Deformations of logarithmic connections and apparent singularities

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Outline

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Notations and introductory remarks

- $\mathbf{k} = \mathbf{C}$: the base field
- $\mathbf{A}^1$: the affine line
- z: the standard coordinate on $\mathbf{A}^1$
- $\mathbf{P}^1$: the projective line
- X: a smooth projective curve of genus $g(X)$
- $\mathcal{O}_X$: the structure sheaf of X
- $K_X, \boldsymbol{\Omega}^1_X$: the canonical sheaf / the sheaf of holomorphic 1-forms of X

Fuchsian equations and their exponents

Let $X = \mathbf{P}^1$, $P = \{p_1, \dots, p_n \in \mathbf{A}^1, p_0 = \infty\}$ be distinct marked points ($n \geq 2$).
Fix $m \geq 2$, and in all p_j a non-resonant system of constants $\mu^j_1, \dots, \mu^j_m$:

$$\mu^j_k - \mu^j_l \notin \mathbf{Z} \quad (\text{for } k \neq l).$$

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$$\mu^j_k - \mu^j_l \notin \mathbf{Z} \quad \text{(for } k \neq l\text{)}.$$

Assume they satisfy Fuchs' relation

$$\sum_{j=0}^{n} \sum_{k=1}^{m} \mu^j_k = \frac{(n-1)m(m-1)}{2}.$$

Fuchsian equations, cont'd

We consider ordinary differential equations for $w = w(z)$

$$\frac{\mathrm{d}^m w}{\mathrm{d}z^m} - R_1(z)\frac{\mathrm{d}^{m-1} w}{\mathrm{d}z^{m-1}} - \cdots - R_m(z)w = 0,$$

where $R_k(z) = \frac{P_k(z)}{Q_k(z)}$ are rational functions.
Such an equation is called a Fuchsian differential equation with exponents $\{\mu_k^j\}$ if the coefficients R_k satisfy:

1. Q_k has a zero of order at most k at any p_j and no other zeros (i.e. $Q_k = \prod_{j=1}^n (z - p_j)^k$);
2. $\deg(P_k) \leq k(n-1)$;
3. $\mu_1^j, \ldots, \mu_m^j$ are the roots of the indicial polynomial

$$\rho(\rho-1)\cdots(\rho-m+1)-\mathrm{res}_{z=p_j} R_1(z)\rho(\rho-1)\cdots(\rho-m+2)+\cdots$$

A question of N. Katz

Let

1. $\mathcal{E}$ be the affine space of Fuchsian differential equations with singularities at the points P, with exponents $\{\mu_k^j\}$;
2. $\mathcal{M}$ be the moduli space of stable logarithmic connections (E, ∇) on $\mathbf{P}^1$ with singularities in P, with $\text{res}_{z=p_j} \nabla$ conjugate to $\text{diag}(\mu_1^j, \ldots, \mu_m^j)$.

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$$\dim(\mathcal{M}) = 2 - 2m^2 + m(m-1)(n+1) = 2\dim(\mathcal{E}).$$

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Question (N. Katz, 1996)

Does there exist a weight 1 Hodge structure on $T\mathcal{M}$ whose $(1,0)$-part is $T\mathcal{E}$?

Hitchin's Teichmüller component

Let $g(X) \geq 2$, and M be the moduli space of stable $\mathrm{PSl}_m(\mathbf{C})$ Higgs bundles on X, with its Dolbeault symplectic structure ω_{Dol}. Consider the Hitchin map

$$p : M \to \bigoplus_{k=2}^{n} H^0(X, K_X^k)$$

which defines a completely integrable system. In particular, the fibers are Lagrangian tori, and the base is of dimension $\dim(M)/2$.

Hitchin's Teichmüller component, cont'd

Then, p admits a section s as follows: given $\alpha_k \in H^0(X, K_X^k)$, set

$$V = K_X^{-(m-1)/2} \oplus \cdots \oplus K_X^{(m-1)/2}$$
$$\theta : V \to V \otimes K_X$$
$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ \alpha_m & \alpha_{m-1} & \cdots & \alpha_2 & 0 \end{pmatrix}$$

In particular, the dimension of the family of Higgs bundles on the fixed vector bundle V is $\dim(M)/2$.

The basic set-up

Let $X = \mathbf{P}^1$, $p_0 = \infty, p_1, \ldots, p_n \in \mathbf{A}^1$, and for all $j \in \{0, \ldots, n\}$ fix a regular adjoint orbit $\mathcal{C}_j \subset \mathrm{Gl}_m(\mathbf{C})$. Denote by $\{\mu^j_1, \ldots, \mu^j_m\}$ the eigenvalues of $\mathcal{C}_j$ repeated according to their multiplicity, and assume that non-resonance and Fuchs' relation hold.
Let

$$\mathcal{M} = \mathcal{M}_{\mathrm{dR}}(P; \mathcal{C}_0, \ldots, \mathcal{C}_n)$$

stand for the moduli space of stable meromorphic connections (E, ∇) on $\mathbf{P}^1$ with logarithmic singularities in P, such that $\mathrm{res}_{p_j} \nabla \in \mathcal{C}_j$. Denote by ω_{dR} the natural de Rham symplectic structure on $\mathcal{M}$.

The frame

Set

$$\psi(z) = \prod_{j=1}^{n} (z - p_j),$$

and consider a Fuchsian equation with exponents $\{\mu_k^j\}$

$$\mathcal{L}(w) = \frac{\mathrm{d}^m w}{\mathrm{d}z^m} - \frac{P_1(z)}{\psi(z)} \frac{\mathrm{d}^{m-1} w}{\mathrm{d}z^{m-1}} - \cdots - \frac{P_m(z)}{\psi^m(z)} w = 0.$$

Introduce a new frame on the affine part $\mathbf{A}^1$

$$\begin{aligned} w_1 &= w \\ w_2 &= \psi w' \\ &\vdots \\ w_m &= \psi^{m-1} w^{(m-1)} \end{aligned}$$

The extension

$\mathcal{L}$ is then equivalent to the connection

$$\nabla_{\mathcal{L}} = \mathrm{d} - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \psi' & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & (m-2)\psi' & 1 \\ P_m & P_{m-1} & \cdots & P_2 & P_1 + (m-1)\psi' \end{pmatrix} \frac{\mathrm{d}z}{\psi}$$

in the frame $(w_1, \ldots, w_m)$. A similar construction at ∞ extends $\nabla_{\mathcal{L}}$ as a logarithmic connection on the vector bundle

$$V = \mathcal{O} \oplus K_{\mathbf{P}^1}(P) \oplus \cdots \oplus K_{\mathbf{P}^1}^{m-1}((m-1)P).$$

The embedding

This yields and embedding

$$\mathcal{E} \to \mathcal{M}$$
$$\mathcal{L} \mapsto (V, \nabla_{\mathcal{L}}).$$

We will use this to think of $\mathcal{E}$ as an algebraic subvariety of $\mathcal{M}$.

Endomorphism sheaves

Denote by $\mathcal{E}nd(E)$ the sheaf of holomorphic endomorphisms of E, and by $\mathcal{E}nd_{\text{iso}}(E)$ the sheaf of locally isomonodromic endomorphisms:

$$\mathcal{E}nd_{\text{iso}}(E)(U) = \{\varphi \in \mathcal{E}nd(E)(U) : \quad \varphi(p_j) \in \text{im}(\text{ad}_{\text{res}_{p_j}} \nabla)\}$$

for an open set U containing p_j as only marked point.
Clearly, one has an exact sequence

$$0 \to \mathcal{E}nd_{\text{iso}}(E) \to \mathcal{E}nd(E) \to \text{coker}(\text{ad}_{\text{res}_P} \nabla) \to 0,$$

where $\text{coker}(\text{ad}_{\text{res}_P} \nabla)$ stands for the sky-craper sheaf with stalk at p_j equal to the vector-space $\text{coker}(\text{ad}_{\text{res}_{p_j}} \nabla)$.
We say that $\mathcal{E}nd_{\text{iso}}(E)$ is the negative Hecke-modification of $\mathcal{E}nd(E)$ along $\text{coker}(\text{ad}_{\text{res}_P} \nabla)$.

The deformation complex

The infinitesimal automorphisms, deformations and the obstruction to the smoothness of $\mathcal{M}$ in its point (E, ∇) are then given by the hypercohomology groups $\mathbf{H}^d$ of degrees $d = 0$, $d = 1$ and $d = 2$ respectively of the complex

$$\mathcal{E}nd(E) \xrightarrow{\nabla} \mathcal{E}nd_{\text{iso}}(E) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}(P), \qquad (\mathcal{D})$$

with non-zero terms lying in degrees 0 and 1 (Biquard 1997).

Some exact sequences

The hypercohomology long exact sequence of $\mathcal{D}$ reads

$$0 \to \mathbf{H}^0(\mathcal{D}) \to H^0(\mathcal{E}nd(E)) \xrightarrow{H^0(\nabla)} H^0(\mathcal{E}nd_{iso}(E) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}(P)) \to$$
$$\to \mathbf{H}^1(\mathcal{D}) \to H^1(\mathcal{E}nd(E)) \xrightarrow{H^1(\nabla)} H^1(\mathcal{E}nd_{iso}(E) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}(P)) \to$$
$$\to \mathbf{H}^2(\mathcal{D}) \to 0.$$

Introducing

$$C = \operatorname{coker}(H^0(\nabla))$$
$$K = \ker(H^1(\nabla)),$$

we obtain

$$0 \to C \to \mathbf{H}^1(\mathcal{D}) \to K \to 0.$$

Duality

Claim

The vector spaces C and K are naturally dual to each other.

Proof.

The dual $\mathcal{D}^\vee$ of $\mathcal{D}$ fits into the exact sequence of complexes

$$0 \to \mathcal{D}^\vee[-1] \to \mathcal{D} \to [\mathrm{coim}(\mathrm{ad}_{\mathrm{res}_P \nabla}) \xrightarrow{\mathrm{ad}_{\mathrm{res}_P \nabla}} \mathrm{im}(\mathrm{ad}_{\mathrm{res}_P \nabla})] \to 0,$$

where the two non-zero terms in the last complex lie in degrees 0 and 1 (i.e. $\mathcal{D}^\vee[-1]$ is a negative Hecke-modification of $\mathcal{D}$). Notice that $\mathrm{ad}_{\mathrm{res}_P \nabla}$ is an isomorphism from its coimage onto its image; apply Serre-duality. □

A remark on parabolic structures

Suppose (E, ∇) is furthermore endowed with a non-trivial (quasi-)parabolic structure. Denote by

- $\mathcal{E}nd_{\text{par}}$ the sheaf of endomorphisms compatible with the parabolic structure at the marked points (parabolic endomorphisms);
- $\mathcal{E}nd_{\text{par iso}}$ the sheaf of locally isomonodromic parabolic endomorphisms.

Then the complex governing the deformations of the parabolic integrable connection is

$$\mathcal{E}nd_{\text{par}}(E) \xrightarrow{\nabla} \mathcal{E}nd_{\text{par iso}}(E) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}(P) \qquad (\mathcal{D}_{\text{par}})$$

Parabolic structures, cont'd

Denote by $\mathfrak{p}_j \subset \mathfrak{gl}(E|_{p_j})$ the parabolic subalgebra, and by $\mathfrak{p}$ the sky-craper sheaf whose stalk at p_j is $\mathfrak{p}_j$. We have exact sequences

$$0 \to \mathcal{E}nd(-P) \to \mathcal{E}nd_{\mathrm{par}} \to \mathfrak{p} \to 0$$

and

$$0 \to \mathcal{E}nd_{\mathrm{iso}}(-P) \to \mathcal{E}nd_{\mathrm{par\,iso}} \to \mathfrak{p} \to 0.$$

Since $\mathrm{res}_{p_j}(\nabla)$ preserves $\mathfrak{p}_j$, it follows as before that the analogous short exact sequence

$$0 \to C_{\mathrm{par}} \to \mathbf{H}^1(\mathcal{D}_{\mathrm{par}}) \to K_{\mathrm{par}} \to 0$$

is naturally isomorphic to the non-parabolic one. In particular, the spaces C_{par} and K_{par} are naturally dual to each other.

For any $\mathcal{L} \in \mathcal{E} \subset \mathcal{M}$, consider the short exact sequence

$$0 \to C \to T_{\mathcal{L}}\mathcal{M} \to K \to 0.$$

Theorem

We have $T_{\mathcal{L}}\mathcal{E} = C$. In particular, $\mathcal{E}$ is Lagrangian for ω_{dR}.

Proof.

Inclusion $T_{\mathcal{L}}\mathcal{E} \subseteq C$: the map $T_{\mathcal{L}}\mathcal{M} \to K$ is restriction of an infinitesimal modification of (E, ∇) to the infinitesimal modification of the underlying holomorphic vector bundle E. Since for deformations in $\mathcal{E}$ we always have $E = V$, this map restricted to $T_{\mathcal{L}}\mathcal{E}$ is 0. $\square$

Apparent singularities

Let $\mathcal{L}$ be an equation with rational coefficients. A singular point p is called an apparent singularity if there is a basis of regular solutions of $\mathcal{L}$ in a small neighborhood of p.

Example

For any $k \in \mathbf{N}_+$, the equation

$$w' - \frac{k}{z} w = 0$$

has an apparent singularity at 0: the solution $w(z) = z^k$ is regular.

Fact

At any apparent singularity p of $\mathcal{L}$, we have $P_1(p) \in \mathbf{N}_+$.

Cyclic vectors

Let $(E, \nabla) \in \mathcal{M}$ be arbitrary and $U \subset \mathbf{A}^1$ be open. A cyclic vector for (E, ∇) on U is a section $v \in H^0(U, E)$ such that

$$v, \nabla_{\partial_z} v, \ldots, \nabla_{\partial_z}^{m-1} v$$

generate E on U (over $\mathcal{O}(U)$).
If a cyclic vector v for (E, ∇) on U exists, then ∇ on U can be written as $\nabla_{\mathcal{L}_v}$ for some equation $\mathcal{L}_v$ with analytic coefficients.

Theorem (N. Katz, 1987)

For any $(E, \nabla) \in \mathcal{M}$, there exists a finite set $S \subset \mathbf{A}^1$ such that on $\mathbf{A}^1 \setminus S$ the connection ∇ admits a cyclic vector.

Properties of the singular locus

In general, we have

$$S = P \cup A,$$

where A is the set of apparent singularities of $\mathcal{L}_v$.

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If moreover $(E(0), \nabla(0)) \in \mathcal{E}$, then for $v = w$ we have $A(0) \subset P$.
Therefore, the points $a(t) \in A(t)$ "split off" analyitically from the set P.

Proof of the inverse inclusion

Inclusion $C \subseteq T_{\mathcal{L}}\mathcal{E}$: assume $(E(t), \nabla(t))$ is tangent to some

$$V \in \ker(T_{(E(0),\nabla(0))}\mathcal{M} \to K).$$

Let $a(t) \in A(t)$ be the position of an apparent singularity of $\mathcal{L}_{v(t)}$. Let $v(t,z)$ denote a cyclic vector for $\nabla(t)$, analytic in t, and such that $v(0,z) = w(z)$. Introduce $\psi(t,z) = (z - a(t))\psi(z)$ and write $\mathcal{L}_t$ locally as

$$\frac{\mathrm{d}^m v(t,z)}{\mathrm{d}z^m} - \frac{P_1(t,z)}{\psi(t,z)}\frac{\mathrm{d}^{m-1} v(t,z)}{\mathrm{d}z^{m-1}} - \cdots - \frac{P_m(t,z)}{\psi^m(t,z)} v(t,z) = 0,$$

with $P_k(t,z)$ analytic in t, z.
Notice that $P_1(t, a(t)) \in \mathbf{N}_+$ is analytic in $t \Rightarrow$ constant.
Furthermore, $P_1(0,z) = (z-p)P_1(z)$. But then $P_1(0,p) = 0$, a contradiction. So $a(t) = p$, and for all t the connection $\nabla(t)$ comes from a Fuchsian equation.

The number of apparent singularities

Let N denote the smallest number such that every $(E, \nabla) \in \mathcal{M}$ can be written as a Fuchsian equation with at most N apparent singularities.

Theorem (M. Ohtsuki, 1982)

We have $N \leq \dim(\mathcal{E})$.

Furthermore, equality was conjectured.

Theorem

We have $N \geq \dim(\mathcal{E})$.

Estimation of N

For any $A = \{a_1, \ldots, a_N\}$ denote by $\mathcal{M}_{a_1,\ldots,a_N}$ the subvariety defined by connections that can be represented by a Fuchsian equation with apparent singularities in A.
It is sufficient to show that

$$\dim(\mathcal{M}_{a_1,\ldots,a_N}) = \dim(\mathcal{E}).$$

The formal dimension count

van der Put and Singer: there is a total of

$$m + \frac{m(m+1)(n+N-1)}{2}$$

parameters. Conditions on them:

- at real singularities: local exponents yield $m(n+1)$ constraints, redundant by the residue theorem, so there remain $m(n+1)-1$ constraints;
- at apparent singularities: local exponents yield Nm constraints, plus $N\frac{m(m-1)}{2}$ additional constraints, so a total of $N\frac{m(m+1)}{2}$ constraints.

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Need to check: these conditions are independent.

A generalised van der Monde determinant

Lemma

For any $r \geq 0$ and $b_1, \dots, b_{r+2} \in \mathbf{C}$, the determinant of

$$\begin{pmatrix} 1 & b_1 & b_1^2 & \cdots & b_1^{2r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & b_{r+2} & b_{r+2}^2 & \cdots & b_{r+2}^{2r+1} \\ 0 & 1 & 2b_1 & \cdots & (2r+1)b_1^{2r} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2b_r & \cdots & (2r+1)b_r^{2r} \end{pmatrix}$$

is (up to a sign) equal to

$$(b_{r+1} - b_{r+2}) \prod_{1 \leq i \leq r, r+1 \leq j \leq r+2} (b_i - b_j)^2 \prod_{1 \leq i < j \leq r} (b_i - b_j)^4.$$

A generalisation of the main Theorem

For any $(E, \nabla) \in \mathcal{M}_{a_1,\dots,a_N}$ write

$$0 \to C \to T_{(E,\nabla)}\mathcal{M} \to K \to 0.$$

Corollary

We have

$$T_{(E,\nabla)}\mathcal{M}_{a_1,\dots,a_N} = C.$$

In particular, $\mathcal{M}_{a_1,\dots,a_N}$ *is Lagrangian with respect to* ω_{dR}*.*

Dimensions agree $\Rightarrow$ sufficient to check $\subseteq$.

Proof of the inculsion

Two cases to check:

1. An apparent singularity a of weight > 1 splitting into two (or more) apparent singularities: similar to the previous argument;
2. An apparent singularity $a(t)$ depending non-trivially with t: the cyclic trivialisations at $t = 0$ and at arbitrary t are related by the gauge transformation

$$\operatorname{diag}\left(1, \frac{z-a(t)}{z-a(0)}, \ldots, \frac{(z-a(t))^{m-1}}{(z-a(0))^{m-1}}\right),$$

whose derivative with respect to t is

$$\operatorname{diag}\left(1, \frac{-a'(t)}{z-a(0)}, \ldots\right)$$

which is not holomorphic at $z = a(0)$ unless $a'(t) = 0$.

The connection on an affine chart

Studied by Jimbo-Miwa, Arinkin-Lysenko, Inaba-Iwasaki-Saito, ...
Let $m = 2$, $n = 3$. Fix $p_1, p_2, p_3 \in \mathbf{A}^1$ and non-zero semisimple orbits $\mathcal{C}_j$ for all j, such that $\sum \operatorname{tr} \mathcal{C}_j = 0$. Then $\dim(\mathcal{E}) = 1$, so there is a unique apparent singularity.
Let $(E, \nabla) \in \mathcal{M}_{dR}(P; \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. On $\mathbf{A}^1$, in a logarithmic trivialisation (e_1, e_2) one can write ∇ as

$$\nabla = \mathrm{d} - \sum_{j=1}^{3} \frac{A^j}{z - p_j} \mathrm{d}z$$

for some matrices $A^j \in \mathcal{C}_j$

$$A^j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$

The locus of the apparent singularity

Apply a constant gauge transformation to make $\gamma_1 + \gamma_2 + \gamma_3 = 0$. (This is always possible — choice of an eigenvector of $\mathrm{res}_{p_0}(\nabla)$!) Then we can apply a Hecke-modification at infinity and extend ∇ to a connection on $E = \mathcal{O}_{\mathbf{P}^1}(p_0) \oplus \mathcal{O}_{\mathbf{P}^1}$.

The only global section of $E^\vee$ (up to a constant) is $e_2^\vee$. We have

$$\nabla^\vee(e_2^\vee) = \sum_j \begin{pmatrix} \gamma_j \\ \delta_j \end{pmatrix} \frac{\mathrm{d}z}{z - p_j},$$

hence the Wronski-determinant is

$$-\sum_j \frac{\gamma_j \mathrm{d}z}{z - p_j}.$$

Therefore the equation (in z) of the apparent singularity is

$$z(\gamma_1(p_2+p_3)+\gamma_2(p_3+p_1)+\gamma_3(p_1+p_2)) = \gamma_1 p_2 p_3 + \gamma_2 p_3 p_1 + \gamma_3 p_1 p_2.$$

Splitting off infinity

In case the solution z converges to ∞, in the limit it is possible to apply another Hecke-modification and extend ∇ on the bundle $\mathcal{O}_{\mathbf{P}^1}(p_0) \oplus \mathcal{O}_{\mathbf{P}^1}(-p_0)$. By a result of A. Bolibruch, ∇ is then associated to a Fuchsian equation.

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Furthermore, if we had started with the other eigenvector of $\mathrm{res}_{p_0}(\nabla)$, then we would have arrived at the bundle $\mathcal{O}_{\mathbf{P}^1}(-p_0) \oplus \mathcal{O}_{\mathbf{P}^1}(p_0)$, which again comes from a Fuchsian equation. So, at ∞ we get two copies of $\mathcal{E} \cong \mathbf{A}^1$ included in $\mathcal{M}$.

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A similar construction holds for the other singular points $\Rightarrow$ we get 8 copies of $\mathbf{A}^1$ in $\mathcal{M}$, 2 over each singular point p_j.

The fibration

The moduli space $\mathcal{P}$ of stable quasi-parabolic bundles on $\mathbf{P}^1$ with parabolic points in p_0, p_1, p_2, p_3 is known to be the non-separated scheme $\mathbf{P}^1$ with p_0, p_1, p_2, p_3 doubled.
There is a natural map

$$\mathcal{M} \to \mathcal{P},$$

whose fibers are isomorphic to $\mathbf{A}^1$.
The tangent to this map is

$$T\mathcal{M} \to K.$$

Open questions

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- Link between the minimal number of apparent singularities and the stratification corresponding to the type of the underlying vector bundle?
- Other structure groups?
- etc.