Quantum spectra of birational surfaces (joint w/ A. Gyenge)

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Dedication

This talk is dedicated to the memory of Professor Yuri Manin (1937–2023).

Notations

- X: nonsingular projective surface over $\mathbb{C}$ with $H^{\text{odd}}(X, \mathbb{C}) = 0$.
- X_{min}: a minimal surface of X
- $H^*(X, \mathbb{C}), H^*(X_{\text{min}}, \mathbb{C})$: cohomology rings of X (and X_{min})
- $t_0,\ q_1,\ \ldots,\ q_m,\ t_p$: variables associated to a basis of $H^*(X_{\text{min}})$
- $q_{m+1}, \ldots, q_{m+r}$: variables associated to exceptional divisors
- $* = *_{q,t}$: quantum product (a deformation of $\cup$ on H^*)
- u: further variable
- $\nabla_X, \nabla_{X_{\text{min}}}$: Dubrovin connection on the trivial H^*-bundles over $\operatorname{Spec} \mathbb{C}\{q, q^{-1}, t\}[u, u^{-1}]$
- K, K_{min}: leading-order term of $\nabla, \nabla_{\text{min}}$ at $u = 0$
- $\{\lambda_0, \ldots, \lambda_{m+1},\}$: spectrum of K_{min}
- $\{\mu_0, \ldots, \mu_{m+r+1}\}$: spectrum of K

Asymptotic behaviour of spectra

Theorem

Let

$$U \subset \operatorname{Spec} \mathbb{C}\{q_1^{\pm}, \dots, q_{m+r}^{\pm}\}$$

be any conical open subset for the analytic topology whose closure intersects the coordinate hyperplanes $\{q_i = 0\}$, $1 \leq i \leq m+r$ *trivially. Then, up to a suitable relabeling,*

$$\lim \frac{\lambda_j(q_1, \dots, q_m, t_p)}{\mu_j(q_1, \dots, q_{m+r}, t_p)} = 1 \qquad (0 \leq j \leq m+1)$$

$$\lim \mu_j(q_1, \dots, q_{m+r}, t_p) q_{j-1} = 1 \quad (m+2 \leq j \leq m+r+1)$$

as $|t_p| \ll \infty$ *and the point* $(q_1, \dots, q_{m+r})$ *converges to* $\vec{0}$ *in* U.

The initial motivation & relationship to PVI

Quantum connection is defined for smooth projective varieties X of higher dimension.

Conjecture (B. Dubrovin, 1998)

For a (Fano) variety X the connection ∇_X is generically semi-simple iff $D^b(\mathrm{Coh}(X))$ admits a full exceptional collection $E_1, \dots, E_n$. Moreover, in this case the Stokes matrix of ∇_X coincides with the Gram matrix of the bilinear form $\chi(E, F) = \sum_k (-1)^k \dim \mathrm{Ext}^k(E, F)$ with respect to this basis.

Proven by Dubrovin (1998) for $\mathbb{C}P^2$ and Guzzetti (1999) for $\mathbb{C}P^n$ with $n \geq 3$.
Moreover, Dubrovin also proved that isomonodromicity of $\nabla_{\mathbb{C}P^2}$ is equivalent to (a special type of) Painlevé VI equation.

A recent motivation

Let X be quasi-projective, $Y \subset X$ be a smooth closed subvariety of codimension $m \geq 2$ and $\widetilde{X}$ the blow-up of X with center Y.

Conjecture (Blow-up conjecture (informal version), M. Kontsevich, 2021)

Under suitable assumptions (for instance, if X is projective), the spectrum of $\nabla_{\widetilde{X}}$ is "close to" the union of the spectrum of ∇_X and $(m-1)$ copies of the spectrum of ∇_Y, shifted by the roots of unity $\sqrt[m-1]{1}$.

There exists a more formal version (stated in terms of isomonodromy) and yet a stronger one relating it to Dubrovin's conjecture.

Moduli space of $g = 0$ stable maps, GW-invariants

For any $\beta \in H_2(X, \mathbb{Q})$ and $k \in \mathbb{Z}_+$, Kontsevich–Manin, 1994:

$$\overline{\mathcal{M}}_k(X, \beta)$$

the moduli space of genus zero k-pointed stable maps $f : C \to X$ with C a connected, reduced, at worst nodal curve of genus 0, such that $f_*[C] = \beta$.

There exist canonical evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_k(X, \beta) \to X, \quad 1 \leq i \leq k$$

For $\beta_1, \dots, \beta_k \in H^*(X)$, the associated (genus zero) Gromov-Witten invariant:

$$I_\beta(\beta_1 \dots \beta_k) = \int_{[\overline{\mathcal{M}}_k(X,\beta)]^{\mathrm{Vir}}} \prod_i \mathrm{ev}_i^*(\beta_i)$$

GW-potential

Let

- $T_0 = [1] \in H^0(X, \mathbb{Q})$, $T_{m+1} = T_p \in H^4(X, \mathbb{Q})$: canonical generators,
- $T_1, \dots, T_m \in H^2(X, \mathbb{Q})$: some basis
- $T_i^\vee$: dual elements in H_*
- t_0, q_1, $\dots, q_m$, t_p (abbreviated as q, t): variables associated to these classes (one can assume $t_0 = 0$)

(Genus zero) Gromov–Witten potential:

$$F(q,t) = \sum_{\substack{n \geq 0 \\ \beta \in B \setminus \{0\}}} I_\beta(T_p^n) q_1^{\int_\beta T_1} \cdots q_m^{\int_\beta T_m} \frac{t_p^n}{n!} \in \mathbb{Q}[\![q, q^{-1}, t]\!]$$

Quantum product, WDVV-relations

Let

$$g_{ij} = \int_X T_i \cup T_j,$$

be the intersection pairing and (g^{ij}) be its matrix inverse.
Let

$$\partial_i = \begin{cases} q_i \frac{\partial}{\partial q_i} & i \notin \{0, p\} \\ \frac{\partial}{\partial t_i} & i \in \{0, p\} \end{cases}$$

Set $F_{ijk} = \partial_i \partial_j \partial_k F$.
Quantum product of T_i and T_j:

$$T_i * T_j = T_i \cup T_j + \sum_{e,f} F_{ije} g^{ef} T_f$$

Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) relations: $*$ is associative.

Convergence of F for surfaces

Theorem (Itenberg–Kharlamov–Shustin, 2004; Gyenge–Sz, 2022)

For any surface X, the formal power series F_X has non-zero radius of convergence.

Sketch of proof: sufficient to treat the case of rational X. Let $E_1, \dots, E_r$ be exceptional divisors of a blow-up of $\mathbb{C}P^2$. Let

$$\beta = dH - \sum_{i=1}^{r} a_i E_i$$

be abbreviated as (d, α). Set

$$n_{d,\alpha} = 3d - |\alpha| - 1$$
$$N_{d,\alpha} = I_{(d,\alpha)}(T_p^{n_{d,\alpha}})$$

Key estimate for convergence

The potential simplifies as

$$F(q,t) = \sum_{(\beta,\alpha)} N_{d,\alpha} q_h^d q^\alpha \frac{t_p^{n_{\beta,a}}}{n_{\beta,a}!}$$

where

$$q^\alpha = q_{m+1}^{a_1} \cdots q_{m+r}^{a_r}$$

and the sum is taken over classes satisfying $n_{d,\alpha} \geq 0$.
We then have

$$N_{d,\alpha} \leq \prod_{i=1}^{r} \frac{1}{a_i!} N_d$$

Itenberg & al prove this bound geometrically. Our proof is combinatorial, relying on WDVV-relations.

Frobenius manifold structure

- Base manifold: $M = H^*(X, \mathbb{C}) \times \operatorname{Spec} \mathbb{C}[u^{\pm}]$
- co-ordinates: $t_0, q_1, \ldots, q_m, t_p$
- Euler vector field: $E = c_1(T_X) + 2t_0 T_0 - 2t_p T_p$
- vector bundle $V \to M$: trivial, with fiber $H^*(X, \mathbb{C})$
- grading operator: $G = \frac{d-2}{2} \cdot \mathrm{Id}$ on $H^d(X)$
- $A_i = T_i *$
- $A = c_1(\mathcal{O}(1)) *$
- $K = E *$

Connection matrices

The formulas

$$\nabla_{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} + \frac{1}{u^2}K + \frac{1}{u}G$$
$$\nabla_{\frac{\partial}{\partial t_i}} = \frac{\partial}{\partial t_i} + \frac{1}{u}A_i$$
$$\nabla_{\frac{\partial}{\partial q}} = \frac{\partial}{\partial q} + \frac{1}{uq}A$$

define a meromorphic connection (Dubrovin- or quantum connection) ∇ in V.

Theorem (Dubrovin, 1998)

WDVV-relations $\Leftrightarrow F_{\nabla} = 0.$

Initial values of recursion

Theorem

1. *The numbers $N_{\beta,\alpha}$ satisfy the following properties.*
   1. $N_{\beta,\alpha} = N_{\beta,(\alpha,0)}$
   2. $N_{0,\alpha} = 1$ *if* $\alpha = -[i]$ *for some* $1 \leq i \leq r$*, and 0 for any other* α
   3. $N_{\beta,\alpha} = 0$ *if* β *is effective and any of the* a_i *is negative*
   4. *If* $n_{\beta,\alpha} > 0$*, then* $N_{\beta,\alpha} = N_{\beta,(\alpha,1)}$.
2. *The numbers $N_{\beta,\alpha}$ can be determined by a recursive algorithm starting from the ones given by (a) and (b) from part* (1).

Partitions

For a surface X with $T_{m+1}, \ldots, T_{m+r}$ classes of exceptional divisors we may rewrite

$$T_i * T_j = T_i \cup T_j + \sum_{k,l=1}^{m} F_{ijk} g^{kl} T_l - \sum_{e=m+1}^{m+r} F_{ije} T_e + F_{ijp} T_0.$$

We use induction on r, and from now on we set $r = 1$.
Let $\vdash (\beta, a)$ denote the set of pairs $((\beta_1, a_1), (\beta_2, a_2))$ satisfying

(i) $(\beta_1, a_1), (\beta_2, a_2) \neq 0$
(ii) $(\beta_1, a_1) + (\beta_2, a_2) = (\beta, a)$
(iii) $n_{\beta_1, a_1}, n_{\beta_2, a_2} \geq 0$

Let $\vdash (\beta, a) \neq 0$ denote the subset of $\vdash (\beta, a)$ for which $\beta_1 \neq 0$, $\beta_2 \neq 0$.

Göttsche–Pandharipande–Hu identity 1

If $n_{\beta,a} \geq 3$ and $g_{ij} \neq 0$ for some $i, j \in \{1, \ldots, m\}$, then

$$N_{\beta,a} = \frac{1}{g_{ij}} \sum_{\vdash(\beta,a)\neq 0} N_{\beta_1,a_1} N_{\beta_2,a_2} \left(\sum_{a,b=1}^{m} T_a(\beta_1) g^{ab} T_b(\beta_2) - a_1 a_2 \right) \cdot \left[T_i(\beta_1) T_j(\beta_2) \binom{n_{\beta,a} - 3}{n_{\beta_1,a_1} - 1} - T_i(\beta_1) T_j(\beta_1) \binom{n_{\beta,a} - 3}{n_{\beta_1,a_1}} \right]$$

Göttsche–Pandharipande–Hu identity 2

If $n_{\beta,a} \geq 0$, then

$$T_i(\beta)T_j(\beta)aN_{\beta,a} = (T_i(\beta)T_j(\beta) - g_{ij}(a-1)^2))N_{\beta,a-1}$$
$$+ \sum_{\vdash(\beta,a-1)\neq 0} N_{\beta_1,a_1}N_{\beta_2,a_2}\left(\sum_{a,b=1}^{m} T_a(\beta_1)g^{ab}T_b(\beta_2) - a_1a_2\right)$$
$$\cdot \left(T_j(\beta_1)T_i(\beta_2)a_1a_2 - T_i(\beta_1)T_j(\beta_1)a_2^2\right)\binom{n_{\beta,a-1}-1}{n_{\beta_1,a_1}}$$

Let

$$T_i'(\beta, a) = \begin{cases} T_i(\beta) & \text{if } i \in \{1, \dots, m\} \\ a & \text{if } i = e \\ 1 & \text{if } i = p \\ 0 & \text{if } i = 0 \end{cases}$$

and set

$$\varepsilon = \varepsilon(i, j) = \delta_{i0} + \delta_{jp} \in \{0, 1, 2\}$$

Let the index e stand for the class of the exceptional divisor.

Behaviour of quantum product by Euler class under blow-up

Let us denote by K quantum product with the Euler class of $\widetilde{X}$ and by $\overline{K}$ the same thing for X, where $\widetilde{X} \to X$ is blow-up at a smooth point.

Proposition

For $i \neq p$ and $j \neq 0$,

$$K_{ij} = \overline{K}_{ij} - q_e^{-1}\delta_{ie}\delta_{je} + \sum_{\substack{\beta,a \\ a>0}} N_{\beta,a} \left(\sum_{k \in \{1,\dots,m,e,p\}} g^{ki} T'_k(\beta) \right) T'_j(\beta)(1 - n_{\beta,a} + 2\varepsilon) \cdot q^{\beta} q_e^{a} \frac{t_p^{n_{\beta,a}-\varepsilon}}{(n_{\beta,a}-\varepsilon)!}$$

Characteristic polynomial

We fix arbitrary $\nu_1, \ldots, \nu_{m+r} \in \mathbb{C} \setminus \{0\}$ and take

$$q_1 = \nu_1 q$$
$$\vdots$$
$$q_{m+r} = \nu_{m+r} q$$

for some $q \neq 0$. Let t_p be some sufficiently small constant. Let λ be an indeterminate and set

$$\chi_X(\lambda) = \det(\lambda \cdot \mathrm{Id} - K_X) \in \mathbb{C}\{q^{\pm}\}[\lambda]$$

Newton polygon

The Newton–Puiseux pairs of $\chi_X(\lambda)$ are the lattice points $(x, y) \in \mathbb{Z}^2$ such that the coefficient of the monomial $\lambda^x q^y$ in χ is non-zero.

The Newton polygon of the characteristic polynomial χ_X is the lower convex hull of the set of Newton pairs in the (x, y)-plane (i. e., the smallest convex set containing the rays parallel to the positive y-axis emanating from the Newton pairs).

Behaviour of Newton polygon under blow-up

Theorem

Let X_{min} be a minimal model and $X = X_r$ is its r-fold blow-up in generic points. Then, for all $\nu_1, \ldots, \nu_{m+r+1} \in \mathbb{C}$ the Newton polygon of $\chi_{X_r}(\lambda)$ is obtained by translating the Newton polygon of $\chi_{X_{\text{min}}}(\lambda)$ by the vector $(0, -r)$, and extending it by a segment of slope 1 and length r on its right.

Sketch of proof

Proposition $\Rightarrow$ each Newton–Puiseux pair of $\chi_{X_{r-1}}(\lambda)$ translated by $(0, -1)$ appears as a Newton pair of $\chi_{X_r}(\lambda)$.
We also need to show that

- any further Newton–Puiseux pair of X_r lies on or above the diagram determined by the pairs of the statement
- the monomials corresponding to the salient vertices of the boundary of the polygon given by the statement are unique.

For these statements, we use the classification of surfaces into the classes:

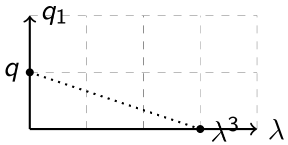
1. $-c_1(X_{\min})$ is nef
2. $X_{\min}$ is a minimal ruled surface over some curve Σ_g
3. $X_{\min} \simeq \mathbb{C}P^2$

Case of $\mathbb{C}P^2$

The minimal entries of K for $X_{\min}$ for the basis $0, h, p$ are

$$\begin{pmatrix} 0 & q_h t_p & 3q_h \\ 3 & -q_h \frac{t_p^2}{2} & q_h t_p \\ 2t_p & 3 & 0 \end{pmatrix}$$

The Newton polygon is

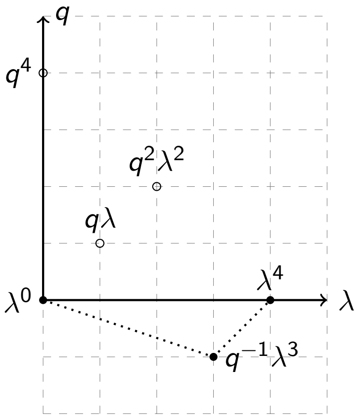


Minimal entries of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$

The minimal entries of K for $\mathbb{C}P^2$ blown-up in a point for the basis $0, h, 2, p$ are

$$\begin{pmatrix} 0 & q_h t_p & 2q_h q_2 & 3q_h \\ 3 & -q_h \frac{t_p^2}{2} & q_h^2 q_2 \frac{t_p^4}{12} & 2q_h t_p \\ -1 & -q_h^2 q_2 \frac{t_p^4}{12} & -q_2^{-1} & -2q_h q_2 \\ 2t_p & 3 & -1 & 0 \end{pmatrix}$$

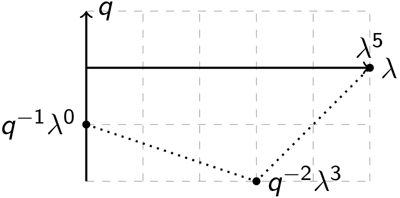
Newton polygon of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$



Minimal entries of $\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$

$$\begin{pmatrix} 0 & q_h t_p & 2q_h q_2 & 2q_h q_3 \\ 3 & -q_h \frac{t_p^2}{2} & q_h q_2 \left(q_h \frac{t_p^4}{12} + q_3 \right) & q_h q_3 \left(q_h \frac{t_p^4}{12} + q_2 \right) \\ -1 & -q_h q_2 \left(q_h \frac{t_p^4}{12} + q_3 \right) & -q_2^{-1} & -q_h q_2 q_3 \\ -1 & -q_h q_3 \left(q_h \frac{t_p^4}{12} + q_2 \right) & -q_h q_2 q_3 & -q_3^{-1} \\ 2t_p & 3 & -1 & -1 \end{pmatrix}$$

Newton polygon of $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$



Thank you for your attention!