Nahm transform for parabolic integrable connections on the Riemann sphere

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Mini-Workshop on Krichever-Novikov algebras
Luxembourg, June 11th 2008

Outline

Motivation – the compact case

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Motivation – the compact case

Preparation

Polar parts

Parabolic structure and Hermitian metrics

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- Polar parts
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Nahm transform for parabolic connections
- Construction
- Properties

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The Deligne-Simpson problem
- The problem
- Katz' middle convolution

Notations

X: an arbitrary Riemann surface of genus $g \geq 2$

G: a connected reductive Lie group over $\mathbf{R}$ or $\mathbf{C}$

$\mathfrak{g}$: its Lie-algebra

$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$

$P = \{p_0 = \infty, p_1, \dots, p_n\}$: a finite set of distinct points in $\mathbf{P}^1$

$\mathcal{O}$: sheaf of holomorphic functions (over X or $\mathbf{P}^1$)

Ω^k: sheaf of smooth k-forms

$\mathbf{\Omega}^1$: sheaf of holomorphic 1-forms

Connections on G-bundles

Let

$$p : E \to X$$

be a principal G-bundle on X, with action

$$R_g : E \to E$$

on the right. Denote by V the "vertical" distribution of TE tangent to the fibers of p.

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A connection on E is a splitting

$$TE = V \oplus H$$

of the tangent bundle of E into V and a "horizontal" part H, which is G-equivariant: for any $e \in E$ and $g \in G$, we have

$$H_{eg} = d(R_g)_e H_e.$$

Integrability and parallel transport

A connection D is said to be integrable if the distribution of horizontal subspaces is integrable (i.e. there exists a foliation such that the tangent space of the leaf containing any e is H_e). The couple (E, D) is then a flat principal G-bundle.

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If D is an integrable connection on E and γ is a loop in X based at x_0, then one can consider parallel transport along γ of an arbitrary $e \in E_{X_0}$. The result is $R_{g_\gamma}(e)$ for some $g_\gamma \in G$.

Let $f = R_h(e)$ be another element of E_{X_0}. Then parallel transport along γ carries f into $R_{h^{-1} g_\gamma h}(f)$.

MONODROMY OF AN INTEGRABLE CONNECTION

Integrability implies that the element g_γ occuring in the parallel transport only depends on the class $[\gamma] \in \pi_1(X, x_0)$.
So, parallel transport induces a monodromy representaion

$$\rho_D : \pi_1(X, x_0) \to G,$$

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Remark

In other words, we get $A_1, B_1, \ldots, A_g, B_g \in G$ satisfying the relation

$$[A_1, B_1] \cdots [A_g, B_g] = \mathbf{1}.$$

Riemann-Hilbert correspondence

Let a representation

$$\rho : \pi_1(X, x_0) \to G$$

be given. It induces a flat principal G-bundle E_ρ on X:

$$E_\rho = \tilde{X} \times_{\pi_1(X,x_0)} G,$$

where $\tilde{X}$ is the universal cover of X and $\pi_1(X, x_0)$ acts by deck transformations on $\tilde{X}$ and by ρ on G.

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where $\tilde{X}$ is the universal cover of X and $\pi_1(X, x_0)$ acts by deck transformations on $\tilde{X}$ and by ρ on G.
Denote by $\mathcal{M}(E)$ the moduli space of irreducible integrable connections on E, and by $\mathcal{R}_E(X, G)$ the space of irreducible representations whose induced flat G-bundle has E as underlying G-bundle, up to conjugation. Then, the above maps define an analytic isomorphism

$$\mathcal{M}(E) \cong \mathcal{R}_E(X, G).$$

Leitmotiv

Finding all (irreducible) solutions of

$$[A_1, B_1] \cdots [A_g, B_g] = 1$$

in G is equivalent to finding all (irreducible) flat G-connections on X.
For various groups G, a lot is known about the space $\mathcal{M}(E)$ of solutions (Atiyah–Bott, Hitchin, ...)

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Aim: extend results to non-compact curves X.

Meromorphic connections

$X \rightsquigarrow \mathbf{P}^1 \setminus P$, $G = \mathrm{Gl}_r(\mathbf{C})$. An integrable connection ∇ splits up into a holomorphic structure operator (its $(0,1)$-part) and a holomorphic connection on the induced holomorphic bundle ($(1,0)$-part). To have solutions, must allow singularities at P.

Meromorphic connections

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$$D : E \longrightarrow \mathbf{\Omega}^1(P) \otimes_{\mathcal{O}} E.$$

We fix the behaviours of D near the singular points as follows.

Assumption on points at finite distance

D is supposed to have a logarithmic singularity at p_j for $j \in \{1, \ldots, n\}$: in a local trivialisation of E near p_j, one has

$$D = d + \frac{A^j(z)}{z - p_j} dz,$$

where A^j is a holomorphic matrix-valued function defined near p_j. Furthermore, the residue

$$A^j(p_j) = \mathrm{diag}(0, \ldots, 0, \mu^j_{r_j+1}, \ldots, \mu^j_r),$$

is diagonal, with μ^j_k non-zero and generic.

Assumption at infinity

D is supposed to have an irregular singularity of Poincaré-rank 1 at infinity: in a local trivialisation of E near ∞, one has

$$D = d + Adz + B\frac{dz}{z} + \text{lower order terms},$$

where

$$A = \operatorname{diag}(\xi_1, \ldots, \xi_1, \ldots\ldots\ldots, \xi_{n'}, \ldots.., \xi_{n'})$$
$$B = \operatorname{diag}(\mu_1^0, \ldots, \mu_{a_2}^0, \ldots\ldots, \mu_{1+a_{n'}}^0, \ldots, \mu_r^0)$$

(the leading order term and residue, respectively). Here the ξ_k are pairwise distinct constants, and the μ_l^0 are generic non-zero. (Notation:$a_1 = 0, a_{n'+1} = r$.)

The standard example

Let E be the trivial holomorphic vector bundle of rank r over $\mathbf{P}^1$ and A, B be $r \times r$ matrices such that A is diagonal with distinct eigenvalues and B is semi-simple without any 0's on the diagonal. Then, the connection

$$D = d + Adz + B\frac{dz}{z}$$

has the desired properties.

Parabolic structure at logarithmic points

At each logarithmic point $p = p_j$ $(1 \leq j \leq n)$, let us fix a diagonalising trivialisation $\{\tau_1^j, \ldots, \tau_r^j\}$. We suppose a compatible parabolic structure is given:

- a filtration

$$E|_p = F_0 E|_p \supset F_1 E|_p \supset \cdots \supset F_{r-r_j} E|_p \supset F_{r-r_j+1} E|_p = 0,$$

such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau^j_{r_j+k}, \ldots, \tau^j_r \rangle$$

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  such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau_{r_j+k}^j, \dots, \tau_r^j \rangle$$

  (the parabolic filtration),
- an r-uple of real numbers

$$0 = \beta_1^j = \cdots = \beta_{r_j}^j < \beta_{r_j+1}^j < \cdots < \beta_r^j < 1$$

  (the parabolic weights).

Parabolic structure at infinity

At infinity, let us fix a diagonalising trivialisation $\{\tau_1^0, \dots, \tau_r^0\}$. The parabolic structure is given by:

- a full flag

$$E|_\infty = F_0 E|_\infty \supset F_1 E|_\infty \supset \cdots \supset F_r E|_\infty = 0,$$

such that for all $k > 0$

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- weights

$$0 < \beta_1^0 < \cdots < \beta_r^0 < 1.$$

Stability

The parabolic degree and slope of E are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta_k^j$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$

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(E, D) is said to be parabolically stable if for all non-trivial proper subbundle $F \subset E$ such that $D|_F \subset \mathbf{\Omega}^1(*P) \otimes F$, one has

$$\text{par-slope}(F) < \text{par-slope}(E).$$

Adapted Hermitian metrics

A Hermitian fiber metric h is adapted to the parabolic structure if near all $p_j \in \mathbf{C}$ in the trivialisation $\{\tau_1^j, \dots, \tau_r^j\}$ it is mutually bounded with

$$\mathrm{diag}(|z - p_j|^{2\beta_k^j})_{k=1,\dots,r},$$

and near ∞ in the trivialisation $\{\tau_1^0, \dots, \tau_r^0\}$ it is mutually bounded with

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Remark

Without the semi-simplicity assumption on the residues, the form of the matrices in the definition is more complicated, involving logarithmic terms.

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Decompose

$$D = D^{+} + \Phi$$

into h-unitary and self-adjoint parts respectively.

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Decompose these parts further according to bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^{+} = \partial^{+} + \bar{\partial}^{+}$$
$$\Phi = \theta + \theta^{*}.$$

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Then, h is said to be harmonic if

$$\bar{\partial}^{+} \theta = 0.$$

Non-abelian Hodge theory

Theorem (C. Sabbah 1999, O. Biquard – P. Boalch 2004)

Let (E, D) be a parabolically stable meromorphic integrable connection of parabolic degree 0. Then, there exists a unique adapted harmonic metric h (up to a constant). Furthermore, the moduli space $\mathcal{M}$ of parabolically stable connections of parabolic degree 0 with prescribed singularity data up to holomorphic gauge transformations is a complete hyper-Kähler manifold.

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From now on, (E, D) is supposed to be a parabolically stable meromorphic integrable connection of parabolic degree 0 with the given singular parts, and h denotes its harmonic Hermitian metric.

Example of moduli space

Fix diagonal matrices A, Λ and let B_0 be a non-degenerate semi-simple matrix. Suppose that A is regular, and the form of D at infinity is

$$d + A dz + \Lambda \frac{dz}{z}$$

and at 0 it is

$$d + B_0 \frac{dz}{z}.$$

Let O be the adjoint orbit of B_0 in $\mathrm{Gl}(r, \mathbf{C})$, and T denote the maximal complex torus. Then, the moduli space of connections on the trivial rank r vector bundle having these local behaviours is the symplectic quotient

$$O//_{\Lambda} T$$

of the orbit acted on by the torus, at the value Λ.

Exponential twist

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.
For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

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For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$

Let D_ξ^* stand for the adjoint operator of D_ξ with respect to h, and define the twisted Laplace operator

$$\Delta_\xi = D_\xi D_\xi^* + D_\xi^* D_\xi$$

as an unbounded operator acting on $L^2(\Omega^1 \otimes E)$.

The kernel of the twisted Laplacian

Theorem

For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, the twisted Laplace operator

$$\Delta_\xi : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

has finite dimensional kernel.

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has finite dimensional kernel.

The vector spaces $\ker(\Delta_\xi)$ form a smooth family of finite-dimensional subspaces of $L^2(\Omega^1 \otimes E)$ of the same dimension, parametrized by $\widehat{\mathbf{C}} \setminus \widehat{P}$.

Transformed vector bundle and metric

Definition

The smooth vector bundle with fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ equal to $\ker(\Delta_\xi)$ is called the transformed smooth vector bundle. We denote it by $\widehat{E}$, and its fiber over ξ by $\widehat{E}_\xi$.

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Let $\varphi(z), \psi(z) \in \widehat{E}_\xi$ for some $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

Definition

The transformed Hermitian metric $\widehat{h}$ is defined on the fiber $\widehat{E}_\xi$ by the formula

$$\widehat{h}(\varphi, \psi) = \int_{\mathbf{C}} h(\varphi(z), \psi(z)).$$

The transformed flat connection

L^2-metric on sections of $\Omega^1 \otimes E$ induces an orthogonal projection

$$\pi_\xi : L^2(\mathbf{P}^1, \Omega^1 \otimes E) \longrightarrow \widehat{E}_\xi.$$

Fix $\xi_0 \in \widehat{\mathbf{C}} \setminus \widehat{P}$ and let $\varphi_1(z), \ldots, \varphi_{r'}(z)$ be a basis of $\widehat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all ξ sufficiently close to ξ_0 in $\widehat{\mathbf{C}} \setminus \widehat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi - \xi_0)z} \varphi_j(z)) \in \widehat{E}_\xi.$$

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Definition

The transformed flat connection $\widehat{D}$ on $\widehat{E}$ is defined by the basis of local parallel sections $\varphi_j(\xi; z)$ for $j \in \{1, \dots, r'\}$.

Metric extension

Definition

The metric extension of $\widehat{E}$ over $\xi_l \in \widehat{P}$ (respectively $\widehat{\infty}$) is the lattice consisting of local holomorphic sections outside of ξ_l (respectively $\widehat{\infty}$) whose $\widehat{h}$-norm is bounded from above by a constant.

Properties of the transform

Theorem

- *$\widehat{D}$ is an integrable connection on $\widehat{E}$, with logarithmic singularities in $\xi_l \in \widehat{P}$ and an irregular singularity of Poincaré-rank 1 at $\widehat{\infty}$.*
- *The metric extension induces a parabolic structure on $\widehat{E}$ at the singular points.*
- *The corresponding eigenvalues and parabolic weights transform according to the diagrams on the next two slides. In particular, $\widehat{E}$ is of rank $\sum_{j=1}^{n}(r - r_j)$ and of parabolic degree 0.*
- *The metric $\widehat{h}$ is harmonic for $\widehat{D}$.*
- *Nahm transform is involutive (up to a sign), and it induces a hyper-Kähler isomorphism between moduli spaces.*

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n
$\xi_1 + z^{-1}\mu_1^0$	0		0
$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$
$\vdots$	0		$\vdots$	$\vdots$
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu_1^0$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$		0
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0		$\mu^0_{1+a_{n'}}$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$		μ^0_r

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu_1^0$	0		0	$-p_1 + \zeta^{-1}\mu_{r_1+1}^1$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu_r^1$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$	$\vdots$	0		$\mu_{1+a_{n'}}^0$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$	$-p_n + \zeta^{-1}\mu_{r_n+1}^n$	μ_1^0		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n	$-p_n + \zeta^{-1}\mu_r^n$	$\mu_{a_2}^0$		μ_r^0

Transform of the weights

∞	p_1	$\cdots$	p_n
β_1^0	0		0
$\vdots$	0		$\vdots$
$\beta_{a_2}^0$	$\vdots$		0
$\vdots$	0		$\beta_{r_n+1}^n$
$\beta_{1+a_{n'}}^0$	$\beta_{r_1+1}^1$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
β_r^0	β_r^1		β_r^n

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
β^0_1	0		0	$\beta^1_{r_1+1}$
$\vdots$	0		$\vdots$	$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
β^0_1	0		0	$\beta^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
— — —	— — —		— — —	— — —	— — —		— — —
β^0_1	0		0	$\beta^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0		$\beta^0_{1+a_{n'}}$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$		β^0_r

The multiplicative Deligne-Simpson problem

For $j = 0, \ldots, n$, fix conjugacy classes C_j in $\mathrm{Sl}(r, \mathbf{C})$.
Let γ_j denote a positively oriented circle around the point p_j.

Problem (Deligne-Simpson)

Does there exist a representation ρ of $\pi_1(\mathbf{P}^1 \setminus P)$ in $\mathrm{Sl}(r, \mathbf{C})$ such that $\rho(\gamma_j) \in C_j$?

That is, we are looking for matrices $M_j \in C_j$ such that

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W. Crawley-Boevey obtained very general results about this problem using Kac-Moody algebras. Here we give an alternative approach, using N. Katz' middle convolution algorithm. This is ongoing joint work with Olivier Biquard.

Additive version

Fact: the monodromy around p_j of $d + \frac{A_j}{z-p_j}dz$ is $\exp(2i\pi A_j)$.

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For $j \in \{1, \dots n\}$ let $c_j \subset \mathfrak{sl}(r, \mathbf{C})$ be an arbitrary logarithm of $C_j / 2i\pi$. Suppose we have a connection on the trivial holomorphic vector bundle

$$D = d + \sum_{j=1}^{n} \frac{A_j}{z - p_j} dz$$

with $A_j \in c_j$. Then, D is logarithmic at infinity, and the residue at infinity is

$$A_0 = -A_1 - \cdots - A_n.$$

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In particular, the monodromy representation ρ_D corresponding to D satisfies the requirements of the D-S problem if and only if

$$\exp(-2i\pi(A_1 + \dots + A_n)) \in C_j.$$

Choice of eigenvalues and tensoring

For $j = 0, \dots, n$, let c_j be adjoint orbits in $\mathfrak{sl}(r, \mathbf{C})$. Let

$$l_j = \min\{\text{rank}(A_j - \mu \mathbf{1}) | \quad A_j \in c_j \quad \text{and} \quad \mu \in \mathbf{C}\},$$

and for all $j > 0$ pick a solution μ_1^j of the minimization problem; it is one of the eigenvalues of A_j appearing with highest multiplicity. In particular, the set of possible choices for μ_1^j only depends on c_j.

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$$A_0 + \cdots + A_n = 0.$$

For all $j > 0$ set

$$A_j' = A_j - \mu_1^j \mathbf{1},$$

then A_j' is of rank $l_j = r - r_j$. Setting $A_0' = A_0 + \sum_{j=1}^n \mu_1^j \mathbf{1}$, we obtain a new solution $(A_0', \dots, A_n')$ of the problem (for different conjugacy classes c_j').

Rank of the transform

Obviously,

$$\mathrm{rank}(A_j') \leq \mathrm{rank}(A_j),$$

so

$$\sum_{j=1}^{n} \mathrm{rank}(A_j') \leq \sum_{j=1}^{n} \mathrm{rank}(A_j).$$

In particular, the rank of the transform of

$$d + \sum_{j=1}^{n} \frac{A_j'}{z - p_j}$$

is smaller than that of

$$d + \sum_{j=1}^{n} \frac{A_j}{z - p_j}.$$

Transforming forth and back

Apply Nahm transform to $d + \sum_{j=1}^{n} \frac{A'_j}{z - p_j}$: get a connection $\widehat{D}$ with a logarithmic singularity at $\widehat{0}$ and a Poincaré rank 1 singularity at $\widehat{\infty}$.

Now repeat the procedure: pick an eigenvalue $\hat{\mu}_1^1$ of the residue at $\widehat{0}$ of maximal multiplicity, subtract $\hat{\mu}_1^1 \frac{d\xi}{\xi}$, and transform $\widehat{D} - \hat{\mu}_1^1 \frac{d\xi}{\xi}$ back to $\mathbf{P}^1$.

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Keep iterating this algorithm until one of the following possibilities occur:

- The rank is small enough to directly prove existence.
- Get a contradiction (e.g., formula for rank becomes negative).

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- Study the middle convolution algorithm.
- Generalize to various other structure groups.
- Other non-compact curves.