Irregular Higgs bundles on curves and sheaves on ruled surfaces

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Outline

Irregular parabolic
Beauville–Narasimhan–Ramanan correspondence

Outline

Notations

C: smooth projective curve over $\mathbf{C}$
$G = \mathrm{Gl}_r(\mathbf{C})$ for some $r \geq 2$
$P = \{p_1, \ldots, p_n\}$: a finite non-empty set of distinct points on C
$m_1, \ldots, m_n \geq 0$: multiplicities

$$D = (m_1 + 1)p_1 + \cdots + (m_n + 1)p_n$$

$$D_{\mathrm{red}} = p_1 + \cdots + p_n$$

$\mathcal{O}$: sheaf of holomorphic functions
K_C: canonical sheaf of holomorphic 1-forms on C
$L = K_C(D)$

Aim

1. describe L-Higgs bundles with fixed semi-simple irregular parts in terms of spectral data on a multiple blow-up of the ruled surface associated to L-valued Higgs bundles ("refined" BNR-correspondence)
2. in a particular case, describe explicitly the Hitchin-fibration

SEMI-SIMPLE IRREGULAR PART

Biquard–Boalch '04, Boalch '04–: for $1 \leq i \leq n$ we fix

- a local coordinate z_i of C near p_i
- for $1 \leq j \leq m_i$ semi-simple endomorphisms $A_i^j \in \mathfrak{gl}_r(\mathbf{C})$ satisfying
$$[A_i^j, A_i^{j'}] = 0.$$
- a semi-simple irregular part at p_i given by
$$Q_i = A_i^{m_i} z_i^{-m_i} + \cdots + A_i^1 z_i^{-1}$$

Irregular Higgs bundle

Irregular Higgs bundle with irregular part Q_i:

- a holomorphic vector bundle $\mathcal{E}$ of rank r over C
- an L-valued Higgs field on $\mathcal{E}$

$$\theta \in H^0(C, \mathcal{E}nd(\mathcal{E}) \otimes_{\mathcal{O}_C} L)$$

such that for any $i \in \{1, \dots, n\}$

$$\theta = \mathrm{d}Q_i + \left(\Lambda_i z_i^{-1} + \text{ holomorphic terms}\right) \mathrm{d}z_i$$

in some local trivialisation of $\mathcal{E}$ near p_i, for some $\Lambda_i \in \mathfrak{gl}_r(\mathbf{C})$.

Up to gauge transformation we may assume

$$\Lambda_i \in \mathfrak{z}_i = \mathfrak{z}_{\mathfrak{gl}_r}(A_i^{m_i}, \dots, A_i^1).$$

Compatible parabolic structure

Quasi-parabolic structure compatible with θ at p_i: a filtration

$$\{0\} \subset F_i^{l_i-1} \subset \cdots \subset F_i^1 \subset F_i^0 = \mathcal{E}_{|p_i}$$

preserved by Λ_i and by $A_i^1, \ldots, A_i^{m_i}$.
Compatible parabolic structure at p_i:

- a compatible quasi-parabolic structure at p_i
- parabolic weights

$$1 > \alpha_i^{l_i-1} > \cdots > \alpha_i^0 \geq 0$$

Fixing the orbit of the residue

Quasi-parabolic filtration $\leftrightarrow$ a parabolic subalgebra $\mathfrak{p}_i \subset \mathfrak{gl}_r(\mathbf{C})$ containing A_i^j, Λ_i.
Let

$$\mathfrak{p}_i \cap \mathfrak{z}_i \to \mathfrak{l}_i$$

be the Levi-quotient.
Fix an adjoint orbit $\mathcal{O}_i$ in $\mathfrak{l}_i$. We assume $\Lambda_i \in \mathcal{O}_i$.

R-PARABOLIC SHEAVES

A generalization of parabolic bundles.
X: an non-singular projective variety over $\mathbf{C}$
Δ: a reduced effective divisor on X
An $\mathbf{R}$-parabolic sheaf over X with parabolic divisor Δ: a family $S_\bullet$ indexed by $\mathbf{R}$ of coherent sheaves of $\mathcal{O}_X$-modules satisfying for every $\alpha \in \mathbf{R}$

- there exists some $\varepsilon > 0$ with $S_{\alpha-\varepsilon} = S_\alpha$
- we have $S_{\alpha+1} = S_\alpha \otimes \mathcal{O}_X(-\Delta)$

R-parabolic sheaves on smooth curves

If $X = C$ is a smooth curve and $\mathcal{E}$ is a parabolic bundle over C with divisor D_{red}, we get an **R**-parabolic sheaf $\mathcal{E}_\bullet$ over C with divisor D_{red} as follows:

- locally near $z \notin D_{\text{red}}$ for every $\alpha \in \mathbf{R}$ we set $\mathcal{E}_\alpha = \mathcal{E}$
- near $p_i \in D$ for $\alpha_i^{l-1} < \alpha \leq \alpha_i^l$ we set

$$\mathcal{E}_\alpha = \ker \left(\mathcal{E} \xrightarrow{eval_{p_i}} \mathcal{E}|_{p_i} = F_i^0 \to F_i^0 / F_i^l \right).$$

This construction gives an equivalence of categories between parabolic bundles and locally free **R**-parabolic sheaves over C with divisor D_{red}.

The ruled surface Z

Set

$$Z = \mathbf{P}_C(\mathcal{O}_C \oplus L^\vee),$$

equipped with

- a natural projection

$$p : Z \to C$$

- a relative hyperplane bundle $\mathcal{O}_Z(1)$
- canonical sections

$$\xi \in H^0(Z, \mathcal{O}_Z(1)), \zeta \in H^0(Z, p^*L \otimes \mathcal{O}_Z(1)).$$

- zero-section C_0

$$(\zeta = 0)$$

- infinity-section C_∞

$$(\xi = 0)$$

EIGENVALUES

Let us simultaneously diagonalize the endomorphisms A_i^m appearing in the irregular part Q_i. Denote the eigenvalue of A_i^m corresponding to the j'th basis vector: $-\zeta_{i,j}^m/m$.
Then the expansion of the eigenvalue of θ corresponding to the j'th basis vector is:

$$\left(\frac{\zeta_{i,j}^{m_i}}{z_i^{m_i+1}} + \cdots + \frac{\zeta_{i,j}^{1}}{z_i^2} + O\left(z_i^{-2+\frac{1}{r}} \right) \right) \mathrm{d}z_i$$

This gives rise to a decomposition of $\mathcal{E}$ over $\mathbf{C}\{z_i\}$ as

$$\mathcal{E} = \mathcal{E}^{i,1} \oplus \cdots \oplus \mathcal{E}^{i,L_i} :$$

given $1 \leq l \leq L_i$ the eigenvalues of θ on $\mathcal{E}^{i,l}$ agree up to $z_i^{-m_i}$.

Further decompositions

For $0 \leq m < m_i$ let

$$\mathcal{E}^{i,j,j',\ldots,j^{(m)}}$$

stand for the eigenbundle of θ for the eigenvalues of the form

$$\left(\frac{\zeta_{i,j}^{m_i}}{z_i^{m_i+1}} + \cdots + \frac{\zeta_{i,j}^{m_i-m}}{z_i^{m_i-m+1}} + O\left(z_i^{-m_i+m-1+\frac{1}{r}} \right) \right) \mathrm{d}z_i .$$

By fixing the values of $\zeta_{i,j}^{m}$ we thus get a nested sequence of decompositions over $\mathbf{C}\{z_i\}$:

$$\mathcal{E} = \bigoplus_{j,j',\ldots,j^{(m)}} \mathcal{E}^{i,j,j',\ldots,j^{(m)}} .$$

By compatibility of θ with the parabolic structure, these restrict to decompositions for any $\alpha \in \mathbf{R}$

$$\mathcal{E}_\alpha = \bigoplus_{j,j',\ldots,j^{(m)}} \mathcal{E}_\alpha^{i,j,j',\ldots,j^{(m)}} .$$

Construction of a new surface – first step

We recursively define a surface $\widetilde{Z}$ together with a birational morphism

$$\tilde{\sigma} : \widetilde{Z} \to Z.$$

Consider the local trivialisation

$$\lambda_i = z_i^{-m_i - 1} \mathrm{d} z_i$$

of L and the local holomorphic function ζ_i defined near $p^{-1}(p_i) \setminus C_\infty$ by

$$\zeta = \zeta_i p^* \lambda_i.$$

We use the chart (z_i, ζ_i) near $p^{-1}(p_i) \setminus C_\infty \subset Z$. Let

$$\sigma_1 : Z^1 \longrightarrow Z$$

be the blow-up for all i and all j of the points

$$z_i = 0, \zeta_i = \zeta_{i,j}^{m_i}.$$

Construction of a new surface – recursion

In the $(m+1)$'th step for all i with $m_i \geq m$ and all eigenvalue

$$\zeta^{m_i - m}_{i,j,j',\dots,j^{(m)}}$$

of $(m-m_i)A_i^{m_i-m}$ on the fiber of $\mathcal{E}^{i,j,j',\dots,j^{(m-1)}}$ at p_i, we let

$$\sigma_{m+1} : Z^{m+1} \longrightarrow Z^m$$

be the blow up of Z^m at the point

$$\zeta'_{i,j,j'\dots,j^{(m-1)}} = \zeta^{m_i - m}_{i,j,j',\dots,j^{(m)}} \in E_{i,j,j'\dots,j^{(m-1)}}$$

and let $E_{i,j,j'\dots,j^{(m)}}$ denote the exceptional divisor of this blow-up, with induced homogeneous coordinates

$$(z'_{i,j,j'\dots,j^{(m)}} : \zeta'_{i,j,j'\dots,j^{(m)}}).$$

Construction of a new surface

We then let

$$\widetilde{Z} = Z^M$$

for the value $M = \max_{1 \leq i \leq n}(m_i)$ and

$$\tilde{\sigma} = \sigma_1 \circ \cdots \circ \sigma_M.$$

Remark

This construction and the next theorem was motivated by Conjecture 8.6.1 of M. Kontsevich and Y. Soibelman: Wall-crossing structures in Donaldson–Thomas invariants, integrable systems and Mirror Symmetry.

Refined BNR–correspondence

Theorem (Sz 2015 arXiv:1502.02003)

There exists an equivalence of categories between the groupoids

1. *parabolic Higgs bundles $(\mathcal{E}, \theta)$ of rank r on C with an irregular singularity with pole of order m_i at p_i with semi-simple commuting endomorphisms A_i^m, endowed with a compatible parabolic structure,*
2. $\mathbf{R}$*-parabolic pure sheaves $S_\bullet$ of dimension 1 and rank 1 with parabolic divisor*

$$(p \circ \tilde{\sigma})^{-1}(D_{\mathrm{red}})$$

on $\widetilde{Z}$, with support $\widetilde{\Sigma}$ satisfying the following properties:

Properties of the support

- $\widetilde{\Sigma}$ is generically r to 1 over C,
- $\widetilde{\Sigma} \cap (\xi = 0) = \emptyset$,
- for all $1 \leq i \leq n$, $\widetilde{\Sigma}$ does not pass through the point at infinity of the exceptional divisor $E_{i,j,j',\ldots,j^{(m_i-1)}}$,
- for all $1 \leq i \leq n$ the algebraic multiplicity of the intersection $\widetilde{\Sigma} \cap E_{i,j,j',\ldots,j^{(m_i-1)}}$ is equal to

$$\dim_{\mathbf{C}} \left(\mathcal{E}^{i,j,j',\ldots,j^{(m_i-1)}}|_{p_i} \right),$$

- for all $1 \leq i \leq n$ and all $m < m_i$ the proper transform of $E_{i,j,j',\ldots,j^{(m-1)}}$ in $\widetilde{Z}$ does not intersect $\widetilde{\Sigma}$.

The functors of the correspondence

The functor

$$S_\bullet \mapsto (\mathcal{E}_\bullet, \theta)$$

is the usual direct image.
The quasi-inverse functor

$$(\mathcal{E}_\bullet, \theta) \mapsto S_\bullet$$

is given by a suitably-defined multiple "proper transform" of the usual cokernel-sheaf.

Proper transform of sheaves of homological dimension 1 on a smooth surface

Aker-Sz '14: let X be a non-singular complex surface and S a coherent sheaf of $\mathcal{O}_X$-modules. Let

$$\sigma : \widetilde{X} \to X$$

be the blow-up of a point $x \in X$. Let E denote the exceptional divisor. Set

$$S^E := \mathcal{T}or_1^{\mathcal{O}_{\widetilde{X}}}(\sigma^* S, \mathcal{O}_{\widetilde{X}}(E)_E).$$

We define the proper transform of S with respect to σ by

$$\tilde{S} = \sigma^* S / S^E.$$

Claim

IF $dh(S_x) = 1$ and S_x is torsion, then $\tilde{S}_y$ is of homological dimension 1 for any $y \in E$.

Choices

Joint work in progress with A. Stipsicz and P. Ivanics.

- $C = \mathbf{C}P^1$
- rang: $r = 2$
- singularities: $n = 1$, $p_1 = q = 0 \in \mathbf{C}P^1$
- order of Q_1: $m_1 = 3$
- irregular part Q_1 with respect to some trivialisation:

$$\theta = \Bigg[z^{-4} \begin{pmatrix} a_+ + b_+ z + c_+ z^2 + \lambda_+ z^3 & 0 \\ 0 & a_- + b_- z + c_- z^2 + \lambda_- z^3 \end{pmatrix} +$$

- compatible parabolic structure: $(\alpha_q^+, \alpha_q^-) \in [0,1)^2$ corresponding to the $\lambda_\pm$-eigenspaces $L_q^\pm$ of the residue.

Assumptions on the data

By the residue theorem we must have

$$\lambda_+ + \lambda_- = -\operatorname{deg}(\mathcal{E}).$$

We define

$$\text{par-deg}(\mathcal{E}) = \operatorname{deg}(\mathcal{E}) + \alpha_{\boldsymbol{q}}^+ + \alpha_{\boldsymbol{q}}^-$$

and impose

$$\text{par-deg}(\mathcal{E}) = 0.$$

Parabolic Higgs sub-bundles

A Higgs sub-bundle of $(\mathcal{E}, \theta)$ is a pair $(\mathcal{F}, \theta|_{\mathcal{F}})$ with $\mathcal{F}$ a holomorphic line-subbundle of $\mathcal{E}$ such that

$$\theta|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes L.$$

If $(\mathcal{F}, \theta|_{\mathcal{F}})$ is a Higgs sub-bundle of $(\mathcal{E}, \theta)$ then

$$\mathcal{F}|_q \cong L_q^+ \quad \text{or} \quad \mathcal{F}|_q \cong L_q^-$$

In particular, if $(\mathcal{E}, \theta)$ is endowed with a compatible parabolic structure then $(\mathcal{F}, \theta|_{\mathcal{F}})$ inherits a parabolic structure:

$$\alpha_q(\mathcal{F}) = \alpha_q^+ \quad \text{or} \quad \alpha_q(\mathcal{F}) = \alpha_q^-$$

according to whether $\mathcal{F}|_q \cong L_q^+$ or L_q^-.

Quotient parabolic Higgs bundles

If $(\mathcal{F}, \theta|_{\mathcal{F}})$ is a Higgs sub-bundle of $(\mathcal{E}, \theta)$ then we set

$$\mathcal{Q} = \mathcal{E}/\mathcal{F},$$

and we denote the resulting Higgs field

$$\theta|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q} \otimes L.$$

We then call $(\mathcal{Q}, \theta|_{\mathcal{Q}})$ a quotient parabolic Higgs bundle.
If $\alpha_{\boldsymbol{q}}(\mathcal{F}) = \alpha_{\boldsymbol{q}}^{\pm}$ then we set

$$\alpha_{\boldsymbol{q}}(\mathcal{Q}) = \alpha_{\boldsymbol{q}}^{\mp}$$

and

$$\text{par-deg}(\mathcal{Q}) = \deg(\mathcal{Q}) + \alpha_{\boldsymbol{q}}(\mathcal{Q}).$$

Stability

We say that $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-semistable if and only if for all quotient parabolic Higgs sub-bundles $(\mathcal{Q}, \theta|_{\mathcal{Q}})$ we have

$$\text{par-deg}(\mathcal{Q}) \geq 0$$

and $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-stable if and only if strict inequality holds.

Irregular Dolbeault moduli space

Consider the moduli space

$$\mathcal{M} = \mathcal{M}(\mathbf{C}P^1, q, a_\pm, b_\pm, c_\pm, \lambda_\pm, \alpha_q^\pm)$$

of $\vec{\alpha}$-stable wild Higgs bundles on $\mathbf{C}P^1$ of 0 parabolic degree with the polar parts at q given by Q_1.
Constructed in general and shown to be a hyperKähler manifold by O. Biquard and P. Boalch (2004).

The fibration

In this case

- $Z = F_2$, the Hirzebruch-surface of index 2
- $\widetilde{Z} = \mathbf{C}P^2 \# 9\overline{\mathbf{C}P}^2$ as a smooth manifold
- any curve $\widetilde{\Sigma}$ satisfying the properties of the refined BNR-correspondence has arithmetic genus 1
- $\widetilde{Z}$ is fibered over $\mathbf{C}P^1$ in such curves $\widetilde{\Sigma}$
- the fibration has sections
- the generic fiber is smooth (except in one very particular case)
- the fiber over $\infty \in \mathbf{C}P^1$ is of type $\widetilde{E}_7$ and
- the set of remaining singular fibres is one of the following list:
  1. a type III fiber
  2. a type II and an I_1 fiber
  3. an I_2 and an I_1 fiber, or
  4. three I_1 fibers.

Picard-variety of regular or integral fibers

Let $\widetilde{\Sigma}$ be a fiber of

$$\pi : \widetilde{Z} \to \mathbf{C}P^1.$$

Let $(\mathbf{C}P^1)^{sm} \subset \mathbf{C}P^1$ parametrize smooth curves $\widetilde{\Sigma}$. Then relative pure sheaves are locally free and since the fibration has a section,

$$\mathrm{Pic}^{\delta}_{\mathrm{rel}}(\pi^{-1}(\mathbf{C}P^1)^{sm}) \cong \pi^{-1}(\mathbf{C}P^1)^{sm}$$

by the relative Abel–Jacobi map; in particular the fibers are projective.
If $\widetilde{\Sigma}$ is of type I_1 or II then torsion-free sheaves on $\widetilde{\Sigma}$ of degree δ are parametrized by the Altman–Kleiman compactification

$$\overline{\mathrm{Pic}}^{\delta}(\widetilde{\Sigma}).$$

Fibers of type I_2 – degrees

Let $\widetilde{\Sigma}$ be a fiber of type I_2, and

$$\sigma : \Sigma_+ \cup \Sigma_- \to \widetilde{\Sigma}$$

its normalization.
For a a torsion-free coherent sheaf S of $\mathcal{O}_{\widetilde{\Sigma}}$-modules of rank 1 let us set

$$\mathcal{L}(S) = \sigma^* S / \mathcal{T}or^{\mathcal{O}_{\widetilde{\Sigma}}}(\sigma^* S)$$

and and for $i \in \{\pm\}$ define

$$\delta_i = \deg(\mathcal{L}(S)|_{\Sigma_i}).$$

Canonical quotients

There exist canonical morphisms

$$S \to \sigma^* S \to \mathcal{L}(S) \to \mathcal{L}(S)|_{\Sigma_i}.$$

They give rise to quotient parabolic Higgs bundles

$$\mathcal{E} \to \mathcal{Q}_i.$$

As $\widetilde{\Sigma}$ has two irreducible components, $\mathcal{Q}_\pm$ are the only quotient parabolic Higgs bundles.

Oda–Seshadri stability for a curve of type I_2

Given a sheaf S as above define

$$J(S) = \left\{ j \in \widetilde{\Sigma}^{sing} : \quad S \text{ is locally free near } j \right\}$$

Fix a constant

$$\phi_+ \in \mathbf{R}$$

and set $\phi_- = -\phi_+$.
Then S is $\vec{\phi}$-semistable if and only if for $i \in \{\pm\}$

$$\delta_i - \phi_i \leq |J(S)| - 1$$

holds, and stability is defined by the corresponding strict inequality.

Oda–Seshadri stability and parabolic stability

Assume

$$\deg(\mathcal{E}) = -2,$$

and set

$$\phi_i = 1 - \alpha^i.$$

Let S and $(\mathcal{E}, \theta)$ correspond to each other under the refined BNR-correspondence.

Claim

S is $\vec{\phi}$-stable if and only if $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-stable.

Remark

Analogous result in the non-parabolic case: D. Schaub 1998.

Sheaves on curves of type *III*

Assume $\widetilde{\Sigma}$ is of type *III* and set

$$\mathcal{E} = p_*(S).$$

Let $x \in \widetilde{\Sigma}$ be the singular point (of type A_3).
Up to isomorphism we may assume

$$\mathcal{O}_{\widetilde{\Sigma},x} \subseteq S_x \subseteq \widetilde{\mathcal{O}}_{\widetilde{\Sigma},x}.$$

The length of S at x is

$$l(S) = \dim_{\mathbf{C}}(S_x / \mathcal{O}_{\widetilde{\Sigma},x})$$

Stability for curves of type III

Let $\Sigma_\pm$ denote the components of the normalization of $\widetilde{\Sigma}$ and $\delta_\pm$ the degrees of $\mathcal{L}(S)|_{\Sigma_+}, \mathcal{L}(S)|_{\Sigma_-}$ respectively.
Then $(\mathcal{E}, \theta)$ is $\vec{\alpha}$-stable if and only if

$$\delta_+ + \alpha^+ + 2l(S) - 2 < \delta_- + \alpha^- + l(S) < \delta_+ + \alpha^+ + 2.$$

In particular, there exist no stable Higgs bundles with $l(S) = 2$.

Invertible sheaves

Let

$$\deg(\mathcal{E}) = -1.$$

Assume first

$$l(S) = 0,$$

i.e. S is an invertible sheaf. Then either

$$\delta_+ = 0, \delta_- = 1$$

or

$$\delta_+ = 1, \delta_- = 0.$$

Such sheaves are parameterized by $\mathbf{C} \coprod \mathbf{C}$.

Length 1 sheaves

Assume now

$$l(S) = 1.$$

Then we must have

$$\delta_+ = 0 = \delta_-$$

and there exists a unique such sheaf up to isomorphism.

Further directions

- Describe explicitly all the 2-dimensional irregular Dolbeault moduli spaces and describe wall-crossing with respect to parabolic weights (in progress).
- study the relationship between parabolic stability for $(\mathcal{E}, \theta)$ and various known stability conditions for S.