L}_\mathcal{E}} \equiv 0 \pmod{\pi\sqrt{-1}}.$$

hold for one choice of a square root Z_+ of Q. Otherwise, $[p_2 : q_2] \to [1 : 0]$.

Asymptotics of complex twist co-ordinate $[p_2 : q_2]$

Specifically, in the first situation we have

$$\frac{p_2}{q_2} \approx \exp\left(-8\sqrt{R}\Re(2\sqrt{s_2\tau_2} - \sqrt{s_2\tau_1} - \sqrt{s_3\tau_3})\right)$$
$$\cdot \exp\left(2\int_{\psi_2}\left(B_{\mathcal{L}_\mathcal{E}} + 2\sqrt{R}\Re Z_+\right)\right).$$

Proof uses $c_j^\pm = \pm\sqrt{-1}$! Idea:

$$q_2 \approx C_1 e^{c_1\sqrt{R}} + C_2 e^{c_2\sqrt{R}}$$
$$p_2 \approx C_1 e^{c_1\sqrt{R}} + C_2 e^{c_2\sqrt{R}} + C_3 e^{c_3\sqrt{R}} + C_4 e^{c_4\sqrt{R}}$$

with $C_1, C_2, C_3, C_4 \in \mathbb{C}^\times$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Observation: if $c_1 < c_2$ and $c_4 = c_2$ and $C_4 = -C_2$ and $c_3 < c_2$ then

$$\lim_{R\to\infty} \frac{p_2}{q_2} = 0.$$

Geometry of period integrals

Define the open subset

$$U_2(s_2) \subset S_1^3$$

by the conditions

$$0 \neq \pi_1(q) - \pi_2(q) \neq \pm 2\sqrt{s_2}(\sqrt{\tau_1}(q) - \sqrt{\tau_2}(q)).$$

For every $q \in U_2$ there exists a unique $\varphi^* \in [0, 2\pi)$ such that

$$\Re(\pi_1(e^{\sqrt{-1}\varphi^*}q) - \pi_2(e^{\sqrt{-1}\varphi^*}q)) = 2\sqrt{s_2}\Re(\sqrt{\tau_1}(e^{\sqrt{-1}\varphi^*}q) - \sqrt{\tau_2}(e^{\sqrt{-1}\varphi^*}q)$$

This provides a smooth section of the Hopf fibration

$$S_2 \colon t(U_2) \to S_1^3$$
$$[a : b] \mapsto e^{\sqrt{-1}\varphi^*(q)}q$$

Finding small $[p_2 : q_2]$

We make the choices

$$t_0 = -\frac{1}{k}, \quad t_1 = 0, \quad t_2 = 1, \quad t_3 = -1, \quad t_4 = \frac{1}{k}$$

for some $0 < k < 1$.

Proposition

Let $q = S_2(t_1)$. Then q belongs to $U_2(s_2)$ for every $s_2 > 0$, and we have $\Re(\pi_1(q) - \pi_2(q)) \neq 0$. Moreover, there exist distinct points $x_2, x_3 \in \mathbb{C}P^1 \setminus D$ and

$$\rho = \rho(q, t_0, \ldots, t_4, x_2, x_3) > 0$$

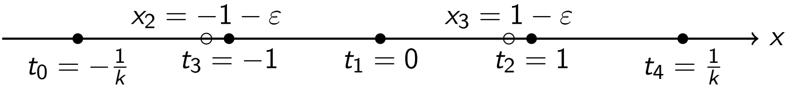
such that for every $0 < s_2, s_3 < \rho$ we have $[p_2 : q_2] \to [0 : 1]$ as $R \to \infty$.

Idea of proof

We find

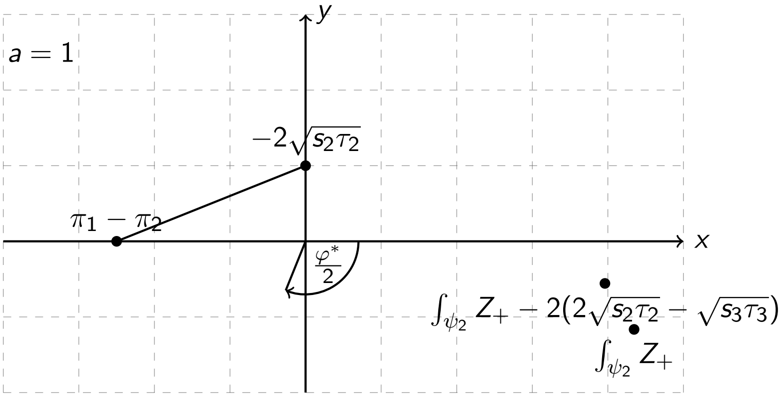
$$Z_+ = \sqrt{a}\frac{\mathrm{d}z}{\sqrt{(z^2-1)\left(z^2-\frac{1}{k^2}\right)}} \qquad \text{with } a = e^{\sqrt{-1}\varphi} \in S^1$$

where φ is the Hopf fiber parameter.
Thus, the periods are (in)complete elliptic integrals of the first kind with eccentricity k.
Choice of base points:

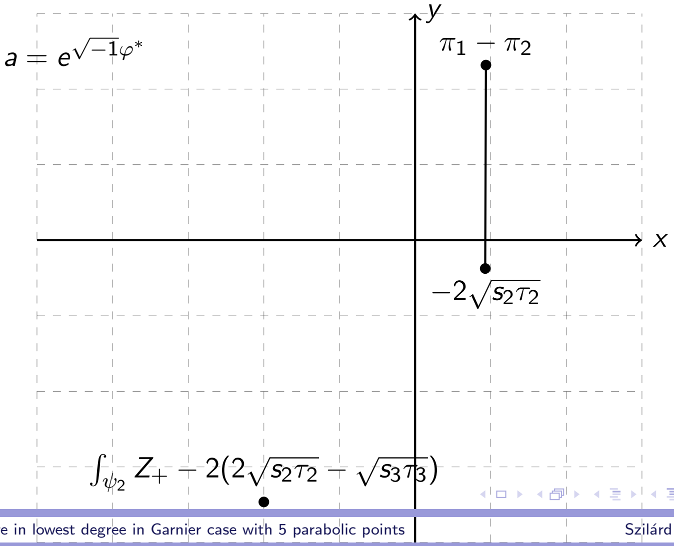


Idea of proof, continued

Rotating triangles. Before:



Idea of proof; after:



Finding small $[p_2 : q_2]$ and $[p_3 : q_3]$ simultaneously

Set $\rho'' = \min(\rho, \rho')$.

Proposition

There exist $0 < s_2, s_3, s_4 < \rho''$ and

$$x_4 \in \mathbb{C}P^1 \setminus (D \cup \{x_2, x_3\})$$

such that $S_2(t_1) = S_3(t_1)$. For the choice $q^* = S_2(t_1)$, we have $[p_2 : q_2] \to [0 : 1]$ and $[p_3 : q_3] \to [0 : 1]$ as $R \to \infty$.

Choice of base point

We choose $x_4 > \frac{1}{k}$ so that

$$\left| \int_{\frac{1}{k}}^{x_4} \frac{\mathrm{d}z}{\sqrt{(z^2-1)\left(z^2 - \frac{1}{k^2}\right)}} \right| = 2 \int_{1-\varepsilon}^{1} \frac{\mathrm{d}z}{\sqrt{(z^2-1)\left(z^2 - \frac{1}{k^2}\right)}}.$$

Such a choice is clearly possible for fixed $\varepsilon \ll 1$, and

$$x_4 \to \frac{1}{k} = t_4 \qquad \text{as} \quad \varepsilon \to 0.$$

Proof of $P = W$ in lowest degree, limit point

Let D_1, D_2, D_3, D_4 stand for the divisor components defined in order by

$$l_2 = \infty, \quad [p_2 \colon q_2] = [0 \colon 1], \quad l_3 = \infty, \quad [p_3 \colon q_3] = [0 \colon 1].$$

We consider the point

$$Q^* = D_1 \cap D_2 \cap D_3 \cap D_4 \in \tilde{D}^4 \subset \overline{\mathcal{M}}_{\mathrm{B}}.$$

The Hitchin fiber $H^{-1}(Rq^*)$ converges to Q^* as $R \to \infty$.

Proof of $P = W$ in lowest degree, phase factors

The phase factors of the Fenchel–Nielsen co-ordinates near Q^* of $\mathrm{RH} \circ \psi(\mathcal{E}, \theta)$ defining D_1, D_2, D_3, D_4 are in this order:

$$-\exp\left(2\left|\int_{t_1}^{t_2} B_{\mathcal{L}(\mathcal{E},\theta)}\right|\right),$$

$$\exp\left(-2\int_{t_3}^{t_2} B_{\mathcal{L}(\mathcal{E},\theta)}\right),$$

$$-\exp\left(2\left|\int_{t_0}^{t_4} B_{\mathcal{L}(\mathcal{E},\theta)}\right|\right),$$

$$\exp\left(-2\int_{t_2}^{t_4} B_{\mathcal{L}(\mathcal{E},\theta)}\right),$$

over the contours $\eta_2 - \eta_1, \psi_2, \eta_4 - \eta_0, \psi_3$.

Proof of $P = W$ in lowest degree, phase factors, bis

These can be recast as

$$
\begin{aligned}
-\exp\left(\int_{A_1} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right),\\
\exp\left(\int_{B_1} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right),\\
-\exp\left(\int_{A_2} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right),\\
\exp\left(\int_{B_2} B_{\mathcal{L}_{(\mathcal{E},\theta)}}\right).
\end{aligned}
$$

where A_1, A_2, B_1, B_2 is a basis of $H_1(X_{q^*}, \mathbb{Z})$.

End of proof of $P = W$ in lowest degree

This shows that $\mathrm{RH} \circ \psi(H^{-1}(Rq^*))$ is homotopic to a torus T^4 generating $H_4(U(Q^*), \mathbb{Z})$.

For every $0 \leq k \leq 4$ and any subset $I \subset \{1, 2, 3, 4\}$ with $|I| = 4 - k$ one may define a k-dimensional subtorus T_I^k in $H^{-1}(Y_{-2})$ by fixing the phases corresponding to the divisor components D_i with $i \in I$.

Moreover, for any I the subtorus T_I^k is mapped by $\mathrm{RH} \circ \mathrm{NAHT}$ to a normal torus at the generic point of the intersection

$$\bigcap_{j \in \{1,2,3,4\} \setminus I} D_j$$

of the remaining k divisor components.

Comparing with the representatives of classes compatible P and W this gives the result.

Further directions

- Generalization to higher number of parabolic points.
- Generalization to higher genus, without parabolic points.
- Generalization to higher rank.
- Relationship to DT-invariants.
- ...