Nahm transform for parabolic integrable connections on the Riemann sphere

Szilárd Szabó

Budapest University of Technology and Rényi Institute of Mathematics
Budapest

Singapore, August 4, 2014

Outline

Notations

X: smooth projective curve over $\mathbf{C}$
$G = \mathrm{Gl}_r(\mathbf{C})$
$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$P = \{p_0, p_1, \ldots, p_n\}$: a finite set of distinct points in X
$\mathcal{O}$: sheaf of holomorphic functions
Ω^k: sheaf of smooth k-forms
$\mathbf{\Omega}^1$: sheaf of holomorphic 1-forms

Meromorphic connections and Higgs bundles on curves

Let E be a holomorphic vector bundle of rank r on X and D a meromorphic connection with singularities in P:

$$D : E \longrightarrow \mathbf{\Omega}^1(*P) \otimes_{\mathbf{C}} E$$

satisfying the Leinbiz-rule.
Paralelly, let $\mathcal{E}$ be a holomorphic vector bundle of rank r on X and

$$\theta : \mathcal{E} \longrightarrow \mathbf{\Omega}^1(*P) \otimes_{\mathcal{O}} \mathcal{E}$$

a meromorphic Higgs field.

Fixing the irregular parts of D

We fix the behaviour of D near the singular points as follows:

$$D = d + A_n \frac{dz}{z^n} + \cdots + A_2 \frac{dz}{z^2} + O(z^{-1})dz$$

with respect to some local analytic coordinate z and some holomorphic trivialisation, where

$$A_2, \ldots, A_n$$

belong to some torus $\mathfrak{t} \subset \mathfrak{gl}_r(\mathbf{C})$. Let

$$H \subset \mathrm{Gl}_r(\mathbf{C})$$

stand for the common centraliser of $A_2, \ldots, A_n$ and $\mathfrak{h}$ for its Lie-algebra.

Fixing the irregular parts of θ

Paralelly, we assume

$$\theta = T_n \frac{dz}{z^n} + \cdots + T_2 \frac{dz}{z^2} + O(z^{-1}) dz$$

with respect to some trivialisation, where

$$T_k = \frac{A_k}{2} \quad (2 \leq k \leq n).$$

Parabolic structure at singular points

A compatible parabolic structure for D is the choice of an element

$$\beta \in \mathfrak{t}_{\mathbf{R}}.$$

Up to conjugation we may assume $\mathfrak{t}$ consists of diagonal matrices, so we have

$$\beta = \mathrm{diag}(\beta_1, \ldots, \beta_r).$$

Similarly, a compatible parabolic structure for θ is the choice of

$$\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_r) \in \mathfrak{t}_{\mathbf{R}}.$$

To α we associate the parabolic subgroup

$$P_\alpha = \{g \in \mathrm{Gl}_r(\mathbf{C}) | \quad z^\alpha g z^{-\alpha} \text{ exists as } z \to 0\}$$

and similarly we get P_β with Lie-algebras $\mathfrak{p}_\alpha, \mathfrak{p}_\beta$ respectively.

Residues

We assume that

$$A_1 \in \mathcal{O} \subset \mathfrak{h} \cap \mathfrak{p}_\beta$$

is in a fixed semi-simple adjoint orbit, defined by eigenvalues

$$\mu_1, \dots, \mu_r$$

and similarly,

$$T_1 \in \mathcal{O}' \subset \mathfrak{h} \cap \mathfrak{p}_\alpha$$

is in a fixed semi-simple adjoint orbit, defined by eigenvalues

$$\lambda_1, \dots, \lambda_r.$$

These parameters are subject to Simpson's relations

$$\alpha_i = \Re(\mu_i), \quad \lambda_i = \frac{\mu_i - \beta_i}{2}.$$

Stability of connections

The parabolic degree and slope of E are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta_k^j$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$

Stability of connections

The parabolic degree and slope of E are defined respectively as

$$\text{par-deg}(E) = \deg(E) + \sum_{j=0}^{n} \sum_{k=1}^{r} \beta_k^j$$

and

$$\text{par-slope}(E) = \frac{\text{par-deg}(E)}{\text{rank}(E)}.$$

(E, D) is said to be parabolically stable if for all non-trivial proper subbundle $F \subset E$ such that $\text{Im}(D|_F) \subset \mathbf{\Omega}^1(*P) \otimes F$, one has

$$\text{par-slope}(F) < \text{par-slope}(E).$$

Stability of Higgs bundles

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_k^j$$

and

$$\text{par-slope}(\mathcal{E}) = \frac{\text{par-deg}(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

Stability of Higgs bundles

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_k^j$$

and

$$\text{par-slope}(\mathcal{E}) = \frac{\text{par-deg}(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

$(\mathcal{E}, \theta)$ is said to be parabolically stable if for all non-trivial proper subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\text{Im}(\theta|_{\mathcal{F}}) \subset \mathbf{\Omega}^1(*P) \otimes \mathcal{F}$, one has

$$\text{par-slope}(\mathcal{F}) < \text{par-slope}(\mathcal{E}).$$

Adapted Hermitian metrics

A Hermitian fiber metric h is adapted to the parabolic structure of (E, D) (respectively $(\mathcal{E}, \theta)$) if near all $p_j \in X$ it is mutually bounded with

$$\mathrm{diag}(|z_j|^{2\beta^j_k})_{k=1,\dots,r},$$

(respectively $\mathrm{diag}(|z_j|^{2\alpha^j_k})$) where z_j is a local holomorphic coordinate of X vanishing at p_j.

Adapted Hermitian metrics

A Hermitian fiber metric h is adapted to the parabolic structure of (E, D) (respectively $(\mathcal{E}, \theta)$) if near all $p_j \in X$ it is mutually bounded with

$$\mathrm{diag}(|z_j|^{2\beta^j_k})_{k=1,\ldots,r},$$

(respectively $\mathrm{diag}(|z_j|^{2\alpha^j_k})$) where z_j is a local holomorphic coordinate of X vanishing at p_j.

Remark

Without the semi-simplicity assumption on the residues, the form of the matrices involves logarithmic terms corresponding to the weight filtration too.

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Consider the decomposition

$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively.

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Consider the decomposition

$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively. Decompose these parts further according to bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

Harmonic metrics

Let (E, D) be a meromorphic connection endowed with a parabolic structure, and h an adapted Hermitian metric on it. Consider the decomposition

$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively. Decompose these parts further according to bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

Then, h is said to be harmonic if

$$\bar{\partial}^+ \theta = 0.$$

Hermitian–Einstein metrics

Let $(\mathcal{E}, \theta)$ be a meromorphic Higgs field endowed with a parabolic structure, and h an adapted Hermitian metric on it. Let

$$D_{Chern}$$

denote the Chern connection associated with $\bar{\partial}^{\mathcal{E}}$ and h and θ^* the adjoint of θ with respect to h. Then, h is said to be Hermitian–Einstein if the connection

$$D = D_{Chern} + \theta + \theta^*$$

is flat. If this holds then (D, h) solve Hitchin's equations

$$\begin{aligned} F_{D^+} + [\theta, \theta^*] &= 0 \\ \bar{\partial}^+ \theta &= 0. \end{aligned}$$

Wild non-abelian Hodge theory

Theorem (O. Biquard – P. Boalch 2004)

1. *Let (E, D) be a parabolically stable meromorphic integrable connection of parabolic degree 0, with polar parts fixed as above. Then, there exists a unique adapted harmonic metric h (up to a constant).*
2. *Let $(\mathcal{E}, \theta)$ be a parabolically stable meromorphic Higgs bundle of parabolic degree 0, with polar parts fixed as above. Then, there exists a unique adapted Hermitian–Einstein metric h (up to a constant).*
3. *For generic values of the parameters the moduli space $\mathcal{M}^{irr}$ of irreducible solutions of Hitchin's equations with prescribed singularity data up to unitary gauge transformations is a smooth complete hyper-Kähler manifold.*

Deformation theory and Riemannian structure

From now on, the parameters are assumed to be generic so that $\mathcal{M}$ is smooth and complete, and $(E, D) \in \mathcal{M}$ with harmonic Hermitian metric h.

The tangent space of $\mathcal{M}$ at (E, D) is given by

$$T_{(E,D)}\mathcal{M} = \{a \in L^2(X, \Omega^1_X \otimes End(E)) : \quad D(a) = 0, D^*(a) = 0\}.$$

The Atiyah–Bott Riemannian structure is given by the natural L^2-metric

$$\sqrt{-1} \int_X \mathrm{tr}(a \wedge a^*).$$

Complex structures

The de Rham and Dolbeault complex structures are respectively given by

$$J(a) = \sqrt{-1}a, \quad I(a) = \sqrt{-1}(a^{0,1})^* - \sqrt{-1}(a^{1,0})^*.$$

Write

$$(D + a)^+ = (\partial^+ - \dot{A}^*) + (\bar{\partial}^+ + \dot{A})$$

with $\dot{A}$ of type $(0, 1)$ and let

$$\Phi + \dot{\Phi} + \dot{\Phi}^*, \quad \dot{\Phi} \in \Omega^{1,0}$$

denote the self-adjoint part of $D + a$. Then we have

$$I(\dot{A}, \dot{\Phi}) = (\sqrt{-1}\dot{A}, \sqrt{-1}\dot{\Phi}).$$

Dolbeault holomorphic symplectic structure

Given a hyper-Kähler manifold (M, g, I, J, K), let

$$\omega_J(.,.) = g(.,J.), \quad \omega_K(.,.) = g(.,K.)$$

be the Kähler forms and

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K.$$

Then (I, Ω_I) defines a holomorphic symplectic structure on M. For $\mathcal{M}$ with g, I, J, K defined as above this structure is given by

$$\Omega_I((\dot{A}, \dot{\Phi}), (\dot{B}, \dot{\Psi})) = \int_X \mathrm{tr}(\dot{\Psi} \wedge \dot{A} - \dot{\Phi} \wedge \dot{B}).$$

Isometries between moduli spaces

Question

Are there isometries between the wild Hitchin moduli spaces ?

Isometries between moduli spaces

Question

Are there isometries between the wild Hitchin moduli spaces ?

Yes, some are given by Nahm transformation.

Assumption on points at finite distance

From now on we let $X = \mathbf{P}^1$, $p_1, \dots, p_n \in \mathbf{C}$, $p_0 = \infty$.
D is supposed to have a logarithmic singularity (i.e., $n = 0$) at p_j for $j \in \{1, \dots, n\}$: in a local trivialisation of E near p_j, one has

$$D = d + \frac{A^j(z)}{z - p_j} dz,$$

where A^j is a holomorphic matrix-valued function defined near p_j.
Furthermore, the residue

$$A^j(p_j) = \mathrm{diag}(0, \dots, 0, \mu^j_{r_j+1}, \dots, \mu^j_r),$$

is diagonal, with μ^j_k non-zero and generic.

Assumption at infinity

D is supposed to have an irregular singularity with $n-1=1$ at infinity: in a local trivialisation of E near ∞, one has

$$D = d + Adz + B\frac{dz}{z} + \text{lower order terms},$$

where

$$A = \operatorname{diag}(\xi_1, \ldots, \xi_1, \ldots\ldots\ldots, \xi_{n'}, \ldots.., \xi_{n'})$$
$$B = \operatorname{diag}(\mu_1^0, \ldots, \mu_{a_2}^0, \ldots\ldots, \mu_{1+a_{n'}}^0, \ldots, \mu_r^0)$$

(the leading order term and residue, respectively). Here the ξ_k are pairwise distinct constants, and the μ_l^0 are generic non-zero. (Notation:$a_1 = 0, a_{n'+1} = r$.)

Exponential twist

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.
For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$

Exponential twist

Let $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{P}}^1$ be another copy of $\mathbf{C}$ and $\mathbf{P}^1$ respectively.
Call $\widehat{P} = \{\xi_1, \ldots, \xi_{n'}\}$ the transformed singular set.
For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, define the twisted connection as

$$D_\xi = D - \xi dz.$$

Let D_ξ^* stand for the adjoint operator of D_ξ with respect to h, and define the twisted Laplace operator

$$\Delta_\xi = D_\xi D_\xi^* + D_\xi^* D_\xi$$

as an unbounded operator acting on $L^2(\Omega^1 \otimes E)$.

The kernel of the twisted Laplacian

Theorem

For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, the twisted Laplace operator

$$\Delta_\xi : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

has finite dimensional kernel.

The kernel of the twisted Laplacian

Theorem

For any $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$, the twisted Laplace operator

$$\Delta_\xi : L^2(\Omega^1 \otimes E) \longrightarrow L^2(\Omega^1 \otimes E)$$

has finite dimensional kernel.

The vector spaces $\ker(\Delta_\xi)$ form a smooth family of finite-dimensional subspaces of $L^2(\Omega^1 \otimes E)$ of the same dimension, parametrized by $\widehat{\mathbf{C}} \setminus \widehat{P}$.

Definition

The smooth vector bundle with fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ equal to $\ker(\Delta_\xi)$ is called the transformed smooth vector bundle. We denote it by $\widehat{E}$, and its fiber over ξ by $\widehat{E}_\xi$.

Transformed vector bundle and metric

Definition

The smooth vector bundle with fiber over $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$ equal to $\ker(\Delta_\xi)$ is called the transformed smooth vector bundle. We denote it by $\widehat{E}$, and its fiber over ξ by $\widehat{E}_\xi$.

Let $\varphi(z), \psi(z) \in \widehat{E}_\xi$ for some $\xi \in \widehat{\mathbf{C}} \setminus \widehat{P}$.

Definition

The transformed Hermitian metric $\widehat{h}$ is defined on the fiber $\widehat{E}_\xi$ by the formula

$$\widehat{h}(\varphi, \psi) = \int_{\mathbf{C}} h(\varphi(z), \psi(z)).$$

The transformed flat connection

L^2-metric on sections of $\Omega^1 \otimes E$ induces an orthogonal projection

$$\pi_\xi : L^2(\mathbf{P}^1, \Omega^1 \otimes E) \longrightarrow \widehat{E}_\xi.$$

Fix $\xi_0 \in \widehat{\mathbf{C}} \setminus \widehat{P}$ and let $\varphi_1(z), \ldots, \varphi_{r'}(z)$ be a basis of $\widehat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all ξ sufficiently close to ξ_0 in $\widehat{\mathbf{C}} \setminus \widehat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi-\xi_0)z}\varphi_j(z)) \in \widehat{E}_\xi.$$

The transformed flat connection

L^2-metric on sections of $\Omega^1 \otimes E$ induces an orthogonal projection

$$\pi_\xi : L^2(\mathbf{P}^1, \Omega^1 \otimes E) \longrightarrow \widehat{E}_\xi.$$

Fix $\xi_0 \in \widehat{\mathbf{C}} \setminus \widehat{P}$ and let $\varphi_1(z), \ldots, \varphi_{r'}(z)$ be a basis of $\widehat{E}_{\xi_0}$. These sections are exponentially decreasing at infinity. In particular, for all ξ sufficiently close to ξ_0 in $\widehat{\mathbf{C}} \setminus \widehat{P}$ one can consider the sections

$$\varphi_j(\xi; z) = \pi_\xi(e^{(\xi-\xi_0)z}\varphi_j(z)) \in \widehat{E}_\xi.$$

Definition

The transformed flat connection $\widehat{D}$ on $\widehat{E}$ is defined by the basis of local parallel sections $\varphi_j(\xi; z)$ for $j \in \{1, \ldots, r'\}$.

Metric extension

Definition

The metric extension of $\widehat{E}$ over $\xi_l \in \widehat{P}$ (respectively $\widehat{\infty}$) is the lattice consisting of local holomorphic sections outside of ξ_l (respectively $\widehat{\infty}$) whose $\widehat{h}$-norm is bounded from above by a constant.

We denote

$$(\widehat{E}, \widehat{D}, \widehat{h}) = \mathcal{N}(E, D, h).$$

Properties of the transform

Theorem (Sz 2008)

- $\widehat{D}$ *is an integrable connection on* $\widehat{E}$*, with logarithmic singularities in* $\xi_l \in \widehat{P}$ *and an irregular singularity of Poincaré-rank* 1 *(*$n = 2$*) at* $\widehat{\infty}$*.*
- *The metric extension induces a parabolic structure on* $\widehat{E}$ *at the singular points.*
- *The corresponding eigenvalues and parabolic weights transform according to the diagrams on the next two slides. In particular,* $\widehat{E}$ *is of rank* $\sum_{j=1}^{n}(r - r_j)$ *and of parabolic degree* 0*.*
- *The metric* $\widehat{h}$ *is harmonic for* $\widehat{D}$*.*
- *Nahm transform* $\mathcal{N}$ *is involutive (up to a sign).*

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n
$\xi_1 + z^{-1}\mu_1^0$	0		0
$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$
	0			
$\vdots$			$\vdots$	$\vdots$
	$\vdots$			
$\xi_1 + z^{-1}\mu^0_{a_2}$			0	$-p_1 + \zeta^{-1}\mu^1_r$
	0		$\mu^n_{r_n+1}$	
$\vdots$				$\vdots$
	$\mu^1_{r_1+1}$		$\vdots$	
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$				$-p_n + \zeta^{-1}\mu^n_{r_n+1}$
	$\vdots$		$\vdots$	
$\vdots$				$\vdots$
	μ^1_r		μ^n_r	
$\xi_{n'} + z^{-1}\mu^0_r$				$-p_n + \zeta^{-1}\mu^n_r$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	$\color{blue}{\xi_1}$
$\color{blue}{\xi_1} + z^{-1}\color{red}{\mu_1^0}$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\color{blue}{\xi_1} + z^{-1}\color{red}{\mu_{a_2}^0}$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu^1_r$	$\vdots$
$\vdots$	0		$\mu^n_{r_n+1}$	$\vdots$	0
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$	$\mu^1_{r_1+1}$		$\vdots$	$-p_n + \zeta^{-1}\mu^n_{r_n+1}$	$\color{red}{\mu_1^0}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\xi_{n'} + z^{-1}\mu^0_r$	μ^1_r		μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$	$\color{red}{\mu_{a_2}^0}$

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu_1^0$	0		0	$-p_1 + \zeta^{-1}\mu_{r_1+1}^1$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\xi_1 + z^{-1}\mu_{a_2}^0$	$\vdots$		0	$-p_1 + \zeta^{-1}\mu_r^1$	$\vdots$		0
$\vdots$	0		$\mu_{r_n+1}^n$	$\vdots$	0		$\mu_{1+a_{n'}}^0$
$\xi_{n'} + z^{-1}\mu_{1+a_{n'}}^0$	$\mu_{r_1+1}^1$		$\vdots$	$-p_n + \zeta^{-1}\mu_{r_n+1}^n$	μ_1^0		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\xi_{n'} + z^{-1}\mu_r^0$	μ_r^1		μ_r^n	$-p_n + \zeta^{-1}\mu_r^n$	$\mu_{a_2}^0$		μ_r^0

Transform of the eigenvalues

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
$\xi_1 + z^{-1}\mu^0_1$	0		0	$-p_1 + \zeta^{-1}\mu^1_{r_1+1}$	0		0
	0				0		
$\vdots$			$\vdots$	$\vdots$			$\vdots$
	$\vdots$				$\vdots$		
$\xi_1 + z^{-1}\mu^0_{a_2}$			0	$-p_1 + \zeta^{-1}\mu^1_r$			0
	0		$\mu^n_{r_n+1}$		0		$\mu^0_{1+a_{n'}}$
$\vdots$				$\vdots$			
	$\mu^1_{r_1+1}$		$\vdots$		μ^0_1		$\vdots$
$\xi_{n'} + z^{-1}\mu^0_{1+a_{n'}}$				$-p_n + \zeta^{-1}\mu^n_{r_n+1}$			
	$\vdots$				$\vdots$		
$\vdots$			$\vdots$	$\vdots$			$\vdots$
	μ^1_r				$\mu^0_{a_2}$		
$\xi_{n'} + z^{-1}\mu^0_r$			μ^n_r	$-p_n + \zeta^{-1}\mu^n_r$			μ^0_r

Transform of the weights

∞	p_1	$\cdots$	p_n
β^0_1	0		0
$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
	β^1_r		β^n_r
β^0_r			

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$
$---$	$---$		$---$	$---$
β^0_1	0		0	$\beta^1_{r_1+1}$
$\vdots$	0		$\vdots$	$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1
$---$	$---$		$---$	$---$	$---$
β^0_1	0		0	$\beta^1_{r_1+1}$	0
$\vdots$	0		$\vdots$	$\vdots$	0
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$

Transform of the weights

∞	p_1	$\cdots$	p_n	$\widehat{\infty}$	ξ_1	$\cdots$	$\xi_{n'}$
β_1^0	0		0	$\beta^1_{r_1+1}$	0		0
$\vdots$	0		$\vdots$	$\vdots$	0		$\vdots$
$\beta^0_{a_2}$	$\vdots$		0	β^1_r	$\vdots$		0
$\vdots$	0		$\beta^n_{r_n+1}$	$\vdots$	0		$\beta^0_{1+a_{n'}}$
$\beta^0_{1+a_{n'}}$	$\beta^1_{r_1+1}$		$\vdots$	$\beta^n_{r_n+1}$	β^0_1		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
β^0_r	β^1_r		β^n_r	β^n_r	$\beta^0_{a_2}$		β^0_r

Isometry

Theorem (Sz 2014)

Nahm transform is a hyper-Kähler isometry.

Strategy of proof: show

$$I \mapsto \widehat{I}$$
$$J \mapsto \widehat{J}$$
$$\Omega_I \mapsto \Omega_{\widehat{I}}$$

Complex structures

The transformation of the complex structure I follows from an algebraic interpretation

$$L^2H^1(D_\xi) \cong \mathbf{H}^1(\mathcal{E} \xrightarrow{\theta - \xi dz \wedge} \mathcal{F})$$

as the hypercohomology of the Dolbeault complex (Aker–Sz 2014).

The transformation of the complex structure J follows from identification with minimal extension followed by Fourier–Laplace transform of the underlying holonomic $\mathcal{D}$-module (Sz 2012).

From now on we will focus on the transformation of Ω_I.

Beauville–Narasimhan–Ramanan correspondence

Set

$$L = \boldsymbol{\Omega}^1_{\mathbf{P}^1}(P) = \boldsymbol{\Omega}^1_{\mathbf{P}^1}(p_1 + \cdots + p_n + 2 \cdot \infty)$$

and consider the ruled surface

$$Z = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus L) \xrightarrow{\pi} \mathbf{P}^1$$

with relatively ample line bundle $\mathcal{O}(1)$ and global sections

$$x \in H^0(Z, \mathcal{O}(1) \otimes \pi^* L), \quad y \in H^0(Z, \pi^* L).$$

Consider the cokernel sheaf $M_{(\mathcal{E},\theta)}$ defined by

$$0 \to \pi^*(\mathcal{E} \otimes L^\vee) \xrightarrow{\pi^*\theta \otimes y - \pi^* Id_{\mathcal{E}} \otimes x} \pi^*\mathcal{E} \otimes \mathcal{O}(1) \to M_{(\mathcal{E},\theta)} \to 0.$$

Beauville–Narasimhan–Ramanan correspondence, cont'd

It is possible to recover $(\mathcal{E}, \theta)$ from $M_{(\mathcal{E},\theta)}$:

$$\mathcal{E} = \pi_* M_{(\mathcal{E},\theta)}, \quad \theta = \pi_*(x : M \to M \otimes \pi_* L \otimes \mathcal{O}(1)).$$

The support $S_{(\mathcal{E},\theta)}$ of $M_{(\mathcal{E},\theta)}$ is called spectral curve.
The above associations induce an equivalence between the categories of

Higgs bundles with integral spectral curve S

and

torsion sheaves of pure dimension 1 and of rank 1 supported away from (y)

Hilbert scheme of curves

Notice: Z is a holomorphic Poisson surface with Liouville symplectic form ω degenerating along

$$D_\infty = \pi^{-1}P + 2 \cdot (y).$$

Given r consider the Hilbert scheme

$$\mathrm{Hilb}(r)$$

of curves $S \subset Z$ having the same Hilbert polynomial as a generic r to 1 cover of $\mathbf{P}^1$ in Z,

$$\mathrm{Hilb}^0(r) \subset \mathrm{Hilb}(r)$$

the connected component of a given S_0, and

$$B \subset \mathrm{Hilb}^0(r)$$

the Zariski open subset parameterising smooth irreducible curves S not contained in D_∞.

Consider moreover the relative Picard bundle

$$\mathrm{Pic}^d(Z) \to B$$

whose fiber over $b \in B$ is the set of isomorphism classes of degree d line bundles over S_b.

Theorem (Donagi, Markman 1996)

B is smooth and $\mathrm{Pic}^d(Z)$ has a canonical Poisson structure whose symplectic leaves are obtained by prescribing the intersection of the curves S with D_∞.

Deformation theory of sheaves

The deformation theory of $\mathrm{Pic}^d(Z)$ at a given sheaf M is given by the global Ext-groups

$$\mathrm{Ext}^*_{\mathcal{O}_Z}(M, M)$$

and the restriction of the Poisson structure Ω_{Mukai} to the symplectic leaves is induced by the Yoneda product

$$\cup : \mathrm{Ext}^1_{\mathcal{O}_Z}(M, M) \times \mathrm{Ext}^1_{\mathcal{O}_Z}(M, M) \to \mathrm{Ext}^2_{\mathcal{O}_Z}(M, M)$$

followed by Serre duality.

Matching the symplectic structures

Consider two 1-parameter families

$$(\mathcal{E}(t), \Phi(t)), \quad (\mathcal{E}(x), \Phi(x))$$

of elements of $\mathcal{M}$ for $t, x \in \mathbf{C}$ both specialising to $(\mathcal{E}, \Phi) \in \mathcal{M}$ at $t = 0$ and $x = 0$ respectively. They give rise to

$$T, X \in T_{(\mathcal{E},\Phi)}\mathcal{M}.$$

The associated families of spectral sheaves

$$M_{(\mathcal{E}(t),\Phi(t))}, \quad M_{(\mathcal{E}(x),\Phi(x))}$$

in $\mathrm{Pic}^d(Z)$ then give rise to tangent vectors

$$\tilde{T}, \tilde{X} \in T_{M_{(\mathcal{E},\Phi)}} \mathrm{Pic}^d(Z).$$

Matching the symplectic structures, cont'd

Then we have the

Key Formula

$$\Omega_I(T, X) = \Omega_{\mathsf{Mukai}}(\tilde{T}, \tilde{X}).$$

Proven in particular cases by Hurtubise (1996) and Hurtubise–Harnad (2008).

End of the proof using Key Formula

Known:

$$(\mathcal{E}, \theta) \quad (\widehat{\mathcal{E}}, \widehat{\theta})$$

have isomorphic spectral sheaves

$$M_{(\mathcal{E}(t), \Phi(t))} \cong M_{(\widehat{\mathcal{E}}(t), \widehat{\Phi}(t))}$$

on the open surface

$$T^*(\mathbf{C} \setminus P).$$

Therefore, the Key Formula applied to the vectors

$$\widehat{T} = T_{(\mathcal{E}, \Phi)}\mathcal{N}(T), \quad \widehat{X} = T_{(\mathcal{E}, \Phi)}\mathcal{N}(X)$$

shows that

$$\Omega_{\widehat{I}}(\widehat{T}, \widehat{X}) = \Omega_{\mathrm{Mukai}}(\tilde{T}, \tilde{X})$$

too.

Computing global Ext groups

By definition, the Ext-groups $\mathrm{Ext}^*_{\mathcal{O}_Z}(M, M)$ are computed by the hypercohomology of the following complex of coherent sheaves on Z:

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^*\mathcal{E} \otimes \mathcal{O}(1), M_{(\mathcal{E},\Phi)}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_C} L^\vee), M_{(\mathcal{E},\Phi)})$$

in degrees $0, 1$ where the arrow is given by

$$\mathcal{H}om(\pi^*\Phi \otimes y - \pi^* Id_{\mathcal{E}} \otimes x, Id_M).$$

The sheaves of this complex are supported on $S_{(\mathcal{E},\Phi)}$ and its push-forward by π is

$$\mathcal{E}nd_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{E}) \xrightarrow{\mathrm{ad}_\Phi} \mathcal{E}nd_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathbf{P}^1}} L.$$

LOCAL BRANCHES AND FRAMES

Let Δ be a small analytic disc in $\mathbf{P}^1 \setminus P_{red}$ such that

$$S \cap \pi^{-1}(\Delta) = S_1 \cup \cdots \cup S_r$$

with

$$\pi_j = \pi|_{S_j} : S_j \to \Delta$$

bianalytic. Let furthermore $z \in \Delta$ be a local holomorphic coordinate and

$$x_1(z), \ldots, x_r(z)$$

the eigenvalues of θ over Δ so that

$$S_j = \mathrm{Im}(x_j).$$

Then

$$M|_{S_j}$$

is a holomorphic line bundle over S_j with some holomorphic trivialisation $\mathbf{m}_j$.

Diagonalisation

Set

$$\mathbf{e}_j = \pi_* \mathbf{m}_j;$$

with respect to the frame $\mathbf{e}_1, \ldots, \mathbf{e}_r$ one has

$$\theta(z) = \mathrm{diag}(x_1(z), \ldots, x_r(z))$$

and

$$\mathbf{m}_j = [\mathbf{e}_j] \in \mathrm{coker}(\theta - x_j)$$

on S_j. A frame for

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^* \mathcal{E} \otimes \mathcal{O}(1), M_{(\mathcal{E},\Phi)})|_{S_j}$$

is then given by

$$\pi^* \mathbf{e}_1^\vee \otimes \mathbf{m}_j, \ldots, \pi^* \mathbf{e}_r^\vee \otimes \mathbf{m}_j.$$

Diagonalisation, cont'd

Let

$$w_j = \pi_j^{-1}(z)$$

be the local holomorphic coordinate on S_j, then a frame for

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_{\mathbf{P}^1}} L^\vee), M_{(\mathcal{E},\Phi)})|_{S_j}$$

is given by

$$\pi^* \mathbf{e}_1^\vee \otimes \mathbf{m}_j dw_j, \ldots, \pi^* \mathbf{e}_r^\vee \otimes \mathbf{m}_j dw_j.$$

Dolbeault representatives of tangent vectors

Let

$$T = [(\dot{A}, \dot{\Phi})], \quad X = [(\dot{B}, \dot{\Psi})] \in T_{(\mathcal{E},\theta)}\mathcal{M} \cong \mathbf{H}^1(X, \mathrm{ad}_\theta)$$

with

$$\dot{A} = a d\bar{z} \qquad \dot{\Phi} = \phi dz$$
$$\dot{B} = b d\bar{z} \qquad \dot{\Psi} = \psi dz$$

where

$$a = (a_{ij}), b, \phi, \psi : \Delta \to \mathfrak{gl}_r(\mathbf{C})$$

are L^2 matrices of endomorphisms of E with respect to the framing $\mathbf{e}_1, \dots, \mathbf{e}_r$ satisfying

$$\bar{\partial}(\phi dz) + [a d\bar{z}, \theta] = 0, \quad \bar{\partial}(\psi dz) + [b d\bar{z}, \theta] = 0.$$

Lifting tangent vectors

Define now

$$(\dot{\tilde{A}}, \dot{\tilde{\Phi}}), \quad (\dot{\tilde{B}}, \dot{\tilde{\Psi}})$$

on S_j as $(\tilde{a}_j d\bar{w}_j, \tilde{\phi}_j dw_j)$ and $(\tilde{b}_j d\bar{w}_j, \tilde{\psi}_j dw_j)$ respectively, where

$$\tilde{a}_j(w_j) : \pi_j^* \mathbf{e}_i \mapsto a_{ij}(\pi(w_j))\mathbf{m}_j$$
$$\tilde{\phi}_j(w_j) : \pi_j^* \mathbf{e}_i \mapsto \phi_{ij}(\pi(w_j))\mathbf{m}_j$$

and

$$\tilde{b}_j(w_j) : \pi_j^* \mathbf{e}_i \mapsto b_{ij}(\pi(w_j))\mathbf{m}_j$$
$$\tilde{\psi}_j(w_j) : \pi_j^* \mathbf{e}_i \mapsto \psi_{ij}(\pi(w_j))\mathbf{m}_j.$$

Lifting tangent vectors, cont'd

The above local definitions then match up to define global L^2 sections

$$(\tilde{\dot{A}}, \tilde{\dot{\Phi}}), \quad (\tilde{\dot{B}}, \tilde{\dot{\Psi}})$$

away from $\pi^{-1}(P_{red} \cup R)$ where R is the branch locus of $\pi : S \to C$.

The push-forwards of these sections by π are equal to $(\dot{A}, \dot{\Phi})$ and $(\dot{B}, \dot{\Psi})$, respectively.

Finally, they are 1-cocycles in the Dolbeault resolution of

$$\mathcal{H}om_{\mathcal{O}_Z}(\pi^*\mathcal{E} \otimes \mathcal{O}(1), M_{(\mathcal{E},\Phi)}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Z}(\pi^*(\mathcal{E} \otimes_{\mathcal{O}_C} L^\vee), M_{(\mathcal{E},\Phi)}).$$

In particular, they define elements

$$\tilde{T}, \tilde{X} \in \mathrm{Ext}^1_{\mathcal{O}_Z}(M_{(\mathcal{E},\theta)}, M_{(\mathcal{E},\theta)}).$$

Identifying Ext^2

A standard spectral sequence argument yields

$$\mathrm{Ext}^2_{\mathcal{O}_Z}(M_{(\mathcal{E},\Phi)}, M_{(\mathcal{E},\Phi)}) \cong H^1(Z, \mathcal{E}xt^1_{\mathcal{O}_Z}(M_{(\mathcal{E},\Phi)}, M_{(\mathcal{E},\Phi)})).$$

The Poisson bivector induces an isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_Z}(M_{(\mathcal{E},\Phi)}, M_{(\mathcal{E},\Phi)}) \cong K_S(-(S \cap \pi^{-1}(P))).$$

where S stands for $S_{(\mathcal{E},\Phi)}$.We infer

$$\begin{aligned} \mathrm{Ext}^2_{\mathcal{O}_Z}(M_{(\mathcal{E},\Phi)}, M_{(\mathcal{E},\Phi)}) &\cong H^1(S, K_S(-(S \cap \pi^{-1}(P)))) \\ &\cong H^0(S, \mathcal{O}_S(S \cap \pi^{-1}(P)))^\vee \\ &\cong H^0(S, \mathcal{O}_S)^\vee \oplus \oplus_{i=1}^r {}_{k=1}^n \mathbf{C}^{m_k+1}_{(z_k, x_i)}. \end{aligned}$$

Yoneda product

The Yoneda product $\tilde{T} \cup \tilde{X}$ of the lifted tangent vectors is given by composition of homomorphisms coupled with wedge product of differential forms.
Therefore in local coordinates on the sheet S_i it can be represented by

$$\sum_j (a_{ij}\psi_{ji} - b_{ij}\phi_{ji}) dw_i \wedge d\bar{w}_i \in \Omega^2(S_i).$$

The evaluation of this Yoneda product on the generator of $H^0(S, \mathcal{O}_S)$ is

$$\sum_{i=1}^{r} \int_{S_i} (a_{ij}\psi_{ji} - b_{ij}\phi_{ji}) dw_i \wedge d\bar{w}_i.$$

Mukai form

The changes of variables $w_i \rightsquigarrow z$ transform the above sum of integrals into

$$\int_\Delta \mathrm{tr}(\dot{\Psi} \wedge \dot{A} - \dot{\Phi} \wedge \dot{B})$$

which is the expression computing Ω_I on Δ.
The Mukai form evaluated on $\tilde{T}, \tilde{X}$ can then be obtained by globalising the above analysis, using a partition of unity argument.
To make sure the formulae converge at P_{red} one uses the estimate

$$|a_{ij}\psi_{ji}| \leq Cr^{-2+2\delta}$$

of Biquard and Boalch for some $\delta > 0$ as $r = |z - p_k| \to 0$.