Nahm transform for parabolic integrable connections on the Riemann sphere

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Outline

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Notations

$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$P = \{p_0 = \infty, p_1, \dots, p_n\}$: a finite set of distinct points in $\mathbf{P}^1$
$\mathcal{O}$: sheaf of holomorphic functions
$\mathbf{\Omega}^1$: sheaf of holomorphic 1-forms
For any sheaf $\mathcal{S}$, denote by $\mathcal{S}(k \cdot P)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of order at most k in P, and by $\mathcal{S}(*P)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order in P.

Meromorphic connections

Let E be a holomorphic vector bundle of rank r on $\mathbf{P}^1$ and D a meromorphic connection on E with singularities in P:

$$D : E \longrightarrow \boldsymbol{\Omega}^1(*P) \otimes_{\mathcal{O}} E$$

We fix the behaviours of D near the singular points as follows.

Assumption on points at finite distance

D is supposed to have a logarithmic singularity at p_j for $j \in \{1, \ldots, n\}$: in a local trivialisation of E near p_j, one has

$$D = d + \frac{A^j(z)}{z - p_j},$$

where A^j is a holomorphic matrix-valued function defined near p_j. Furthermore, the residue

$$A^j(p_j) = \mathrm{diag}(0, \ldots, 0, \mu^j_{r_j+1}, \ldots, \mu^j_r),$$

is diagonal, with μ^j_k non-zero and generic.

Assumption at infinity

D is supposed to have an irregular singularity of Poincaré-rank 1 at infinity: in a local trivialisation of E near ∞, one has

$$D = d + Adz + B\frac{dz}{z} + \text{lower order terms},$$

where

$$A = \operatorname{diag}(\xi_1, \ldots, \xi_1, \ldots\ldots\ldots, \xi_{n'}, \ldots.., \xi_{n'})$$
$$B = \operatorname{diag}(\mu_1^0, \ldots, \mu_{a_2}^0, \ldots\ldots, \mu_{1+a_{n'}}^0, \ldots, \mu_r^0)$$

(the leading order term and residue, respectively). Here the ξ_k are pairwise distinct constants, and the μ_l^0 are generic non-zero. (Notation:$a_1 = 0, a_{n'+1} = r$.)

The standard example

Let E be the trivial holomorphic vector bundle of rank r over $\mathbf{P}^1$ and A, B be $r \times r$ matrices such that A is diagonal with distinct eigenvalues and B is semi-simple without any 0's on the diagonal. Then, the connection

$$D = d + Adz + B\frac{dz}{z}$$

has the desired properties.

Parabolic structure at logarithmic points

At each logarithmic point $p = p_j$ $(1 \leq j \leq n)$, let us fix a diagonalising trivialisation $\{\tau_1^j, \ldots, \tau_r^j\}$. We suppose a compatible parabolic structure is given:

- a filtration

$$E|_p = F_0 E|_p \supset F_1 E|_p \supset \cdots \supset F_{r-r_j} E|_p \supset F_{r-r_j+1} E|_p = 0,$$

such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau^j_{r_j+k}, \ldots, \tau^j_r \rangle$$

(the parabolic filtration),

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  such that for all $k > 0$

$$F_k E|_p = \mathbf{C}\langle \tau^j_{r_j+k}, \ldots, \tau^j_r \rangle$$

  (the parabolic filtration),
- an r-uple of real numbers

$$0 = \beta_1^j = \cdots = \beta_{r_j}^j < \beta_{r_j+1}^j < \cdots < \beta_r^j < 1$$

  (the parabolic weights).

Parabolic structure at infinity

At infinity, let us fix a diagonalising trivialisation $\{\tau_1^0, \dots, \tau_r^0\}$. The parabolic structure is given by:

- a full flag

$$E|_\infty = F_0 E|_\infty \supset F_1 E|_\infty \supset \cdots \supset F_r E|_\infty = 0,$$

  such that for all $k > 0$

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- weights

$$0 < \beta_1^0 < \cdots < \beta_r^0 < 1.$$

Stability

The parabolic degree and slope of E are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta_k^j$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$

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(E, D) is said to be parabolically stable if for all non-trivial proper subbundle $F \subset E$ such that $D|_F \subset \mathbf{\Omega}^1(*P) \otimes F$, one has

$$\text{par-slope}(F) < \text{par-slope}(E).$$

Adapted Hermitian metrics

A Hermitian fiber metric h is adapted to the parabolic structure if near all $p_j \in \mathbf{C}$ in the trivialisation $\{\tau_1^j, \ldots, \tau_r^j\}$ it is mutually bounded with

$$\mathrm{diag}(|z - p_j|^{2\beta_k^j})_{k=1,\ldots,r},$$

and near ∞ in the trivialisation $\{\tau_1^0, \ldots, \tau_r^0\}$ it is mutually bounded with

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Remark

Without the semi-simplicity assumption on the residues, the form of the matrices in the definition is more complicated, involving logarithmic terms.

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Decompose

$$D = D^+ + \Phi$$

into h-unitary and self-adjoint parts respectively.

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Decompose these parts further according to bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

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Decompose these parts further according to bidegree:

$$\Omega^{1} = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^{+} = \partial^{+} + \bar{\partial}^{+}$$
$$\Phi = \theta + \theta^{*}.$$

Then, h is said to be harmonic if

$$\bar{\partial}^{+}\theta = 0.$$

Non-abelian Hodge theory

Theorem (C. Sabbah 1999, O. Biquard – P. Boalch 2004)

Let (E, D) be a parabolically stable meromorphic integrable connection of parabolic degree 0. Then, there exists a unique adapted harmonic metric h (up to a constant). Furthermore, the moduli space $\mathcal{M}$ of parabolically stable connections of parabolic degree 0 with prescribed singularity data up to holomorphic gauge transformations is a complete hyper-Kähler manifold.

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From now on, (E, D) is supposed to be a parabolically stable meromorphic integrable connection of parabolic degree 0 with the given singular parts, and h denotes its harmonic Hermitian metric. By definition, to this data there is an associated Higgs bundle $(\mathcal{E}, \theta)$, with Hermitian-Einstein metric h.

Definitions

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.
For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$

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The positive and negative spinor bundles are defined respectively as

$$S^+ = \Omega^0(\mathbf{C} \setminus P) \oplus \Omega^2(\mathbf{C} \setminus P)$$
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$$S^+ = \Omega^0(\mathbf{C} \setminus P) \oplus \Omega^2(\mathbf{C} \setminus P)$$
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The twisted Dirac operators are

$$\partial\!\!\!/_\xi : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E)$$
$$\partial\!\!\!/_\xi = D_\xi - D_\xi^*,$$

where D_ξ^* is the adjoint operator of D_ξ with respect to h.

Fredholm theory

Introduce on $\mathcal{C}_0^\infty(\mathbf{C} \setminus P, S^\pm \otimes E)$ the norm

$$\|f\|_{H^1}^2 = \int_{\mathbf{C}} \left(|f|^2 + |D^+ f|^2 + |\Phi f|^2\right) |dz|^2,$$

where $|.|$ is computed with respect to the harmonic metric h and the standard Euclidean metric $|dz|^2$ on $\mathbf{C}$.
Let H^1 be the completion of $\mathcal{C}_0^\infty(\mathbf{C} \setminus P)$ with respect to this norm.

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Theorem

For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, the twisted Dirac operator

$$\not{\partial}_\xi : H^1(S^+ \otimes E) \longrightarrow L^2(S^- \otimes E)$$

is Fredholm, with vanishing kernel.

Transformed vector bundle – first definition

The vector spaces

$$\mathrm{coker}(\not{\partial}_\xi)$$

form a smooth family of finite-dimensional subspaces of $L^2(S^- \otimes E)$ of the same dimension.

Definition

The smooth vector bundle with fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ equal to $\mathrm{coker}(\not{\partial}_\xi)$ is called the transformed smooth vector bundle. We denote it by $\widehat{E}$.

Hodge theory

Denote by

$$\not{\partial}_\xi^* : S^- \otimes E \longrightarrow S^+ \otimes E$$

the adjoint twisted Dirac operator, and by

$$\Delta_\xi = \not{\partial}_\xi \circ \not{\partial}_\xi^* : S^- \otimes E \longrightarrow S^- \otimes E$$

the twisted Dirac Laplacian.

Theorem

The space $\widehat{E}_\xi$ is isomorphic to the L^2-kernel of Δ_ξ.

From now on, whenever we speak of an element of $\widehat{E}_\xi$, we will think of this harmonic representative.

The transformed metric and connection

Let $\varphi(z), \psi(z) \in \widehat{E}_\xi$ for some $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

Definition

The transformed Hermitian metric $\widehat{h}$ is defined on the fiber $\widehat{E}_\xi$ by the formula

$$\widehat{h}(\varphi, \psi) = \int_{\mathbf{C}} h(\varphi(z), \psi(z)).$$

This is of course defined for arbitrary sections of E, and hence induces an orthogonal projection

$$\pi_\xi : L^2(\mathbf{P}^1, S^- \otimes E) \longrightarrow \widehat{E}_\xi.$$

The transformed flat connection

Fix $\xi_0 \in \widehat{\mathbf{C}} \setminus \widehat{P}$ and let

$$\varphi_1(z), \ldots, \varphi_{r'}(z)$$

be a basis of $\widehat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all ξ sufficiently close to ξ_0 in $\widehat{\mathbf{C}} \setminus \widehat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi-\xi_0)z}\varphi_j(z)) \in \widehat{E}_\xi.$$

Definition

The transformed flat connection $\widehat{D}$ on $\widehat{E}$ is defined by the basis of local parallel sections $\varphi_j(\xi; z)$ for $j \in \{1, \ldots, r'\}$.

Metric extension

Definition

The metric extension of $\widehat{E}$ over $\xi_l \in \widehat{P}$ (respectively $\widehat{\infty}$) is the lattice consisting of local holomorphic sections outside of ξ_l (respectively $\widehat{\infty}$) whose $\widehat{h}$-norm is bounded from above by a constant.

Analytic transform

Theorem

- *$\widehat{D}$ is an integrable connection on $\widehat{E}$, with logarithmic singularities in $\xi_l \in \widehat{P}$ and an irregular singularity of Poincaré-rank 1 at $\widehat{\infty}$.*
- *The metric extension induces a parabolic structure on $\widehat{E}$ at the singular points.*
- *The corresponding eigenvalues and parabolic weights transform according to the diagrams on the next two slides. In particular, $\widehat{E}$ is of rank $\sum_{j=1}^{n}(r - r_j)$ and of parabolic degree 0.*
- *The metric $\widehat{h}$ is harmonic for $\widehat{D}$.*

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n
$\xi_1 + z^{-1}\mu_1^0$	0		0
$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$
	0			
$\vdots$			$\vdots$	$\vdots$
	$\vdots$			
$\xi_1 + z^{-1}\mu^0_{a_2}$			0	$-p_1 + \zeta^{-1}\mu^1_r$
	0		$\mu^n_{r_n+1}$	
$\vdots$				$\vdots$
	$\mu^1_{r_1+1}$		$\vdots$	
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$				$-p_n + \zeta^{-1}\mu^n_{r_n+1}$
	$\vdots$		$\vdots$	
$\vdots$				$\vdots$
	μ^1_r		μ^n_r	
$\xi_{n'} + z^{-1}\mu^0_r$				$-p_n + \zeta^{-1}\mu^n_r$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$

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∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu^0_{a_2}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$		0
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0		$\mu^0_{1+a_{n'}}$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	μ^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\mu^0_{a_2}$		μ^0_r

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
	0				0		
$\vdots$			$\vdots$	$\vdots$			$\vdots$
	$\vdots$				$\vdots$		
$\xi_1 + z^{-1}\mu^0_{a_2}$			0	$-p_1 + \zeta^{-1}\mu^1_r$			0
	0		$\mu^n_{r_n+1}$		0		$\mu^0_{1+a_{n'}}$
$\vdots$				$\vdots$			
	$\mu^1_{r_1+1}$		$\vdots$		μ^0_1		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$				$-p_n + \zeta^{-1}\mu^n_{r_n+1}$			
	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$\vdots$				$\vdots$			
	μ^1_r		μ^n_r		$\mu^0_{a_2}$		
$\xi_{n'} + z^{-1}\mu^0_r$				$-p_n + \zeta^{-1}\mu^n_r$			μ^0_r

Transform of the weights

∞	p_1	$\cdots$	p_n
β^0_1	0		0
$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0
	0		$\beta^n_{r_n+1}$
$\vdots$	$\beta^1_{r_1+1}$		$\vdots$
$\beta^0_{1+a_{n'}}$	$\vdots$		$\vdots$
$\vdots$	β^1_r		β^n_r
β^0_r			

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
β^0_1	0		0	$\beta^1_{r_1+1}$
$\vdots$	0		$\vdots$	$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
β^0_1	0		0	$\beta^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$---$	$---$		$---$	$---$	$---$		$---$
β_1^0	0		0	$\beta^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0		$\beta^0_{1+a_{n'}}$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$		β^0_r

$\mathcal{D}$-modules from connections

$\mathcal{D}$: the sheaf of holomorphic differential operators on $\mathbf{P}^1$

$$\sum_i a_i(t)\partial_t^i,$$

with t a local holomorphic coordinate and a_i holomorphic functions.
A holomorphic bundle with an integrable connection (E, D) induces a holonomic $\mathcal{D}$-module on $\mathbf{C} \setminus P$ in the obvious way:

$$\partial_t \cdot e = D_{\partial_t}(e).$$

Minimal extension

Given a logarithmic connection, it is possible to define a minimal extension $\mathcal{M}_{\mathrm{min}}$ over the singularities, i.e. one without any submodule or quotient module supported in a point: it is the inductive limit of sections

$$D_{\partial_t}^k(e)$$

where e is a holomorphic section of E.
Under suitable conditions, a minimal extension can be found at the rank-one irregular singularity at infinity too.
Parabolic structures can be generalised for such $\mathcal{D}$-modules.

Weyl-algebra modules

$\mathbf{C}[x]\langle\partial_x\rangle$: the Weyl algebra in the variable x, algebra of polynomial differential operators over $\mathbf{C}$, generated by x and ∂_x and subject to the relation

$$[x, \partial_x] = -1.$$

Globalising the action of $\mathcal{D}$, the global sections of a $\mathcal{D}$-module $\mathcal{M}$ form a module M over $\mathbf{C}[x]\langle\partial_x\rangle$.

Laplace transform

Let ξ denote a variable dual to x, $\mathbf{C}[\xi]\langle\partial_\xi\rangle$: Weyl algebra in ξ.
We have a natural isomorphism

$$\mathbf{C}[x]\langle\partial_x\rangle \cong \mathbf{C}[\xi]\langle\partial_\xi\rangle$$

by setting

$$x \mapsto -\partial_\xi \quad \text{and} \quad \partial_x \mapsto \xi.$$

Given a module M over $\mathbf{C}[x]\langle\partial_x\rangle$, this isomorphism induces a $\mathbf{C}[\xi]\langle\partial_\xi\rangle$-module structure on M. We call this $\mathbf{C}[\xi]\langle\partial_\xi\rangle$-module the Laplace transform of M, denoted $\widehat{M}$.

Alternative description of Laplace transform

Denote by $\mathbf{C}[x,\xi]\langle\partial_x,\partial_\xi\rangle$ the Weyl algebra in the variables x and ξ. A $\mathbf{C}[x]\langle\partial_x\rangle$-module M induces canonically a $\mathbf{C}[x,\xi]\langle\partial_x,\partial_\xi\rangle$-module

$$\mathbf{M} = M \otimes_{\mathbf{C}} \mathbf{C}[\xi],$$

where ξ and ∂_ξ act trivially on M and canonically on $\mathbf{C}[\xi]$. The kernel of

$$\mathbf{M} \xrightarrow{\partial_x - \xi} \mathbf{M}$$

vanishes and its cokernel is in bijection with M. Moreover, the cokernel inherits a $\mathbf{C}[\xi]\langle\partial_\xi\rangle$-module structure, where the action of ∂_ξ is induced by $\partial_\xi - x$. This $\mathbf{C}[\xi]\langle\partial_\xi\rangle$-module structure on M is equal to $\widehat{M}$.

Nahm and Laplace

Theorem

Nahm transform is carried into Laplace transform under the minimal extension functor.

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Nahm transform is carried into Laplace transform under the minimal extension functor.

Idea of the proof

L^2-theory gives $\mathcal{N}$ an interpretation in terms of first hypercohomology of a sheaf map

$$D_\xi : E \to F.$$

Taking minimal extension leads to the interpretation of Laplace transform as a cokernel.

Twisting the Higgs bundle

Let $(\mathcal{E}, \theta)$ be the Higgs bundle corresponding to (E, D) via non-abelian Hodge theory.

Define the following holomorphic bundle $\mathcal{E} \otimes \mathcal{P}$ on $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$:

- As a smooth bundle, $\mathcal{E} \otimes \mathcal{P}$ is isomorphic to $\pi_2^* \mathcal{E}$,
- With holomorphic structure operator given by

$$\pi_2^* \bar{\partial}^{\mathcal{E}} + \frac{\bar{\xi}}{2} d\bar{z} + \frac{\bar{z}}{2} d\bar{\xi}.$$

Consider the sheaf map

$$\Theta : \mathcal{E} \otimes \mathcal{P} \longrightarrow \mathcal{E}(P + 2\infty) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1} \otimes \mathcal{P}$$

defined on each fiber $\mathbf{P}^1 \times \{\xi\}$ by

$$\Theta = (\theta - \frac{\xi}{2} dz) \otimes \mathbf{1}_{\mathcal{P}_\xi}.$$

Theorem

The first hypercohomology space of the complex

$$\theta - \frac{\xi}{2}dz : \mathcal{E} \longrightarrow \mathcal{E}(P + 2\infty) \otimes \mathbf{\Omega}^1_{\mathbf{P}^1}$$

is finite dimensional for all ξ*. The derived direct image*

$$\mathbf{R}_1(\pi_2)_*(\Theta)$$

defines a holomorphic vector bundle on $\widehat{\mathbf{P}}^1$*. The map*

$$-\frac{z}{2}d\xi$$

descends to a holomorphic endomorphism.

Denote the obtained Higgs bundle by $(\widehat{\mathcal{E}}, \widehat{\theta})$.

Theorem

The Higgs bundle corresponding to $(\widehat{E}, \widehat{D})$ via non-abelian Hodge theory is $(\widehat{\mathcal{E}}, \widehat{\theta})$.

Definition

The variety in $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$ defined by $\det(\Theta)$ is called the spectral curve, and is denoted by Σ. The sheaf $\text{coker}(\Theta)$ is called the spectral sheaf, denoted M.

M is clearly supported on Σ.

Theorem

The bundle $\widehat{\mathcal{E}}$ is equal to $(\pi_2)_* M$, and $\widehat{\theta}$ is induced by multiplication by $-\frac{z}{2}d\xi$ on the fibers.

The map between moduli spaces

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Lemma

If (E, D) is stable of degree 0, then so is $(\widehat{E}, \widehat{D})$.

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Therefore, Nahm transform defines a map on corresponding moduli spaces:

$$\mathcal{N} : \mathcal{M} \longrightarrow \widehat{\mathcal{M}}.$$

Theorem

$\mathcal{N}$ is a hyper-Kähler isomorphism.

Sketch of the proof

From the spectral interpretation, we see that

$$\mathcal{N}^2 = (-1)^*,$$

where (-1) is the "opposite" map on $\mathbf{C}$. Therefore, $\mathcal{N}$ is invertible, with inverse equal to $(-1)^* \circ \mathcal{N}$.
It also follows that $\mathcal{N}$ respects the Dolbeault complex structure I corresponding to the Higgs bundle point of view on $\mathcal{M}$.
Compatibility with the de Rham complex structure J is a consequence of the interpretation as Laplace transform.
Finally, compatibility with the Riemannian metrics is a consequence of conformal invariance of the L^2-norm of 1-forms. □

The hypergeometric system

Looking for a logarithmic connection on $\mathbf{P}^1$ with singularities in $0, 1, \infty$ and eigenvalues of the residue given by the scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ \hline 0 & 0 & z^{-1}\mu_1^{\infty} \\ \mu^0 & \mu^1 & z^{-1}\mu_2^{\infty} \end{pmatrix}$$

with $\mu^0 + \mu^1 + \mu_1^{\infty} + \mu_2^{\infty} = 1$ sufficiently generic.

Applying Nahm transform, we are looking for system of with a logarithmic singularity at $\widehat{0}$ and an irregular singularity of Poincaré-rank 1 at $\widehat{\infty}$ with eigenvalues:

$$\begin{pmatrix} \widehat{0} & \widehat{\infty} \\ \hline \mu_1^{\infty} & \zeta^{-1}\mu^0 \\ \mu_2^{\infty} & -1+\zeta^{-1}\mu^1 \end{pmatrix}$$

Tensoring by the rank 1 logarithmic connection with singularities in $\widehat{0}$ and $\widehat{\infty}$

$$\mathrm{d} - \mu_1^\infty \frac{\mathrm{d}\xi}{\xi},$$

we are led to the system

$$\begin{pmatrix} \dfrac{\widehat{0}}{\begin{matrix} 0 \\ \mu_2^\infty - \mu_1^\infty \end{matrix}} & \dfrac{\widehat{\infty}}{\begin{matrix} \zeta^{-1}(\mu^0 + \mu_1^\infty) \\ -1 + \zeta^{-1}(\mu^1 + \mu_1^\infty) \end{matrix}} \end{pmatrix}$$

Transforming this system back to $\mathbf{P}^1$ yields a system of rank 1 with logarithmic singularities in $0, 1, \infty$

$$\begin{pmatrix} 0 & 1 & \infty \\ \hline \mu^0 + \mu_1^\infty & \mu^1 + \mu_1^\infty & z^{-1}(\mu_2^\infty - \mu_1^\infty) \end{pmatrix}$$

The sum of the residues is by assumption equal to 1, hence there exists a unique such logarithmic connection on the bundle $\mathcal{O}(-1)$. Invertibility of Nahm transform and of tensor product with a rank 1 connection implies existence and uniqueness of the initial hypergeometric system.

What's next?

- Extension of the transform to non semi-simple residues.
- Generalisation to higher-order poles.
- Generalisation to various other structure groups.
- Study of the middle convolution algorithm.

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	$\widehat{0}$
$z^{-1}\mu_1^0$	0		0	$-p_1 + \zeta^{-1}\mu_{r_1+1}^1$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\vdots$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu_r^1$	$\vdots$
$\vdots$	0		$\mu_{r_n+1}^n$	$\vdots$	0
	$\mu_{r_1+1}^1$		$\vdots$	$-p_n + \zeta^{-1}\mu_{r_n+1}^n$	μ_1^0
$z^{-1}\mu_r^0$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
	μ_r^1		μ_r^n	$-p_n + \zeta^{-1}\mu_r^n$	μ_r^0