Nahm transform for parabolic integrable connections on the Riemann sphere

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Outline

1. Introduction
   - Definitions
   - Polar parts
   - Parabolic structure
   - Hermitian metrics

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Notations

$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$P = \{p_0 = \infty, p_1, \ldots, p_n\}$: a finite set of distinct points in $\mathbf{P}^1$
$\mathcal{O}$: sheaf of holomorphic functions
$\mathbf{\Omega}^1$: sheaf of holomorphic 1-forms
For any sheaf $\mathcal{S}$, denote by $\mathcal{S}(k \cdot P)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of order at most k in P, and by $\mathcal{S}(*P)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order in P.

Meromorphic connections

A meromorphic connection on $\mathbf{P}^1$ with singularities in P is a couple (E, D), where E is a holomorphic bundle on $\mathbf{P}^1$, and

$$D : E \longrightarrow \mathbf{\Omega}^1(*P) \otimes_{\mathcal{O}} E$$

is a sheaf map satisfying the Leinbiz-rule: for any open set $U \subset \mathbf{P}^1$, any $f \in \Gamma(U, \mathcal{O})$ and any $e \in \Gamma(U, E)$ one has

$$D(fe) = (df)e + f(De).$$

Let

$$r = \operatorname{rank}(E).$$

Assumption on points at finite distance

D is supposed to have a logarithmic singularity at p_j for $j \in \{1, \dots, n\}$: in a local trivialisation of E near p_j, one has

$$D = d + \frac{A^j(z)}{z - p_j},$$

where A^j is a holomorphic matrix-valued function defined near p_j. Furthermore, the residue

$$A^j(p_j) = \mathrm{diag}(0, \dots, 0, \mu^j_{r_j+1}, \dots, \mu^j_r),$$

is diagonal, with μ^j_k non-zero and generic.

Assumption at infinity

D is supposed to have an irregular singularity of Poincaré-rank 1 at infinity: in a local trivialisation of E near ∞, one has

$$D = d + Adz + B\frac{dz}{z} + \text{lower order terms},$$

where

$$A = \mathrm{diag}(\xi_1, \ldots, \xi_1, \ldots\ldots\ldots, \xi_{n'}, \ldots.., \xi_{n'})$$
$$B = \mathrm{diag}(\mu_1^0, \ldots, \mu_{a_2}^0, \ldots\ldots, \mu_{1+a_{n'}}^0, \ldots, \mu_r^0)$$

(the leading order term and residue, respectively). Here the ξ_k are pairwise distinct constants, and the μ_l^0 are generic non-zero. (Notation:$a_1 = 0, a_{n'+1} = r$.)

The standard example

Let E be the trivial holomorphic vector bundle of rank r over $\mathbf{P}^1$ and A, B be $r \times r$ matrices such that A is diagonal with distinct eigenvalues and B is semi-simple without any 0's on the diagonal. Then, the connection

$$D = d + Adz + B\frac{dz}{z}$$

has the desired properties.

Parabolic structure at logarithmic points

At each logarithmic point $p = p_j$ $(1 \leq j \leq n)$, let us fix a diagonalising trivialisation $\{\tau_1^j, \ldots, \tau_r^j\}$. We suppose a compatible parabolic structure is given:

- a filtration

$$E|_p = F_0 E|_p \supset F_1 E|_p \supset \cdots \supset F_{r-r_j} E|_p \supset F_{r-r_j+1} E|_p = 0,$$

such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau_{r_j+k}^j, \ldots, \tau_r^j \rangle$$

(the parabolic filtration),

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  such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau^j_{r_j+k}, \ldots, \tau^j_r \rangle$$

  (the parabolic filtration),

- an r-uple of real numbers

$$0 = \beta_1^j = \cdots = \beta_{r_j}^j < \beta_{r_j+1}^j < \cdots < \beta_r^j < 1$$

  (the parabolic weights).

Parabolic structure at infinity

At infinity, let us fix a diagonalising trivialisation $\{\tau_1^0, \ldots, \tau_r^0\}$. The parabolic structure is given by:

- a full flag

$$E|_\infty = F_0 E|_\infty \supset F_1 E|_\infty \supset \cdots \supset F_r E|_\infty = 0,$$

such that for all $k > 0$

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  such that for all $k > 0$

$$F_k E|_\infty = \mathbf{C}\langle \tau_{k+1}^0, \ldots, \tau_r^0 \rangle,$$

- weights

$$0 < \beta_1^0 < \cdots < \beta_r^0 < 1.$$

Remark

These are not the general definitions of a parabolic structure, but instead our assumptions.

Stability

The parabolic degree and slope of E are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta_k^j$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$

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(E, D) is said to be parabolically stable if for all non-trivial proper subbundle $F \subset E$ such that $D|_F \subset \mathbf{\Omega}^1(*P) \otimes F$, one has

$$\text{par-slope}(F) < \text{par-slope}(E).$$

Adapted Hermitian metrics

A Hermitian fiber metric h is adapted to the parabolic structure if near all $p_j \in \mathbf{C}$ in the trivialisation $\{\tau_1^j, \ldots, \tau_r^j\}$ it is mutually bounded with

$$\operatorname{diag}(|z - p_j|^{2\beta_k^j})_{k=1,\ldots,r},$$

and near ∞ in the trivialisation $\{\tau_1^0, \ldots, \tau_r^0\}$ it is mutually bounded with

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Remark

Without the semi-simplicity assumption on the residues, the form of the matrices in the definition is more complicated, involving logarithmic terms.

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Decompose

$$D = D^+ + \Phi$$

into h-unitary and self-adjoint parts respectively.

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$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

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$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
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Then, h is said to be harmonic if

$$\bar{\partial}^+ \theta = 0.$$

Non-abelian Hodge theory

Theorem (C. Sabbah 1999, O. Biquard – P. Boalch 2004)

Let (E, D) be a parabolically stable meromorphic integrable connection of parabolic degree 0. Then, there exists a unique adapted harmonic metric h (up to a constant). Furthermore, the moduli space $\mathcal{M}$ of parabolically stable connections of parabolic degree 0 with prescribed singularity data up to holomorphic gauge transformations is a complete hyper-Kähler manifold.

Non-abelian Hodge theory

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From now on, (E, D) is supposed to be a parabolically stable meromorphic integrable connection of parabolic degree 0, and h its harmonic Hermitian metric.

By definition, to this data there is an associated Higgs bundle $(\mathcal{E}, \theta)$, with Hermitian-Einstein metric h.

Definitions

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.
For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$

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The positive and negative spinor bundles are defined respectively as

$$S^+ = \Omega^0(\mathbf{C} \setminus P) \oplus \Omega^2(\mathbf{C} \setminus P)$$
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The twisted Dirac operators are

$$\not{\partial}_\xi : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E)$$
$$\not{\partial}_\xi = D_\xi - D_\xi^*,$$

where D_ξ^* is the adjoint operator of D_ξ with respect to h.

Fredholm theory

Introduce on $\mathcal{C}_0^\infty(\mathbf{C} \setminus P, S^\pm \otimes E)$ the norm

$$\|f\|_{H^1}^2 = \int_{\mathbf{C}} \left(|f|^2 + |D^+ f|^2 + |\Phi f|^2\right) |dz|^2,$$

where $|.|$ is computed with respect to the harmonic metric h and the standard Euclidean metric $|dz|^2$ on $\mathbf{C}$.
Let H^1 be the completion of $\mathcal{C}_0^\infty(\mathbf{C} \setminus P)$ with respect to this norm.

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Theorem

For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, the twisted Dirac operator

$$\not{\partial}_\xi : H^1(S^+ \otimes E) \longrightarrow L^2(S^- \otimes E)$$

is Fredholm, with vanishing kernel.

Transformed vector bundle – first definition

The vector spaces

$$\operatorname{coker}(\not{\partial}_\xi)$$

form a smooth family of finite-dimensional subspaces of $L^2(S^- \otimes E)$ of the same dimension.

Definition

The smooth vector bundle with fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ equal to $\operatorname{coker}(\not{\partial}_\xi)$ is called the transformed smooth vector bundle. We denote it by $\widehat{E}$.

Hodge theory

Denote by

$$\not{\partial}_\xi^* : S^- \otimes E \longrightarrow S^+ \otimes E$$

the adjoint twisted Dirac operator, and by

$$\Delta_\xi = \not{\partial}_\xi \circ \not{\partial}_\xi^* : S^- \otimes E \longrightarrow S^- \otimes E$$

the twisted Dirac Laplacian.

Theorem

The space $\widehat{E}_\xi$ is isomorphic to the L^2-kernel of Δ_ξ.

The transformed connection and metric

Consider the trivial Hilbert bundle

$$L^2(\mathbf{P}^1, S^- \otimes E)$$

over $\widehat{\mathbf{C}} \setminus \widehat{P}$ with its trivial connection $\widehat{d}$ with respect to ξ. Let

$$\iota_\xi : \widehat{E}_\xi \hookrightarrow L^2(\mathbf{P}^1, S^- \otimes E)$$

be the canonical injection, and

$$\pi_\xi : L^2(\mathbf{P}^1, S^- \otimes E) \longrightarrow \widehat{E}_\xi$$

the L^2-othogonal projection.

Definition

The transformed flat connection $\widehat{D}$ on $\widehat{E}$ is defined by the formula

$$\widehat{D} = \pi_\xi \circ (\widehat{d} - z d\xi) \circ \iota_\xi$$

on the fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

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on the fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

Let $\varphi(z), \psi(z) \in \widehat{E}_\xi$ for some $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

Definition

The transformed Hermitian metric $\widehat{h}$ is defined on the fiber $\widehat{E}_\xi$ by the formula

$$\widehat{h}(\varphi, \psi) = \int_{\mathbf{C}} h(\varphi(z), \psi(z)).$$

Metric extension

Definition

The metric extension of $\widehat{E}$ over $\xi_l \in \widehat{P}$ (respectively $\widehat{\infty}$) is the lattice consisting of local holomorphic sections outside of ξ_l (respectively $\widehat{\infty}$) whose $\widehat{h}$-norm is bounded from above by a constant.

Main results

Theorem

- $\widehat{D}$ *is an integrable connection on* $\widehat{E}$*, with logarithmic singularities in* $\xi_l \in \widehat{P}$ *and an irregular singularity of Poincaré-rank* 1 *at* $\widehat{\infty}$*.*
- *The metric extension induces a parabolic structure on* $\widehat{E}$ *at the singular points.*
- *The corresponding eigenvalues and parabolic weights transform according to the diagrams on the next two slides. In particular,* $\widehat{E}$ *is of rank* $\sum_{j=1}^{n}(r - r_j)$ *and of parabolic degree* 0*.*
- *The metric* $\widehat{h}$ *is harmonic for* $\widehat{D}$*.*

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n
$\xi_1 + z^{-1}\mu_1^0$	0		0
$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$
	0			
$\vdots$			$\vdots$	$\vdots$
	$\vdots$			
$\xi_1 + z^{-1}\mu^0_{a_2}$			0	$-p_1 + \zeta^{-1}\mu^1_{r}$
	0		$\mu^n_{r_n+1}$	
$\vdots$				$\vdots$
	$\mu^1_{r_1+1}$		$\vdots$	
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$				$-p_n + \zeta^{-1}\mu^n_{r_n+1}$
	$\vdots$		$\vdots$	
$\vdots$				$\vdots$
	μ^1_r		μ^n_r	
$\xi_{n'} + z^{-1}\mu^0_r$				$-p_n + \zeta^{-1}\mu^n_r$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$

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∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$		0
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0		$\mu^0_{1+a_{n'}}$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$		μ^0_r

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu_1^0$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$		0
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0		$\mu^0_{1+a_{n'}}$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$		μ^0_r

Transform of the weights

∞	p_1	$\cdots$	p_n
β^0_1	0		0
$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
β^0_r	β^1_r		β^n_r

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
β^0_1	0		0	$\beta^1_{r_1+1}$
$\vdots$	0		$\vdots$	$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
— — —	— — —	— —	— — —	— — —	— — —
β^0_1	0		0	$\beta^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
β^0_1	0		0	$\beta^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0		$\beta^0_{1+a_{n'}}$
	$\beta^1_{r_1+1}$		$\vdots$		β^0_1		$\vdots$
$\beta^0_{1+a_{n'}}$				$\beta^n_{r_n+1}$			
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$		β^0_r

Let $(\mathcal{E}, \theta)$ be the Higgs bundle corresponding to (E, D) via non-abelian Hodge theory.

Define the following holomorphic bundle $\mathcal{E} \otimes \mathcal{P}$ on $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$:

- As a smooth bundle, $\mathcal{E} \otimes \mathcal{P}$ is isomorphic to $\pi_2^* \mathcal{E}$,
- With holomorphic structure operator given by

$$\pi_2^* \bar{\partial}^{\mathcal{E}} + \frac{\bar{\xi}}{2} d\bar{z} + \frac{\bar{z}}{2} d\bar{\xi}.$$

Consider the sheaf map

$$\Theta : \mathcal{E} \otimes \mathcal{P} \longrightarrow \mathcal{E}(P + 2\infty) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1} \otimes \mathcal{P}$$

such that

$$\Theta = (\theta - \frac{\xi}{2} dz) \otimes \mathbf{1}_{\mathcal{P}_\xi}$$

on $\mathbf{P}^1 \times \{\xi\}$.

Theorem

The first hypercohomology space of the complex

$$\theta - \frac{\xi}{2}dz : \mathcal{E} \longrightarrow \mathcal{E}(P + 2\infty) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}$$

is finite dimensional for all ξ*. The derived direct image*

$$\mathbf{R}_1(\pi_2)_*(\Theta)$$

defines a holomorphic vector bundle on $\widehat{\mathbf{P}}^1$*. The map*

$$-\frac{z}{2}d\xi$$

descends to a holomorphic endomorphism.

Denote the obtained Higgs bundle by $(\widehat{\mathcal{E}}, \widehat{\theta})$.

Theorem

The Higgs bundle corresponding to $(\widehat{E}, \widehat{D})$ via non-abelian Hodge theory is $(\widehat{\mathcal{E}}, \widehat{\theta})$.

Definition

The variety in $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$ defined by $\det(\Theta)$ is called the spectral curve, and is denoted by Σ. The sheaf $\operatorname{coker}(\Theta)$ is called the spectral sheaf, denoted M.

M is clearly supported on Σ.

Theorem

The bundle $\widehat{\mathcal{E}}$ is equal to $(\pi_2)_* M$, and $\widehat{\theta}$ is induced by multiplication by $-\frac{z}{2} d\xi$ on the fibers.

Remark

A Higgs bundle on the line is naturally a $\mathbf{C}[z,\theta]$*-module. The previous theorem says that on such a module, Nahm transform acts as:*

$$z \mapsto -\widehat{\theta}$$
$$\theta \mapsto \xi.$$

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A Higgs bundle on the line is naturally a $\mathbf{C}[z,\theta]$-module. The previous theorem says that on such a module, Nahm transform acts as:

$$z \mapsto -\widehat{\theta}$$
$$\theta \mapsto \xi.$$

This is therefore a commutative analog of Fourier(-Laplace) transform, acting on real functions of 1 variable by

$$x \mapsto -\partial_\xi$$
$$\partial_x \mapsto \xi.$$

The map between moduli spaces

This is joint work with K. Aker.

Lemma

If (E, D) is stable of degree 0, then so is $(\widehat{E}, \widehat{D})$.

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Lemma

If (E, D) is stable of degree 0, then so is $(\widehat{E}, \widehat{D})$.

Therefore, Nahm transform defines a map on corresponding moduli spaces:

$$\mathcal{N} : \mathcal{M} \longrightarrow \widehat{\mathcal{M}}.$$

Theorem

$\mathcal{N}$ is a hyper-Kähler isomorphism.

Sketch of the proof

From the spectral interpretation, we see that

$$\mathcal{N}^2 = (-1)^*,$$

where (-1) is the "opposite" map on $\mathbf{C}$. Therefore, $\mathcal{N}$ is invertible, with inverse equal to $(-1)^* \circ \mathcal{N}$.
It also follows that $\mathcal{N}$ respects the complex structure I corresponding to the Higgs bundle point of view on $\mathcal{M}$.
For compatibility with the complex structure J corresponding to the integrable connection point of view, there exists a similar algebraic interpretation.
Finally, compatibility with the Riemannian metrics is a computation. □

Kostov's condition

For $j = 0, \dots, n$, let c_j be conjugacy classes in $Sl(r, \mathbf{C})$. Denote by l_j the smallest rank of a matrix $A_j - \mu$, with $A_j \in c_j$ and $\mu \in \mathbf{C}$. We say that an $(n+1)$-tuple of matrices A_j is irreducible if it admits no non-trivial invariant subspace.

Theorem (V. Kostov, 1994)

Suppose one of the conjugacy classes c_0 is regular semi-simple, and the conjugacy classes $c_0, \dots, c_n$ have generic eigenvalues. Then, there exists an irreducible $(n+1)$-tuple of matrices $A_j \in c_j$ such that

$$A_0 + \dots + A_n = 0$$

if and only if $\sum \dim(C_j) \geq 2n^2 - 2$ *and* $l_1 + \dots + l_n \geq r$.

Alternative proof for necessity

Proof(necessity)

This is work in progress, joint with O. Biquard.
We assume that all the classes c_j are semi-simple. Suppose such $(A_0, \dots, A_n)$ exist. For each $j \in \{1, \dots, n\}$, choose μ_1^j to be a solution of the minimization problem of $\text{rank}(A_j - \mu)$; it is one of the eigenvalues of A_j appearing with highest multiplicity. Then, the matrices

$$A_j' = A_j - \mu_1^j \mathbf{1}$$

are of rank $l_j = r - r_j$. Setting $A_0' = A_0 + \sum_{j=1}^n \mu_1^j \mathbf{1}$, we obtain a new solution $(A_0', \dots, A_n')$ of the problem (for different conjugacy classes c_j').

Proof(continued)

Think of A'_0 as the residue of an integrable connection with only first-order poles at infinity (i.e., all $\xi_k = 0$), and apply Nahm transform. Then, the transformed integrable connection has poles only at 0 and infinity. The rank of the transformed bundle is equal to $\sum_{j=1}^{n} l_j$. On the other hand, the rank of the residue at 0 is equal to r. Hence,

$$\sum_{j=1}^{n} l_j \geq r.$$

On the other hand, the dimension of the Zariski tangent space of the moduli of such tuples of matrices up to simultaneous conjugation is $\sum \dim(C_j) - 2n^2 + 2$, so if the eigenvalues are chosen generically then this number has to be non-negative. □

What's next?

- Extension of the transform to non semi-simple residues.
- Generalisation to higher-order poles.
- Generalisation to various other structure groups.
- Study of the middle convolution algorithm.

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	$\widehat{0}$
$z^{-1}\mu_1^0$	0		0	$-p_1+\zeta^{-1}\mu^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\vdots$	$\vdots$		0	$-p_1+\zeta^{-1}\mu^1_{r}$	$\vdots$
	0		$\mu^n_{r_n+1}$	$\vdots$	0
$\vdots$	$\mu^1_{r_1+1}$		$\vdots$		μ_1^0
$z^{-1}\mu_r^0$				$-p_n+\zeta^{-1}\mu^n_{r_n+1}$	
	$\vdots$		$\vdots$	$\vdots$	$\vdots$
	μ_r^1		μ_r^n		μ_r^0
				$-p_n+\zeta^{-1}\mu^n_{r}$	