Nahm transform for parabolic harmonic bundles on the Riemann sphere with regular residues

Szilárd Szabó

Budapest University of Technology and Rényi Institute of Mathematics
Budapest

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Outline

1. Parabolic harmonic bundles

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2. Construction of Nahm transform

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2. Construction of Nahm transform

3. Hyper-Kähler isometry property

Notations

$G = \mathrm{Gl}_r(\mathbf{C})$
$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$U \subset \mathbf{P}^1$: an analytic open set
$P = \{z_0 = \infty, z_1, \ldots, z_n\}$: a finite set of distinct points in $\mathbf{P}^1$
$\mathcal{O}$: sheaf of holomorphic functions on $\mathbf{P}^1$
Ω^k: sheaf of smooth k-forms on $\mathbf{P}^1$
K: sheaf of holomorphic 1-forms on $\mathbf{P}^1$
V: smooth vector bundle over $\mathbf{P}^1$
$E, \mathcal{E}$: holomorphic vector bundles with underlying smooth vector bundle V

Motivation

Nahm's transformation is an integral transform for solutions of the dimensionally reduced Yang–Mills (ASD) equations

$$*F_A = -F_A$$

for a connection A in a Hermitian vector bundle over $(\mathbf{R}^4, g_{Eucl})$.
It maps invariant solutions with respect to some additive subgroup $\Lambda \subset \mathbf{R}^4$ to invariant solutions with respect to the dual subgroup $\Lambda^* \subset (\mathbf{R}^4)^*$.
$\mathbf{R}^2$-invariant solutions $\rightsquigarrow$ Hitchin's equations over $\mathbf{R}^2 = \mathbf{C}$.
Possible application: Katz' middle convolution algorithm.
Related work: push-forward of parabolic structure from the Dolbeault point of view by R. Donagi–T. Pantev–C. Simpson ('16), using C. Sabbah's and T. Mochizuki's work on pure wild twistor $\mathcal{D}$-modules.

Examples

- YM-instantons over $\mathbf{R}^4 \leftrightarrow$ representations of a quiver (ADHM-transform)
- Instantons on a torus $T^4 \leftrightarrow$ instantons on the dual torus $(T^4)^*$ (P. Braam, P. van Baal)
- Monopole equations over $\mathbf{R}^3 \leftrightarrow$ Nahm's equations on an interval (N. Hitchin, S. Donaldson, H. Nakajima,...)
- Calorons $\leftrightarrow$ Nahm's equations on a circle (W. Nahm)
- doubly-periodic instantons $\leftrightarrow$ singular solutions of Hithin's equations (M. Jardim, O. Biquard, T. Mochizuki)
- Spatially periodic instantons $\leftrightarrow$ singular monopoles (B. Charbonneau)
- periodic monopoles $\leftrightarrow$ Hitchin's equations on the cylinder (S. Cherkis, A. Kapustin)
- YM-instantons over a multi-Taub–NUT space $\leftrightarrow$ Cherkis' bow varieties

Decomposition of flat connections

Over any $U \subset \mathbf{P}^1 \setminus P$ open let

- D be a flat connection in V,
- and h a Hermitian metric on V.

Consider the decomposition

$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively.

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of D into h-unitary and self-adjoint parts respectively.
Decompose these parts further according to their bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

Then, (D, h) is said to satisfy Hitchin's equations if

$$\bar{\partial}^{+}\theta = 0.$$

Notation

We denote the holomorphic vector bundle $(V, \bar{\partial}^{+})$ *by* $\mathcal{E}$*. We also set*

$$\bar{\partial}^{+} = \bar{\partial}^{\mathcal{E}}.$$

We call such a triple (V, D, h) a harmonic bundle over U.

Parabolic structures

A quasi-parabolic structure of $\mathcal{E}$ at $z_i \in P$:

$$\{0\} = F_i^{l_i} \subset F_i^{l_i - 1} \subset \cdots \subset F_i^1 \subset F_i^0 = V|_{z_i}$$

for some $1 \leq l_i \leq r$.
A parabolic structure of $\mathcal{E}$ at $z_i \in P$: a quasi-parabolic structure and real numbers

$$1 > \alpha_i^{l_i - 1} > \cdots > \alpha_i^0 \geq 0$$

called parabolic weights.

Consider harmonic bundles such that

$$\theta \in H^0(\mathcal{E}nd(\mathcal{E}) \otimes K(2 \cdot z_0 + z_1 + \cdots + z_n)).$$

For all $1 \leq i \leq n$ we require that $\mathrm{res}_{z=z_i}(\theta)$ respects the filtration $F_i^\bullet$. We derive graded vector spaces

$$\mathrm{Gr}_i^j = \mathrm{Gr}_{F_i}^j = F_i^j / F_i^{j+1}$$

and linear maps

$$\mathrm{Gr}^j \, \mathrm{res}_{z=z_i}(\theta) \in \mathrm{End}(\mathrm{Gr}_i^j).$$

Consider the decomposition

$$\mathrm{Gr}^j \, \mathrm{res}_{z=z_i}(\theta) = S_i^j + N_i^j$$

into semi-simple and nilpotent components.

Weight filtration

There exists an increasing filtration

$$0 = W^j_{i,-r} \subseteq W^j_{i,1-r} \subseteq \cdots \subseteq W^j_{i,r} = \mathrm{Gr}^j_{F_i}$$

satisfying

- for all $k \in \mathbf{Z}$ the endomorphism N^j_i maps $W^j_{i,k}$ into $W^j_{i,k-2}$
- for all $k \in \mathbf{N}$ we have an isomorphism

$$(N^j_i)^k = N^j_i \circ \cdots \circ N^j_i : \mathrm{Gr}_k^{W^j_i} \mathrm{Gr}^j_{F_i} \cong \mathrm{Gr}_{-k}^{W^j_i} \mathrm{Gr}^j_{F_i} .$$

Let $z_i \in U_i$ be a small neighborhood. We say that a trivialization $\{\mathbf{e}_i^s\}_{s=1}^r$ of $\mathcal{E}$ over U_i is compatible with the quasi-parabolic structure if for all j we have

$$F_i^j = \mathbf{C} \left\langle \mathbf{e}_i^1(p_i), \ldots, \mathbf{e}_i^{\dim F_i^j}(p_i) \right\rangle.$$

For $1 \leq s \leq r$ we let $j(s)$ stand for the largest $j \in \{0, \ldots, l_i - 1\}$ such that $\mathbf{e}_i^s \in F_i^j$ and $k(s)$ stand for the smallest $k \in \mathbf{Z}$ such that $\mathbf{e}_i^s \in W_{i,k}^{j(s)}$.
A Hermitian metric h in $\mathcal{E}$ over U_i is said to be compatible with θ if and only if with respect to some compatible $\{\mathbf{e}_i^s\}_{s=1}^r$ it is mutually bounded with the diagonal metric

$$h_0 = \mathrm{diag}(|z - z_i|^{2\alpha_i^{j(s)}}(-\log|z - z_i|)^{k(s)})_{s=1}^r.$$

Main assumption

Assumption

- *if* $\alpha_i^0 = 0$ *then the nilpotent part* N_i^0 *acts trivially on the generalized* 0*-eigenspace of* $\mathrm{Gr}^0 \operatorname{res}_{z=z_i}(\theta)$*;*
- *for* $\alpha_i^j > 0$ *the kernel of* $\mathrm{Gr}^0 \operatorname{res}_{z=z_i}(\theta)$ *vanishes:*

$$\ker \mathrm{Gr}^j \operatorname{res}_{z_i}(\theta) = 0.$$

Remark

Necessary for our construction at this stage.
Ongoing joint work with Takuro Mochizuki: lift these restrictions.

Local form at z_0

Near $z_0 = \infty$ we assume that with respect to some compatible $\{\mathbf{e}_0^s\}_{s=1}^r$ the Higgs field has the form

$$\theta = \frac{A}{2}\mathrm{d}z + B\frac{\mathrm{d}z}{z} + \text{lower order terms},$$

for some diagonal $A \in \mathfrak{gl}_r(\mathbf{C})$ and arbitrary $B \in \mathfrak{gl}_r(\mathbf{C})$, preserving the filtration $F_0^\bullet$.

Let us denote by $\mathfrak{h}$ the Lie-algebra of the centralizer of A in $\mathrm{Gl}_r(\mathbf{C})$. Then we may assume that $B \in \mathfrak{h}$.

A Hermitian metric h in $\mathcal{E}$ over U_0 is said to be compatible with θ if it is mutually bounded with the diagonal metric

$$h_0 = \mathrm{diag}(|z|^{-2\alpha_0^j(s)}(\log|z|)^{k(s)})_{s=1}^r.$$

Assumption

$\ker \mathrm{Gr}^j \operatorname{res}_{z_0}(\theta) = 0.$

Moduli spaces

Fix

- $r, z_1, \dots, z_n$,
- eigenvalues of A and their multiplicities,
- dimensions of $\mathrm{Gr}^j_{F_i}$,
- parabolic weights α^j_i such that $\sum_{i,j} \alpha^j_i \dim \mathrm{Gr}^j_{F_i} \in \mathbf{Z}$,
- regular coadjoint orbits $\mathcal{O}^j_i \subset \mathrm{Gr}^j_{F_i}$.

Simpson ('90), Biquard–Boalch ('04): Solutions of Hitchin's equations with the above asymptotics and such that

$$\mathrm{Gr}^j \operatorname{res}_{z_i}(\theta) \in \mathcal{O}^j_i$$

form a complete hyper-Kähler moduli space

$$\mathcal{M}(\mathbf{P}^1, r, \{z_i\}, A, \{\dim \mathrm{Gr}^j_{F_i}\}, \{\alpha^j_i\}, \{\mathcal{O}^j_i\}).$$

Aim: to describe some isometries between such moduli spaces corresponding to various data.

Exponential twist

For $\zeta \in \widehat{\mathbf{C}} \setminus \widehat{P}$ we consider the twisted flat connection

$$D_\zeta = D - \zeta \mathrm{d}z.$$

and the associated elliptic complex $(L^2(V \otimes \Omega^\bullet), D_\zeta)$:

$$0 \to L^2(V) \xrightarrow{D_\zeta} L^2(V \otimes \Omega^1) \xrightarrow{D_\zeta} L^2(V \otimes \Omega^2) \to 0.$$

Easy to check that:

$$H^0(L^2(V \otimes \Omega^\bullet), D_\zeta) = 0 = H^2(L^2(V \otimes \Omega^\bullet), D_\zeta).$$

We define

$$\widehat{V}_\zeta = H^1(L^2(V \otimes \Omega^\bullet), D_\zeta).$$

Kodaira–Spencer: $\widehat{V}$ is a smooth vector bundle over $\widehat{\mathbf{C}} \setminus \widehat{P}$.

We introduce the Laplace operator

$$\Delta_\zeta = -D_\zeta D_\zeta^* - D_\zeta^* D_\zeta : \Omega^1 \otimes V \to \Omega^1 \otimes V.$$

Proposition

We have $\widehat{V}_\zeta = \ker(\Delta_\zeta)$ on its L^2-domain.

Paralelly, set

$$\theta_\zeta = \theta - \frac{\zeta}{2}\mathrm{d}z \quad \text{and} \quad D''_\zeta = \bar{\partial}^{\mathcal{E}} + \theta_\zeta.$$

Hodge relation:

$$\ker(\Delta_\zeta) \cong \ker(-D''_\zeta (D''_\zeta)^* - (D''_\zeta)^* D''_\zeta) = H^1(L^2(V \otimes \Omega^\bullet), D''_\zeta).$$

The algebraic Dolbeault complex

Proposition

There exists explicit subsheaves $\mathcal{F}, \mathcal{G}$ of $\mathcal{E}$ so that we have

$$H^1(L^2(V \otimes \Omega^\bullet), D''_\zeta) = \mathbf{H}^1\left(\mathcal{F} \xrightarrow{\theta_\zeta} \mathcal{G} \otimes K_{\mathbf{P}^1}(2 \cdot z_0 + z_1 + \cdots + z_n)\right).$$

(Analog for the Poincaré metric: C. Sabbah, T. Mochizuki.)
This endows $\widehat{V}$ with a holomorphic structure, denoted $\widehat{\mathcal{E}}$.

At logarithmic points z_i $(i > 0)$ a local frame $\{\mathbf{g}_i^s\}_{s=1}^r$ for $\mathcal{G}$ may be defined from a compatible trivialization of $\mathcal{E}$ as follows:

- if $\alpha_i^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_i^s = (z - z_i)\mathbf{e}_i^s$;
- otherwise set $\mathbf{g}_i^s = \mathbf{e}_i^s$.

At z_0, a local frame is defined by

- if $\alpha_0^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_0^s = z^{-2}\mathbf{e}_0^s$;
- otherwise set $\mathbf{g}_0^s = z^{-1}\mathbf{e}_0^s$.

Let λ_i^s stand for the eigenvalue of $\mathrm{Gr}^{j(s)} \mathrm{res}_{z_i}(\theta)$ corresponding to the vector $\mathbf{e}_i^s$.
At z_i with $i > 0$ we define a local frame $\{\mathbf{f}_i^s\}_{s=1}^r$ for $\mathcal{F}$ as:

- if $\alpha_i^{j(s)} = 0 \neq \lambda_i^s$ and $k(s) \geq -1$: set $\mathbf{f}_i^s = (z - z_i)\mathbf{e}_i^s$;
- otherwise set $\mathbf{f}_i^s = \mathbf{e}_i^s$.

At z_0, we let $\mathbf{f}_0^s = \mathbf{g}_0^s$.

Extension over the singularities

Let $s_0, s_\infty \in H^0(\widehat{\mathbf{P}}^1, \mathcal{O}_{\widehat{\mathbf{P}}^1}(1))$ be sections such that over $\widehat{\mathbf{C}}$ we have

$$s_0(\zeta) = \zeta, \quad s_\infty(\zeta) = 1.$$

In the product $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$ let π_m stand for projection on the m^{th} factor. We consider the following extension of the Higgs field over $\widehat{\mathbf{P}}^1$

$$\theta_\zeta = \theta \otimes s_\infty - \frac{1}{2}\mathrm{Id}_{\mathcal{E}} \mathrm{d}z \otimes s_0 : \pi_1^* \mathcal{F} \to \pi_1^* \mathcal{G} \otimes K_{\mathbf{P}^1}(2 \cdot z_0 + z_1 + \cdots + z_n) \otimes \pi_2^* \mathcal{O}_{\widehat{\mathbf{P}}^1}(1)$$

Proposition

The hypercohomology groups of degree 0 *and* 2 *of this complex vanish for all* $\zeta \in \widehat{\mathbf{P}}^1$.

We get an extension of $\widehat{\mathcal{E}}$ as a locally free sheaf on $\widehat{\mathbf{P}}^1$ given by

$$\widehat{\mathcal{E}} = \mathbf{R}^1 (\pi_2)_* (\theta_\zeta).$$

Topology of $\widehat{\mathcal{E}}$

A computation using Grothendieck–Hirzebruch formula shows that

$$\mathrm{rank}(\widehat{\mathcal{E}}) = \hat{r} = \sum_{i=1}^{n} \left(r - \delta_{0,\alpha_i^0} \dim \ker \mathrm{Gr}_i^0 \right),$$

$$\deg(\widehat{\mathcal{E}}) = \deg(\mathcal{F}) + r + \hat{r}.$$

Filtered Dolbeault complex

Equivalent definition of parabolic structure for $\mathcal{E}$ along a reduced effective divisor D: a decreasing family $\mathcal{E}_\bullet$ of coherent sheaves on $\mathbf{P}^1$ indexed by $\mathbf{R}$ so that for all $\alpha \in \mathbf{R}$

- left-continuity: there exists some $\varepsilon > 0$ with $\mathcal{E}_{\alpha-\varepsilon} = \mathcal{E}_\alpha$;
- quasi-periodicity: we have $\mathcal{E}_{\alpha+1} = \mathcal{E}_\alpha \otimes \mathcal{O}_X(-D)$.

For all $\alpha \in [0,1)$ we consider a filtered version of the Dolbeault complex

$$\theta_{\alpha,\zeta} : \mathcal{F}_\alpha \to \mathcal{G}_\alpha \otimes K_{\mathbf{P}^1}(z_1 + \cdots + z_n + 2 \cdot z_0),$$

and set

$$\widehat{\mathcal{E}}_\alpha = \mathbf{R}^1(\pi_2)_*(\theta_{\alpha,\zeta}).$$

Definition of $\mathcal{F}_\alpha, \mathcal{G}_\alpha$ locally near z_i for $i > 0$:

- if $\alpha_i^{j(s)} = 0 = \lambda_i^s$ we set

$$\mathbf{f}_i^s(\alpha) = \mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = \mathbf{g}_i^s;$$

- otherwise, if $\alpha \leq \alpha_i^{j(s)}$ we set

$$\mathbf{f}_i^s(\alpha) = \mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = \mathbf{g}_i^s;$$

- otherwise, if $\alpha_i^{j(s)} < \alpha$ we set

$$\mathbf{f}_i^s(\alpha) = (z - z_i)\mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = (z - z_i)\mathbf{g}_i^s.$$

For $i = 0$, same definitions up to replacing $(z - z_i)$ by z^{-1}.

Extend $\widehat{\mathcal{E}}_\alpha$ to all $\alpha \in \mathbf{R}$ by quasi-periodicity and set

$$\widehat{D} = \mathrm{div}(\widehat{P}) + \infty.$$

Proposition

$\widehat{\mathcal{E}}_\alpha$ is a parabolic bundle with parabolic divisor $\widehat{D}$.

Transformed Higgs field

Multiplication by $-z/2d\zeta$ induces a morphism

$$\widehat{\theta} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}} \otimes K_{\widehat{\mathbf{P}}^1}(*\widehat{D}).$$

Theorem (Sz '16)

1. *The transformed Higgs field $\widehat{\theta}$ is meromorphic, has a first-order pole at the points of $\widehat{P}$ and a second-order pole at $\infty \in \widehat{\mathbf{P}}^1$ with semi-simple leading-order term, and no other poles.*
2. *The residues of $\widehat{\theta}$ at the parabolic points are compatible with the parabolic structure corresponding to the $\mathbf{R}$-parabolic structure $\widehat{\mathcal{E}}_\bullet$, and the leading-order term of $\widehat{\theta}$ at $\infty \in \widehat{\mathbf{P}}^1$ preserves the parabolic filtration.*
3. *The transformation is holomorphic for the Dolbeault complex structures of the moduli spaces.*

Stationary phase formula

For any $\zeta_\iota \in \widehat{P}$ denote by $(\mathcal{E}|_\infty)_\iota$ the ζ_ι-eigenspace of A and set

$$B_\iota = \mathrm{res}_{z=\infty} \theta|_{(\mathcal{E}|_\infty)_\iota}.$$

By compatibility with the parabolic structure, B_ι induces graded morphisms

$$\mathrm{Gr}^j B_\iota \in \mathrm{End}(\mathrm{Gr}^j(\mathcal{E}|_\infty)_\iota).$$

Assumption

For all ι, j the endomorphism $\mathrm{Gr}^j B_\iota$ is regular and for all $\iota, j \neq j'$ the eigenvalues of $\mathrm{Gr}^j B_\iota$ and $\mathrm{Gr}^{j'} B_\iota$ are distinct.

Theorem (Sz '17)

For all $\alpha_0^j > 0$ we have $\mathrm{Gr}^j \mathrm{res}_{\zeta=\zeta_\iota} \widehat{\theta} = -\mathrm{Gr}^j B_\iota$.
If $\alpha_0^0 = 0$ then we have $\mathrm{Gr}^0 \mathrm{res}_{\zeta=\zeta_\iota} \widehat{\theta} = -\mathrm{Gr}^0 B_\iota \oplus 0$, where 0 stands for the identically 0 endomorphism of appropriate dimension.

Transformed harmonic metric

For two elements $\hat{f}_1, \hat{f}_2 \in \widehat{V}_\zeta = \ker \Delta_\zeta$ let us set

$$\widehat{h}(\hat{f}_1, \hat{f}_2) = \int_{\mathbf{C}} h(\hat{f}_1(z), \hat{f}_2(z)). \tag{1}$$

This formula defines a Hermitian metric on $\widehat{V}$.

Theorem (Sz '07)

The triple $(\widehat{V}, \widehat{\theta}, \widehat{h})$ satisfies Hitchin's equations over $\widehat{\mathbf{P}}^1 \setminus \widehat{P}$.

Compatibility with the parabolic structure

Theorem (Sz '17)

At any parabolic point ζ_ι *(including* $\zeta_0 = \infty$*) and for any* $\alpha \in (0,1)$ *the parabolic weight induced by* $\widehat{h}$ *on* $\mathrm{Gr}_\alpha \widehat{\mathcal{E}}$ *is* $\alpha - 1$.
The parabolic weight induced by $\widehat{h}$ *on* $\mathrm{Gr}_0 \widehat{\mathcal{E}}$ *is*

- 0 *on* $\ker \mathrm{Gr}_0 \operatorname{res}_{\zeta_\iota} \widehat{\theta}$
- *otherwise* 0 *on the weight* $k \geq -1$ *part, and*
- -1 *on the weight* $k < -1$ *part.*

Idea of Proof

Following an idea of Biquard–Jardim, it is sufficient to show that the parabolic weights induced by $\widehat{h}$ are bounded from below by the above stated quantities. One proves these estimates by providing explicit representatives of cohomology classes.

Namely, let $\varepsilon > 0$ be chosen very small and fix a cut-off function

$$\chi : \mathbf{C} \to [0, 1]$$

satisfying

- the support of $d\chi$ is contained in the annulus $1/3 < |w| < 2/3$
- χ is identically 1 on the disc $|w| \leq 1/3$
- χ is identically 0 on the complement of the disc $|w| < 2/3$.

For simplicity assume $z_i = 0$. Let ς be any local section of $\mathcal{E}$ near $z = 0$ such that $\varsigma(0) \in \mathrm{Gr}_\alpha \mathcal{E}|_0$. Assume that $\varsigma(0)$ is a generalized eigenvector of $\mathrm{Gr}_\alpha \mathrm{res}_{z=0}\, \theta$ for some eigenvalue λ. It induces a local holomorphic section $\hat{\varsigma}$ of $\widehat{\mathcal{E}}_\alpha$ near $\zeta = \infty$ as follows. We let

$$v(z,\zeta)\mathrm{d}z = \chi\left(\varepsilon^{-1}|\zeta|(z - \lambda\zeta^{-1})\right)\varsigma(z)\zeta\frac{\mathrm{d}z}{z}.$$

and find $t(z,\zeta)\mathrm{d}\bar{z} \in C^\infty(\mathbf{P}^1, V \otimes \Omega^{0,1})$ such that

$$\bar{\partial}^{\mathcal{E}} v(z)\mathrm{d}z + \theta_\zeta t(z)\mathrm{d}\bar{z} = 0.$$

Then as ζ varies the section

$$v(z,\zeta)\mathrm{d}z + t(z,\zeta)\mathrm{d}\bar{z}$$

represents a local section $\hat{\varsigma}$ of $\widehat{\mathcal{E}}_\alpha$ near $\zeta = \infty$.

Estimates

Proposition

For $\alpha \in (0,1)$ *there exists a polynomial* R *such that we have*

$$\int_{\mathbf{C}} |v(z,\zeta)|_h^2 \leq |\zeta|^{2-2\alpha} \cdot R(\log|\zeta|)$$

and

$$\int_{\mathbf{C}} |t(z,\zeta)|_h^2 \leq |\zeta|^{2-2\alpha} \cdot R(\log|\zeta|).$$

Isometry

In what follows, we write

$$\mathcal{M} = \mathcal{M}(\mathbf{P}^1, r, \{z_i\}, A, \{\dim \mathrm{Gr}^j_{F_i}\}, \{\alpha^j_i\}, \{\mathcal{O}^j_i\}).$$

Theorem (Sz '14)

If the orbits $\mathcal{O}^j_i$ are regular semi-simple then Nahm transform is a hyper-Kähler isometry between moduli spaces

$$\mathcal{M} \to \widehat{\mathcal{M}}$$

corresponding to singularity behaviours as specified above.

Strategy of proof: show

$$I \mapsto \widehat{I}$$
$$J \mapsto \widehat{J}$$
$$\Omega_I \mapsto \Omega_{\widehat{I}}$$

Complex structures

The transformation of the Dolbeault complex structure I follows by the algebraic definition of $(\widehat{\mathcal{E}}, \widehat{\theta})$.

The transformation of the complex structure J follows from identification with minimal extension followed by Fourier–Laplace transform of the underlying holonomic $\mathcal{D}$-module (Sz '12).

It remains to show the transformation of Ω_I.

Consider the ruled surface

$$Z = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus L) \xrightarrow{\pi} \mathbf{P}^1$$

and for fixed r consider the Hilbert scheme

$$\mathrm{Hilb}(r)$$

of curves $S \subset Z$ having the same Hilbert polynomial as a generic r to 1 cover of $\mathbf{P}^1$ in Z, and let

$$B \subset \mathrm{Hilb}^0(r)$$

be the Zariski open subset parameterising smooth irreducible curves S not contained in D_∞.

Consider moreover the relative Picard bundle

$$\mathrm{Pic}^d(Z) \to B$$

whose fiber over $b \in B$ is the set of isomorphism classes of degree d line bundles M over S_b.
$\mathrm{Pic}^d(Z)$ has a canonical Poisson structure. The orbits $\mathcal{O}_i^j$ naturally single out points on Z, whose ideal sheaf is denoted by $\mathcal{I}$. There is a symplectic leaf

$$\mathcal{L}(\{z_i\}, A, \{\mathcal{O}_i^j\}) \subset \mathrm{Pic}^d(Z)$$

consisting of sheaves on curves passing through the points defining $\mathcal{I}$.
The deformation space of $\mathcal{L}$ at a point is given by

$$\mathrm{Ext}^1_{\mathcal{I}}(M, M).$$

The induced Mukai symplectic structure Ω_{Muk} on $\mathcal{L}$ is defined by the Yoneda product

$$\Omega_{\mathrm{Muk}} : \mathrm{Ext}^1_{\mathcal{J}}(M, M) \times \mathrm{Ext}^1_{\mathcal{J}}(M, M) \to \mathrm{Ext}^2_{\mathcal{J}}(M, M) \cong \mathbf{C}.$$

There exists a Zariski open subset $\mathcal{M}^0 \subset \mathcal{M}$ such that we have a biholomorphism

$$\Pi : \mathcal{L}(\{z_i\}, A, \{\mathcal{O}_i^j\}) \to \mathcal{M}^0.$$

It is given as follows: given $M \in \mathrm{Pic}^d(Z)$ associate to it

$$\Pi(M) = (\pi_* M, \pi_* \zeta).$$

Proposition

We have $\Pi^*\Omega_I = \Omega_{Muk}$.

A similar result: J. Harnad, J Hurtubise (2008).
Idea of proof: Π induces an isomorphism on deformation spaces

$$\mathrm{Ext}^1_{\mathcal{J}}(M, M) \to \mathbf{H}^1(\mathbf{P}^1, \mathrm{ad}(\theta)).$$

One may compute the Yoneda product using any projective resolution of the sheaves M. The definition of the spectral sheaf

$$0 \to \pi^*\mathcal{E} \otimes L^\vee \xrightarrow{\theta_\zeta} \pi^*\mathcal{E} \to M \to 0$$

provides one such resolution. Using this resolution we get the claim.

The symplectomorphism between Dolbeault structures

Proposition

The Higgs bundles $(\mathcal{E}, \theta)$ and $(\widehat{\mathcal{E}}, \widehat{\theta})$ have isomorphic spectral sheaves:

$$M_{(\mathcal{E},\theta)} \cong M_{(\widehat{\mathcal{E}},\widehat{\theta})}.$$

Applying the previous Proposition to both of these resolutions, we get

$$\Pi^* \Omega_I = \Omega_{\mathrm{Muk}} = \widehat{\Pi}^* \Omega_{\widehat{I}}.$$

An example

(Aker–Sz. '14) On the trivial holomorphic vector bundle $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ of rank 2 set

$$\theta = \begin{pmatrix} \frac{1}{z} & 1 \\ -1 & -\frac{1}{z} \end{pmatrix}.$$

We have $P = \{0\}$ with residue

$$\mathrm{res}_0(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respective parabolic weights $\alpha_1^+ \neq \alpha_1^-$. At $z = \infty$ we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

semisimple with eigenvalues $\pm i$, and $\mathrm{res}_\infty(\theta) = 0$.

Example, continued

We get $\widehat{\mathcal{E}} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ with

$$\widehat{\theta} = -\frac{1}{\zeta^2+1}\begin{pmatrix}\zeta & 1\\ 1 & -\zeta\end{pmatrix}.$$

Furthermore, the parabolic weights at $\zeta = \infty$ are α_1^+, α_1^-.