P = W conjecture and its geometric version for Painlevé spaces

Szilárd Szabó

Budapest University of Technology and Rényi Institute of Mathematics
Budapest

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Outline

Hodge theory, Riemann–Hilbert

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Filtrations

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Hodge theory, Riemann–Hilbert

Filtrations

Dolbeault

Outline

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Data of wild non-abelian Hodge theory (NAHT)

Simpson '90, Sabbah '99, Biquard–Boalch '04: fix

- C: smooth projective curve over $\mathbb{C}$
- $r \geqslant 2$ rank (i.e., $G = \mathrm{Gl}_r(\mathbb{C})$)
- $p_1, \ldots, p_n \in C$ irregular singularities (with local charts z_j), and for each p_j:
- a parabolic subalgebra $\mathfrak{p}_j \subset \mathfrak{gl}_r$ with associated Levi $\mathfrak{l}_j$
- parabolic weights $\{\alpha_j^i\}_i$
- an unramified irregular type $Q_j \in \mathfrak{t} \otimes \mathbb{C}((z_j))/\mathbb{C}[[z_j]]$ with centralizer $\mathfrak{h}_j$
- an adjoint orbit $\mathcal{O}_j$ in $\mathfrak{l}_j \cap \mathfrak{h}_j$.

Hitchin's equations and wild NAHT

The space of solutions of Hitchin's equations

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection D on a rank r smooth Hermitian vector bundle (V, h) and a field $\theta : V \to V \otimes \Omega^{1,0}_C$ having prescribed irregular part and residue in $\mathcal{O}_j$ near $p_j \rightsquigarrow$ hyper-Kähler moduli space $\mathcal{M}_{\text{Hod}}$.

de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\mathsf{Hod}}$ have a geometric meaning:

- de Rham: $\mathcal{M}_{\mathsf{dR}}$ parameterising certain poly-stable parabolic connections with irregular singularities
- Dolbeault: $\mathcal{M}_{\mathsf{Dol}}$ parameterising certain poly-stable parabolic Higgs bundles with higher-order poles.

By non-abelian Hodge theory, $\mathcal{M}_{\mathsf{dR}}$ and $\mathcal{M}_{\mathsf{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\mathsf{Hod}}$).

Irregular Riemann–Hilbert correspondence

- Birkhoff, Mebkhout, Kashiwara, Deligne, Malgrange, Jimbo–Miwa–Ueno...: equivalence between the categories of irregular connections and Stokes-filtered local systems.
- Boalch '07: algebraic geometric construction of wild character varieties $\mathcal{M}_{\mathrm{B}}$ parameterising Stokes data.
- Irregular Riemann–Hilbert correspondence (RH): bi-analytic map

$$\mathrm{RH} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}.$$

Conclusion: $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are all diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}$), in particular

$$H^\bullet(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \cong H^\bullet(\mathcal{M}_{\mathrm{B}}, \mathbb{Q}).$$

Painlevé spaces

From now on, we set $C = \mathbb{C}P^1$ and we assume $r = 2$ and $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Hod}} = 4$. There exists a finite list

$$PI, PII, PIII(D6), PIII(D7), PIII(D8), PIV, PV_{\mathrm{deg}}, PV, PVI$$

of irregular types with this property, called Painlevé cases. From now on, we let

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

and we write PX to refer to one of the above Painlevé cases. We therefore have smooth non-compact Kähler surfaces

$$\mathcal{M}_{\mathrm{dR}}^{PX}, \quad \mathcal{M}_{\mathrm{Dol}}^{PX}, \quad \mathcal{M}_{\mathrm{B}}^{PX}$$

diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}^{PX}$) for any fixed X.

Singularity type of Painlevé cases

X	$D = \sum n_i p_i$
VI	$p_1 + p_2 + p_3 + p_4$
V	$2p_1 + p_2 + p_3$
$III(D6) = V_{\text{deg}}$	$2p_1 + 2p_2; \frac{3}{2}p_1 + p_2 + p_3$
$III(D7)$	$\frac{3}{2}p_1 + 2p_2$
$III(D8)$	$\frac{3}{2}p_1 + \frac{3}{2}p_2$
IV	$3p_1 + p_2$
II	$4p_1; \frac{5}{2}p_1 + p_2$
I	$\frac{7}{2}p_1$

Example: nilpotent PVI

- $n = 4$, logarithmic singularities: $0, 1, t, \infty$
- for each $j \in \{0, 1, t, \infty\}$ parabolic algebra $\mathfrak{p}_j = \mathfrak{b}_j$ a Borel, with $\mathfrak{l}_j$ a Cartan,
- generic parabolic weights,
- $Q_j = 0$
- eigenvalues of $\text{res}_{p_j}(\theta)$ in $\mathfrak{l}_j$ equal to 0 (i.e., nilpotent residue).

Example: PIII(D7)

$n = 2$, singularities:

- Poincaré-Katz invariant $\frac{1}{2}$ at $z = 0$, i.e. of the form

$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}z}{z^2} + \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}z}{z} + O(1)\mathrm{d}z$$

  with $b_1 \neq 0$ fixed;

- Poincaré-Katz invariant 1 at $z = \infty$, i.e. of the form

$$\theta = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} dz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \frac{dz}{z} + O(1)\frac{dz}{z^2}$$

  with $a \neq 0$, $b \in \mathbb{C}$ fixed.

Middle perversity t-structure

Given an algebraic variety Y, consider the derived category

$$D^b(Y, \mathbb{Q})$$

of bounded complexes of $\mathbb{Q}$-vector spaces K on Y with constructible cohomology sheaves of finite rank.
Beilinson–Bernstein–Deligne '82: truncation functors

$$^{\mathfrak{p}}\tau_{\leqslant i} : D^b(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leqslant i}(Y, \mathbb{Q})$$

encoding the support condition for the middle perversity function, giving rise to a system of truncations

$$0 \to \cdots \to {}^{\mathfrak{p}}\tau_{\leqslant -p} K \to {}^{\mathfrak{p}}\tau_{\leqslant -p+1} K \to \cdots \to K$$

Perverse filtration on Dolbeault spaces

Hitchin '87: for $\mathcal{M}_{\mathrm{Dol}}$ a Dolbeault moduli space there exists a surjective map

$$h : \mathcal{M}_{\mathrm{Dol}} \to Y = \mathbb{C}^N.$$

Consider

$$K = \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}} \in D^b(Y, \mathbb{Q}).$$

The perverse filtration P on

$$\mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) \cong H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

is defined as

$$P^p \mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = \mathrm{Im}(\mathbf{H}^*(Y, {}^{\mathfrak{p}}\tau_{\leqslant -p} \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathbf{H}^*(Y, \mathbf{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}})).$$

Perverse Leray spectral sequence and perverse polynomial

There exists a spectral sequence

$$ {}^{\mathfrak{p}}_{L}E_r^{k,l} = {}^{\mathfrak{p}}H^k(Y, {}^{\mathfrak{p}}R^l h_* \underline{\mathbb{Q}}_{\mathcal{M}}) \Rightarrow H^{k+l}(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) $$

degenerating at page 2. With this, we have

$$ P^p \mathbf{H}^*(Y, \mathbb{R}h_* \underline{\mathbb{Q}}_{\mathcal{M}}) = \oplus_{l \leqslant p} {}^{\mathfrak{p}}_{L}E_2^{k,l}. $$

We define the perverse Hodge polynomial of $\mathcal{M}_{\mathrm{Dol}}$ by

$$ PH(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) q^i t^k. $$

Weight filtration on Betti spaces

As $\mathcal{M}_{\mathrm{B}}$ is an affine algebraic variety, Deligne's Hodge II. ('71) shows that $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ carries a weight filtration W. We derive a polynomial

$$WH(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}^W_{2i} H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{C}) q^i t^k.$$

Hausel–Rodriguez-Villegas '08: WH is indeed a polynomial in q, t.

$P = W$ conjecture

Theorem (de Cataldo–Hausel–Migliorini '12)

If C is compact and $r = 2$, then for the Dolbeault and Betti spaces corresponding to each other under non-abelian Hodge theory and the Riemann–Hilbert correspondence, the filtrations P and W get mapped into each other. In particular, we have

$$PH(q,t) = WH(q,t).$$

Conjecture (de Cataldo–Hausel–Migliorini '12)

The same assertion holds for any rank r.

Numerical $P = W$ in the Painlevé cases

Let us set

$$PH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) q^i t^k,$$

$$WH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}_{2i}^W H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) q^i t^k.$$

Theorem (Sz '18)

For each

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

we have $PH^{PX}(q,t) = WH^{PX}(q,t)$.

Simpson's Geometric $P = W$ conjecture

We assume $X = VI$, with nilpotent residue condition on θ. Let $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PVI}$ be a smooth compactification of $\mathcal{M}_{\mathrm{B}}^{PVI}$ by a simple normal crossing divisor D and denote by $\mathcal{N}^{PVI}$ the nerve complex of D.

Theorem (Sz '19)

For some sufficiently large compact set $K \subset \mathcal{M}_{\mathrm{B}}^{PVI}$ there exists a homotopy commutative square

$$
\begin{array}{ccc}
\mathcal{M}_{\mathrm{Dol}}^{PVI} \setminus K & \xrightarrow{\ \psi\ } & \mathcal{M}_{\mathrm{B}}^{PVI} \setminus K \\
\Big\downarrow h & & \Big\downarrow \phi \\
D^{\times} & \longrightarrow & |\mathcal{N}^{PVI}|.
\end{array}
$$

Here, $D^{\times} = \mathbb{C} - B_R(0) \subset Y$ and $\psi = RH \circ NAHT$.

Geometric $P = W$ conjecture implies $P = W$ in Painlevé cases

In 2015, L. Katzarkov, A. Noll, P. Pandit and C. Simpson conjectured in higher generality a similar homotopy commutativity property. In 2015, A. Komyo proved that for a complex 4-dimensional moduli space of logarithmic Higgs bundles over $\mathbb{C}P^1$, the body of $\mathcal{N}^{PX}$ is homotopy equivalent to S^3 (and that for two complex 2-dimensional moduli spaces it is S^1). Still in 2015, C. Simpson generalized the homotopy equivalence assertion to logarithmic Higgs bundles of rank 2 over $\mathbb{C}P^1$, and called the homotopy commutativity assertion "Geometric P = W conjecture".

Fact

For all Painlevé cases, the Geometric $P = W$ conjecture implies the (highest graded part of) $P = W$ conjecture.

Hitchin fibration

Irregular Hitchin map

$$h : \mathcal{M}^{PX}_{\mathrm{Dol}} \to Y = \mathbb{C}.$$

Theorem (Ivanics–Stipsicz–Szabó '17)

For generic parabolic weights, there exists an embedding

$$\mathcal{M}^{PX}_{\mathrm{Dol}} \hookrightarrow E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$$

and an elliptic fibration

$$\tilde{h} : E(1) \to \mathbb{C}P^1$$

extending h.

Denote by F^{PX}_{∞} the non-reduced curve $E(1) \setminus \mathcal{M}^{PX}_{\mathrm{Dol}} = \tilde{h}^{-1}(\infty)$.

Euler characteristic and perverse polynomial

Proposition

We have

$$\begin{aligned}
\dim_{\mathbb{Q}} \mathrm{Gr}_0^P H^0(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) &= 1 \\
\dim_{\mathbb{Q}} \mathrm{Gr}_1^P H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) &= 10 - \chi(F_\infty^{PX}) \\
\dim_{\mathbb{Q}} \mathrm{Gr}_2^P H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) &= 1.
\end{aligned}$$

In other words, we have

$$PH^{PX}(q,t) = 1 + (10 - \chi(F_\infty^{PX}))qt^2 + q^2t^2.$$

Table of perverse polynomials

X	F_∞^{PX}	$PH^{PX}(q,t)$
VI	$D_4^{(1)}$	$1 + 4qt^2 + q^2t^2$
V	$D_5^{(1)}$	$1 + 3qt^2 + q^2t^2$
V_{deg}	$D_6^{(1)}$	$1 + 2qt^2 + q^2t^2$
$III(D6)$	$D_6^{(1)}$	$1 + 2qt^2 + q^2t^2$
$III(D7)$	$D_7^{(1)}$	$1 + qt^2 + q^2t^2$
$III(D8)$	$D_8^{(1)}$	$1 + q^2t^2$
IV	$E_6^{(1)}$	$1 + 2qt^2 + q^2t^2$
II	$E_7^{(1)}$	$1 + qt^2 + q^2t^2$
I	$E_8^{(1)}$	$1 + q^2t^2$

Idea of proof of Proposition

Analysis of perverse Leray spectral sequence ${}^{\mathfrak{p}}_{L}E_2^{k,l}$ of h:

$k=2$	0	0	0
$k=1$	0	$H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}})$	0
$k=0$	$\mathbb{C}$	$\mathbb{C}^{b_1(\mathcal{M})}$	$\mathbb{C}$
	$l=0$	$l=1$	$l=2$

Standard algebraic topology shows that

- $b_1(\mathcal{M}) = 0$,
- $\dim_{\mathbb{C}} H^1(Y, R^1 h_* \underline{\mathbb{C}}_{\mathcal{M}}) = 10 - \chi(F_\infty^{PX})$,
- for a generic point $Y_{-1} \in Y$, the following map is surjective

$$H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) \to H^2(h^{-1}(Y_{-1}), \mathbb{C}) = \mathbb{C}.$$

End of proof of the Proposition

We get

$$\begin{aligned}
\mathrm{Gr}_2^P\, H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) &\cong \mathrm{Im}(\mathbf{H}^2(Y, \mathbf{R}h_*\underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbf{R}h_*\underline{\mathbb{C}}|_{Y_{-1}})) \\
&\cong \mathbb{C}, \\
\mathrm{Gr}_1^P\, H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) &= \mathrm{Ker}(\mathbf{H}^2(Y, \mathbf{R}h_*\underline{\mathbb{C}}) \to \mathbf{H}^2(Y_{-1}, \mathbf{R}h_*\underline{\mathbb{C}}|_{Y_{-1}})) \\
&\cong \mathbb{C}^{10-\chi(F_\infty^{PX})}.
\end{aligned}$$

Betti spaces and affine cubic surfaces

P. Boalch (2007): General construction of wild character varieties using quasi-Hamiltonian reduction.
Fricke–Klein 1926, ... , van der Put–Saito '09: for each X there exists a quadric

$$Q^{PX} \in \mathbb{C}[x_1, x_2, x_3]$$

such that

$$\mathcal{M}_{\mathrm{B}}^{PX} = (f^{PX}) \subset \mathbb{C}^3$$

where

$$f^{PX}(x_1, x_2, x_3) = x_1 x_2 x_3 + Q^{PX}(x_1, x_2, x_3).$$

Compactifications of Betti spaces

Let

$$F^{PX} \in \mathbb{C}[x_0, x_1, x_2, x_3]_3$$

be the homogenization of f^{PX} and set

$$\overline{\mathcal{M}}_{\mathrm{B}}^{PX} = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]/(F^{PX})).$$

Projective cubic surface (possibly) with singularities at

$$P_1 = [0:1:0:0], \quad P_2 = [0:0:1:0], \quad P_3 = [0:0:0:1].$$

Let

$$\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX} \to \overline{\mathcal{M}}_{\mathrm{B}}^{PX}$$

denote the minimal resolution of singularities.

Total Milnor number and weight polynomial

Define the total Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ as

$$N^{PX} = \sum_{j=1}^{3} \mu(P_j)$$

where $\mu(P_j)$ is the Milnor number of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ at P_j.

Proposition

We have

$$WH^{PX}(q,t) = 1 + (4 - N^{PX})qt^2 + q^2t^2.$$

Table of weight polynomials

X	Singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$	$WH^{PX}(q,t)$
VI	$\emptyset$	$1 + 4qt^2 + q^2t^2$
V	A_1	$1 + 3qt^2 + q^2t^2$
V_{deg}	A_2	$1 + 2qt^2 + q^2t^2$
$III(D6)$	A_2	$1 + 2qt^2 + q^2t^2$
$III(D7)$	A_3	$1 + qt^2 + q^2t^2$
$III(D8)$	A_4	$1 + q^2t^2$
IV	$A_1 + A_1$	$1 + 2qt^2 + q^2t^2$
II	$A_1 + A_1 + A_1$	$1 + qt^2 + q^2t^2$
I	$A_2 + A_1 + A_1$	$1 + q^2t^2$

Compactifying divisors

The divisor at infinity of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$D = L_1 \cup L_2 \cup L_3$$

where L_i are lines pairwise intersecting each other in P_1, P_2, P_3.
The nerve complex of the divisor at infinity of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ is

$$\mathcal{N}^{PX} = A^{(1)}_{N^{PX}+2} = I_{N^{PX}+3}.$$

The first page of the weight spectral sequence

Deligne: spectral sequence ${}_W E_r$ abutting to $H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C})$ with ${}_W E_1^{-n,k+n}$ given by

$k+n=4$	$\oplus_{p \in \mathcal{N}_1^{PX}} H^0(p, \mathbb{C})$	$\oplus_{L \in \mathcal{N}_0^{PX}} H^2(L, \mathbb{C})$	$H^4\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$
$k+n=3$	0	0	0
$k+n=2$	0	$\oplus_{L \in \mathcal{N}_0^{PX}} H^0(L, \mathbb{C})$	$H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$
$k+n=1$	0	0	0
$k+n=0$	0	0	$H^0\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)$
	$-n=-2$	$-n=-1$	$-n=0$

The first differentials of the weight spectral sequence

The only non-trivial differentials d_1 on ${}_W E_1$ are:

$$\bigoplus_{p\in\mathcal{N}_1^{PX}} H^0(p,\mathbb{C}) \xrightarrow{\delta} \bigoplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \xrightarrow{\delta_4} H^4\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right)$$
$$\bigoplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \xrightarrow{\delta_2} H^2\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right).$$

Algebraic topology of cubic surfaces shows that

$$\delta_4 : \bigoplus_{L\in\mathcal{N}_0^{PX}} H^2(L,\mathbb{C}) \twoheadrightarrow H^4\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right)$$
$$\delta_2 : \bigoplus_{L\in\mathcal{N}_0^{PX}} H^0(L,\mathbb{C}) \hookrightarrow H^2\left(\widetilde{\mathcal{M}}_B^{PX},\mathbb{C}\right) \cong \mathbb{C}^7.$$

Dimensions of graded pieces for the weight filtration

We derive

$$
\begin{aligned}
\mathrm{Gr}_0^W H^0(\mathcal{M}_{\mathrm{B}}^{PX}) &\cong \mathbb{C} \\
\mathrm{Gr}_2^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) &= \mathrm{Coker}\left(\delta_2 : \oplus_i H^0(L_i, \mathbb{C}) \to H^2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C}\right)\right) \\
&\cong \mathbb{C}^{4-N^{PX}} \\
\mathrm{Gr}_4^W H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) &= \mathrm{Ker}(\delta) \\
&\cong \mathbb{C}.
\end{aligned}
$$

Hitchin base and standard spectral curve

(Sz. 1906.01856) We have $\mathrm{tr}(\theta) \in H^0(\mathbb{C}P^1, K) = 0$, Hitchin base:

$$H^0(\mathbb{C}P^1, K^2(0 + 1 + t + \infty)) \cong \mathbb{C},$$

spanned by

$$\frac{(\mathrm{d}z)^{\otimes 2}}{z(z-1)(z-t)}.$$

Set $L = K(0 + 1 + t + \infty)$, and take the canonical section

$$\zeta \frac{\mathrm{d}z}{z(z-1)(z-t)}$$

of $p_L^* L$ over $\mathrm{Tot}(L)$. In $\mathrm{Tot}(L)$ we consider the curve

$$\tilde{X}_{1,0} = \{(z, \zeta) : \quad \zeta^2 + z(z-1)(z-t) = 0\}.$$

Rescaling of spectral curve

For $R >> 0, \varphi \in \mathbb{R}/2\pi\mathbb{Z}$ let $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ be a rank 2 logarithmic Higgs bundle over $\mathbb{C}P^1$ with

$$\det(\theta_{R,\varphi}) = -Re^{\sqrt{-1}\varphi} \in H^0(\mathbb{C}P^1, K^2(0 + 1 + t + \infty)).$$

Its spectral curve is

$$\tilde{X}_{R,\varphi} = \left\{ (z, \zeta) : \quad \det \left(\theta_{R,\varphi} - \zeta \frac{\mathrm{d}z}{z(z-1)(z-t)} \right) = 0 \right\} \subset \mathrm{Tot}(L),$$

with natural projection given by

$$p : \tilde{X}_{R,\varphi} \to \mathbb{C}P^1$$
$$(z, \zeta) \mapsto z.$$

We have

$$(z, \zeta) \in \tilde{X}_{R,\varphi} \Leftrightarrow (z, \sqrt{-1}R^{-\frac{1}{2}} e^{-\sqrt{-1}\varphi/2}\zeta) \in \tilde{X}_{1,0}.$$

Abelianization

Set

$$\omega = \frac{\mathrm{d}z}{\sqrt{z(z-1)(z-t)}}.$$

T. Mochizuki (2016): on simply connected open sets $U \subset \mathbb{C} \setminus \{0, 1, t\}$ there is a gauge $e_1(z), e_2(z)$ of $\mathcal{E}$ with respect to which

$$\theta_{R,\varphi}(z) - \begin{pmatrix} \sqrt{R} e^{\sqrt{-1}\varphi/2} & 0 \\ 0 & -\sqrt{R} e^{\sqrt{-1}\varphi/2} \end{pmatrix} \omega \to 0$$

as $R \to \infty$, and the Hermitian–Einstein metric h is close to an abelian model h_{ab}.
Observe that as ω has ramification at $0, 1, t, \infty$, along a simple loop γ around these points, the local sections $e_1(z), e_2(z)$ get interchanged.

Non-abelian Hodge theory at large R

The connection matrix associated to $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ is

$$
\begin{aligned}
a_{R,\varphi}(z,\bar{z}) &= \theta_{R,\varphi}(z) + \overline{\theta_{R,\varphi}(z)} + b_{R,\varphi} \\
&\approx \sqrt{R} \begin{pmatrix} e^{\sqrt{-1}\varphi/2}\omega + e^{-\sqrt{-1}\varphi/2}\bar{\omega} & 0 \\ 0 & -e^{\sqrt{-1}\varphi/2}\omega - e^{-\sqrt{-1}\varphi/2}\bar{\omega} \end{pmatrix} \\
&+ b_{R,\varphi}
\end{aligned}
$$

where $d + b_{R,\varphi}$ is the Chern connection associated to the holomorphic structure of $\mathcal{E}$ and h_{ab}.

Monodromy matrices at large R

The monodromy matrices of the connection $\mathrm{d} + a_{R,\varphi}$ along a simple loop γ_j around $j \in \{0, 1, t\}$ are

$$B_j(R,\varphi) = \exp \oint_{\gamma_j} -a_{R,\varphi}(z,\bar{z}) = TA_j(R,\varphi)$$

$$\exp \sqrt{R} \begin{pmatrix} -e^{\sqrt{-1}\varphi/2}\pi_j - e^{-\sqrt{-1}\varphi/2}\overline{\pi_j} & 0 \\ 0 & e^{\sqrt{-1}\varphi/2}\pi_j + e^{-\sqrt{-1}\varphi/2}\overline{\pi_j} \end{pmatrix}$$

where we have set

$$\pi_j = \oint_{\gamma_j} \omega,$$

T is the transposition matrix and $A_j(R,\varphi) \in \mathrm{SU}(2)$ is the monodromy of the Chern connection.

Products of monodromy matrices at large R

Setting

$$A_j(R,\varphi) = \begin{pmatrix} e^{\sqrt{-1}\mu_j} & 0 \\ 0 & e^{-\sqrt{-1}\mu_j} \end{pmatrix}$$

and

$$d_{01}(R,\varphi) = \exp\left(\sqrt{-1}(\mu_1 - \mu_0) + 2\sqrt{R}\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))\right)$$

it follows that

$$B_0(R,\varphi)B_1(R,\varphi) \approx \begin{pmatrix} d_{01}(R,\varphi) & 0 \\ 0 & d_{01}(R,\varphi)^{-1} \end{pmatrix}.$$

Affine coordinates on the Betti space

Let us set

$$x_1(R,\varphi) = \mathrm{tr}(B_0(R,\varphi)B_1(R,\varphi))$$
$$x_2(R,\varphi) = \mathrm{tr}(B_t(R,\varphi)B_0(R,\varphi))$$
$$x_3(R,\varphi) = \mathrm{tr}(B_1(R,\varphi)B_t(R,\varphi)).$$

Fricke–Klein cubic relation:

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - s_1x_1 - s_2x_2 - s_3x_3 + s_4 = 0$$

for some constants $s_1, s_2, s_3, s_4 \in \mathbb{C}$. We embed $\mathbb{C}^3 \to \mathbb{C}P^3$ by

$$x_1, x_2, x_3 \mapsto [1 : x_1 : x_2 : x_3].$$

Compactifying divisor:

$$D = (x_1x_2x_3) \subset \mathbb{C}P^2_\infty.$$

Dual boundary complex

The nerve complex $\mathcal{N}$ of D has vertices v_1, v_2, v_3 corresponding to line components

$$L_1 = [0 : 0 : x_2 : x_3], \quad L_2 = [0 : x_1 : 0 : x_3], \quad L_3 = [0 : x_1 : x_2 : 0]$$

of D and edges

$$[v_1 v_2], \quad [v_2 v_3], \quad [v_3 v_1]$$

corresponding to intersection points

$$[0 : 0 : 0 : 1], \quad [0 : 1 : 0 : 0], \quad [0 : 0 : 1 : 0]$$

of the components.

Simpson's map

Let T_i be an open tubular neighbourhood of L_i in $\widetilde{\mathcal{M}}_{\mathrm{B}}$ and set

$$T = T_1 \cup T_2 \cup T_3.$$

Let $\{\phi_i\}$ be a partition of unity subordinate to the cover of T by $\{T_i\}$. Define the map

$$\phi : T \to \mathbb{R}^3$$
$$x \mapsto \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix}.$$

Then,

$$\mathrm{Im}(\phi) = [v_1 v_2] \cup [v_2 v_3] \cup [v_3 v_1] \cong S^1.$$

Asymptotic of Riemann–Hilbert correspondence at large R

Fix $R >> 0$ and let $\varphi \in [0, 2\pi)$ vary. Need to show: the loop

$$\phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$$

generates $\pi_1(\mathrm{Im}(\phi)) \cong \mathbb{Z}$.
Key fact: for $d \in \mathbb{C}$ with $|\Re(d)| >> 0$ we have

$$|2\cosh(d)| \approx e^{|d|}.$$

This implies

$$|x_1(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))|\right),$$

$$|x_2(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_t - \pi_0))|\right),$$

$$|x_3(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_1 - \pi_t))|\right).$$

Rotating triangle

Let $\Delta \subset \mathbb{C}$ be the triangle with vertices π_0, π_1, π_t, assume Δ is non-degenerate. Denote its sides by

$$a = \pi_0 - \pi_1, \quad b = \pi_t - \pi_0, \quad c = \pi_1 - \pi_t.$$

Let us denote by $e^{\sqrt{-1}\varphi/2}\Delta$ the triangle obtained by rotating Δ by angle $\varphi/2$ in the positive direction, with sides $e^{\sqrt{-1}\varphi/2}a$, $e^{\sqrt{-1}\varphi/2}b$, $e^{\sqrt{-1}\varphi/2}c$.

Critical angles

Lemma

For each side a, b, c there exists exactly one value $\varphi_a, \varphi_b, \varphi_c \in [0, 2\pi)$ such that $e^{\sqrt{-1}\varphi_a/2} a$ (respectively $e^{\sqrt{-1}\varphi_b/2} b$, $e^{\sqrt{-1}\varphi_c/2} c$) is purely imaginary. The function

$$\Re(e^{\sqrt{-1}\varphi/2} b) - \Re(e^{\sqrt{-1}\varphi/2} c)$$

changes sign at $\varphi = \varphi_a$. Similar statements hold with a, b, c permuted.

Definition

$\varphi_a, \varphi_b, \varphi_c$ are the critical angles associated to the sides a, b, c respectively.

Arc decomposition of the circle

The critical angles decompose S^1 into three closed arcs

$$S^1 = I_1 \cup I_2 \cup I_3$$

satisfying:

$$\max(|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))|, |\Re(e^{\sqrt{-1}\varphi/2}(\pi_t - \pi_0))|, |\Re(e^{\sqrt{-1}\varphi/2}(\pi_1 - \pi_t))|)$$

is realized

- by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))|$ for $\varphi \in I_1$,
- by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_t - \pi_0))|$ for $\varphi \in I_2$,
- and by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_1 - \pi_t))|$ for $\varphi \in I_3$.

Limiting value of Riemann–Hilbert map

We deduce

- for $\varphi \in \mathrm{Int}(I_1)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 1 : 0 : 0],$$
- for $\varphi \in \mathrm{Int}(I_2)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 0 : 1 : 0],$$
- for $\varphi \in \mathrm{Int}(I_3)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 0 : 0 : 1].$$

Limiting value of Simpson's map

Applying Simpson's map ϕ to the previous limits we get that

- for $\varphi \in \text{Int}(I_1)$, we have
$$\phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_2 v_3],$$
- for $\varphi \in \text{Int}(I_2)$, we have
$$\phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_3 v_1],$$
- for $\varphi \in \text{Int}(I_3)$, we have
$$\phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_1 v_2].$$

Thus, ϕ sends a generator of $\pi_1(S^1_\varphi)$ into a generator of $\pi_1(\text{Im}(\phi))$.

Implication between the conjectures

Recall:

- $P_1 H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) = \mathrm{Ker}\left(H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) \xrightarrow{j^*} H^2(h^{-1}(Y_{-1}), \mathbb{Q}) \right)$
- $W_2 H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong$
  $\mathrm{Coker}\left(\delta_2 : \oplus_i H^0(L_i, \mathbb{C}) \to H^2\left(\tilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C} \right)\right)$.

"$P = W$" $\Leftrightarrow$ NAHT maps $H^2\left(\tilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C} \right)$ isomorphically onto $\mathrm{Ker}(j^*)$ $\Leftrightarrow$ the composition

$$H^2\left(\tilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{C} \right) \xrightarrow{i^*} H^2(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) \cong H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{C}) \xrightarrow{j^*} H^2(h^{-1}(Y_{-1}), \mathbb{C})$$

is the 0-map $\Leftrightarrow$

$$[h^{-1}(Y_{-1})] = 0 \in H_2\left(\tilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{Z} \right).$$

Matching singular cycles

Let the components of the compactifying divisor D^{PX} of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ be denoted by L_j (j understood $\pmod 3$), and set

$$p_j = L_j \cap L_{j+1}.$$

Let

$$z_1 = r_1 e^{\sqrt{-1}\theta_1}, \quad z_2 = r_2 e^{\sqrt{-1}\theta_2}$$

be local coordinates defining the two divisors crossing at p_j. We define a 2-cycle C_j by

$$C_j(\varepsilon) = \{r_1 = \varepsilon = r_2\}$$

for some sufficiently small $0 < \varepsilon << 1$. Then, by Geometric $P = W$ conjecture, for $\varphi \in \mathrm{Int}(I_1)$, we have

$$[h^{-1}(Y_{-1})] = [C_1(\varepsilon)].$$

On the other hand, we obviously have $[C_1(\varepsilon)] = 0 \in H_2\left(\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}, \mathbb{Z}\right)$.