Sheaf-counting and wall-crossing in Painlevé moduli spaces / Kéve-leszámlálás és wall-crossing Painlevé modulustereken

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Outline

Hodge theory, Dolbeault

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Data of wild non-abelian Hodge theory (NAHT)

Hitchin '87, Simpson '90, Sabbah '99, Biquard–Boalch '04: fix

- C: smooth projective curve over $\mathbb{C}$
- $r \geqslant 2$ rank (i.e., $G = \mathrm{Gl}_r(\mathbb{C})$)
- $p_1, \ldots, p_n \in C$ irregular singularities (with local charts z_j), and for each p_j:
- a parabolic subalgebra $\mathfrak{p}_j \subset \mathfrak{gl}_r$ with associated Levi $\mathfrak{l}_j$
- parabolic weights $\{\alpha_j^i\}_i$
- an unramified irregular type $Q_j \in \mathfrak{t} \otimes \mathbb{C}((z_j))/\mathbb{C}[[z_j]]$ with centralizer $\mathfrak{h}_j$; we set $n_j = \deg(Q_j) + 1$;
- an adjoint orbit $\mathcal{O}_j$ in $\mathfrak{l}_j \cap \mathfrak{h}_j$.

Hitchin's equations and wild NAHT

Fix a smooth vector bundle $V \to C$ of rank r.
The space of solutions of Hitchin's equations

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for

- a Hermitian metric h on V with behaviour $h \approx \mathrm{diag}(|z|^{2\alpha_j^1}, \dots, |z|^{2\alpha_j^r})$ near p_j,
- a unitary connection D in (V, h),
- and a tensor field $\theta \colon V \to V \otimes \Omega^{1,0}_C$ over $C \setminus \{p_1, \dots, p_n\}$ having prescribed boundary values (irregular part Q_j and residue in $\mathcal{O}_j$) at p_j

$\rightsquigarrow$ hyper-Kähler Hodge moduli space $\mathcal{M}_{\mathrm{Hod}}$.

Dolbeault complex structure

We will study one of the Kähler structures on $\mathcal{M}_{\mathrm{Hod}}$ called Dolbeault complex structure. $\mathcal{M}_{\mathrm{Hod}}$ equipped with this complex structure will be denoted by $\mathcal{M}_{\mathrm{Dol}}$. We denote by

$$D = \sum_{j=1}^{n} n_j p_j \quad \text{with} \quad n_j = \deg(Q_j) + 1,$$

and set $K(D) = K_C \otimes \mathcal{O}(D)$.
By definition, $\mathcal{M}_{\mathrm{Dol}}$ parametrizes S-equivalence classes of certain semistable parabolic Higgs bundles $(\mathcal{E}, \theta)$ with higher-order poles, where

- $\mathcal{E}$ is the holomorphic vector bundle structure on V defined by $D^{0,1}$,
- $\theta \colon \mathcal{E} \to \mathcal{E} \otimes K(D)$ is the meromorphic Higgs field.

Parabolic (semi)stability

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_k^j$$

and

$$\text{par-slope}(\mathcal{E}) = \frac{\text{par-deg}(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

$(\mathcal{E}, \theta)$ is said to be parabolically semistable if for all non-trivial proper subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\text{Im}(\theta|_{\mathcal{F}}) \subset \mathcal{F} \otimes K(D)$, one has

$$\text{par-slope}(\mathcal{F}) \leqslant \text{par-slope}(\mathcal{E}).$$

It is said to be parabolically stable if $\leqslant$ can be replaced by $<$, and strictly parabolically semistable if it is parabolically semistable but not parabolically stable.

Jordan–Hölder filtration, S-equivalence

Let $(\mathcal{E}, \theta)$ be a parabolically semistable meromorphic Higgs bundle. Then, there exists a filtration of $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_{L-1} \supset \mathcal{E}_L = 0$$

by θ-invariant subbundles such that the action $\mathrm{Gr}_l\theta$ induced on the graded pieces $\mathrm{Gr}_l\mathcal{E} = \mathcal{E}_{l-1}/\mathcal{E}_l$ by θ is parabolically stable.
Such a filtration is called a Jordan–Hölder filtration; it is not unique, however the set of graded Higgs bundles $(\mathrm{Gr}_l\mathcal{E}, \mathrm{Gr}_l\theta)$ is unique up to isomorphism and order.
If $(\mathcal{E}, \theta)$ is parabolically stable, then this filtration is trivial.
Two meromorphic Higgs bundles $(\mathcal{E}, \theta)$ and $(\mathcal{E}', \theta')$ are said to be S-equivalent if and only if $L' = L$ and

$$\oplus_{l=1}^{L}(\mathrm{Gr}_l\mathcal{E}, \mathrm{Gr}_l\theta) \cong \oplus_{l'=1}^{L}(\mathrm{Gr}_{l'}\mathcal{E}', \mathrm{Gr}_{l'}\theta').$$

Painlevé cases

From now on, we set $C = \mathbb{C}P^1$ and we assume $r = 2$ and $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Hod}} = 4$. There exists a finite list

$$PI, PII, PIII(D6), PIII(D7), PIII(D8), PIV, PV_{\mathrm{deg}}, PV, PVI$$

of irregular types with this property (depending on some parameters), called Painlevé cases. From now on, we let

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

and we write PX to refer to one of the above Painlevé cases. We therefore have a (family of) smooth non-compact Kähler surface(s) $\mathcal{M}_{\mathrm{Dol}}^{X}$ for any fixed X.

Singularity type of Painlevé cases

X	$D = \sum n_i p_i$
VI	$p_1 + p_2 + p_3 + p_4$
V	$2p_1 + p_2 + p_3$
$III(D6) = V_{\mathrm{deg}}$	$2p_1 + 2p_2$; $\frac{3}{2}p_1 + p_2 + p_3$
$III(D7)$	$\frac{3}{2}p_1 + 2p_2$
$III(D8)$	$\frac{3}{2}p_1 + \frac{3}{2}p_2$
IV	$3p_1 + p_2$
II	$4p_1$; $\frac{5}{2}p_1 + p_2$
I	$\frac{7}{2}p_1$

Half-integer coefficients refer to ramified irregular type.

Example: nilpotent PVI

- $n = 4$, logarithmic singularities: $0, 1, t, \infty$ (i.e., $Q_j = 0$ for $1 \leqslant j \leqslant 4$)
- for each $j \in \{0, 1, t, \infty\}$ parabolic algebra $\mathfrak{p}_j = \mathfrak{b}_j$ a Borel, with $\mathfrak{l}_j$ a Cartan,
- generic parabolic weights,
- eigenvalues of $\mathrm{res}_{p_j}(\theta)$ in $\mathfrak{l}_j$ equal to 0 (i.e., nilpotent residue).

Example: PIV

$n = 2$, singularities:

- at $z = 0$: logarithmic (i.e., $n_0 = 1$), with full flag parabolic filtration
$$0 \subset \ell \subset \mathbb{C}^2$$
such that ℓ is preserved by $\mathrm{res}_0(\theta)$, and the eigenvalues of $\mathrm{res}_0(\theta)$ on ℓ and on $\mathbb{C}^2/\ell$ are equal to μ_+, μ_-;
- at $z = \infty$: Poincaré rank 2 (i.e., $n_\infty = 3$), with trivial parabolic filtration and local form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} z dz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} dz + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{dz}{z} + \text{lower order terms}$$

If one fixes values of the parameters $a \in \mathbb{C}^\times, \mu_\pm, b, c \in \mathbb{C}$, one gets a smooth complex analytic surface $\mathcal{M}_{\mathrm{Dol}}^{IV}(\mu_\pm, a, b, c)$.

Forward to elimination.

Hitchin fibration

Hitchin '87: for the Dolbeault moduli space $\mathcal{M}_{\mathsf{Dol}}$ over a compact curve C there exists a proper surjective map to an affine space

$$\begin{aligned} h\colon \mathcal{M}_{\mathsf{Dol}} &\to \mathcal{B} = H^0(C, K_C) \oplus \cdots \oplus H^0(C, K_C^r) \\ (\mathcal{E}, \theta) &\mapsto (s_1, \ldots, s_r) \end{aligned}$$

with generic fiber an abelian variety, where

$$\det(\zeta - \theta) = \zeta^r + s_1 \zeta^{r-1} + \cdots + s_r.$$

Irregular version of the Hitchin fibration in the Painlevé cases:

$$\begin{aligned} h\colon \mathcal{M}_{\mathsf{Dol}}^X &\to H^0(\mathbb{C}P^1, K(D)) \oplus H^0(\mathbb{C}P^1, K^2(2D)) \cong \mathbb{C}^8 \\ (\mathcal{E}, \theta) &\mapsto (\operatorname{tr}\theta, \det\theta), \end{aligned}$$

with image a 1-dimensional affine subspace and generic fiber an elliptic curve.

Generic Hitchin fiber

For a generic point

$$\vec{s} = (s_1, \ldots, s_r) \in \mathcal{B},$$

the equation

$$\zeta^r + s_1\zeta^{r-1} + \cdots + s_r = 0$$

defines a smooth projective curve $X_{\vec{s}}$ in $\mathrm{Tot}(K_C(D))$. We then have

$$h^{-1}(\vec{s}) = \mathrm{Pic}^{\delta}(X_{\vec{s}}),$$

for some $\delta \in \mathbb{Z}$. Here, $\mathrm{Pic}^{\delta}(X_{\vec{s}})$ is the Picard variety of isomorphism classes of line bundles $L \to X_{\vec{s}}$ of degree δ.

Extension of h to a compact fibration

Theorem (Ivanics–Stipsicz–Szabó '18 J. Geom. Phys., '19 J. Pure Appl. Alg.)

For generic parabolic weights, there exists an embedding

$$\mathcal{M}_{\mathrm{Dol}}^{X} \hookrightarrow E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$$

and an elliptic fibration

$$\tilde{h} : E(1) \to \mathbb{C}P^1$$

extending h.

Denote by F_{∞}^{X} the non-reduced curve $E(1) \setminus \mathcal{M}_{\mathrm{Dol}}^{X} = \tilde{h}^{-1}(\infty)$.

Types of singular fibers

Kodaira's classification '63: the degenerations of elliptic curves corresponding to the affine root systems: (a) $I^*_{n-4} = D^{(1)}_n$, (b) $E^{(1)}_8$, (c) $E^{(1)}_7$, (d) $E^{(1)}_6$. The label of a node records multiplicity of the corresponding divisor component.

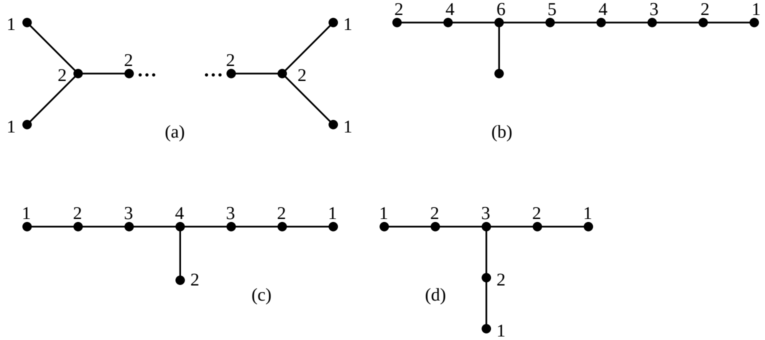
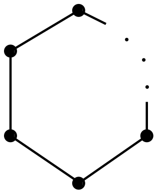


Table of fibers at infinity

X	F_{∞}^{X}
VI	$I_0^* = D_4^{(1)}$
V	$I_1^* = D_5^{(1)}$
V_{deg}	$I_2^* = D_6^{(1)}$
$III(D6)$	$I_2^* = D_6^{(1)}$
$III(D7)$	$I_3^* = D_7^{(1)}$
$III(D8)$	$I_4^* = D_8^{(1)}$
IV	$E_6^{(1)}$
II	$E_7^{(1)}$
I	$E_8^{(1)}$

Singular fibers I_n and II

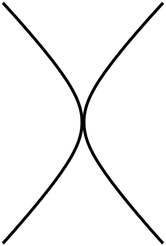
The I_n fiber ($n \geqslant 2$) is a collection of n rational curves of self-intersection -2, all with multiplicity one, intersecting each other transversally in a circular manner, as shown on the figure. The case $n = 1$ corresponds to a single nodal $\mathbb{C}P^1$.



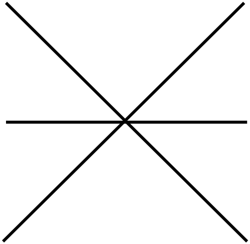
The II fiber is an S^2 with a single cuspidal singular point.

Singular fibers *III* and *IV*

Figure (a): type *III*, two rational curves of self-intersection -2, having a tangency of order 2 (tacnode singularity) at a point.
Figure (b): type *IV*, three rational curves of self-intersection -2 intersecting each other pairwise transversally in a common point.



(a)



(b)

Refined version of the Extension Theorem

For sake of concreteness, let $X = IV$ and assume $\mu_+ = \mu_-$. In addition to $F_\infty^{IV} = E_6^{(1)}$, there are other singular fibers of h.

Theorem (Ivanics–Stipsicz–Szabó '19 JPAA)

The set of further singular fibers of h is one of the following:

1. *a type IV curve*
2. *a type II curve and a type I_2 curve*
3. *a type I_3 curve and a type I_1 curve*
4. *a type III curve and a type I_1 curve*
5. *a type I_2 curve and two type I_1 curves.*

There exist explicit conditions in terms of the parameters a, b, c that determine which one of the above possibilities holds.

Spectral data

Consider the rational ruled surface

$$p\colon \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus K(D)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2)) \to \mathbb{C}P^1,$$

endowed with the canonical section ζ of $p^*K(D)$.
The spectral sheaf of an irregular Higgs bundle $(\mathcal{E}, \theta)$ is the rank 1 pure sheaf M defined by

$$0 \to p^*\mathcal{E} \otimes K^\vee(-D) \xrightarrow{\zeta - p^*\theta} p^*\mathcal{E} \to M \to 0.$$

The support of M is called the spectral curve of $(\mathcal{E}, \theta)$.
Beauville–Narasimhan–Ramanan correspondence:

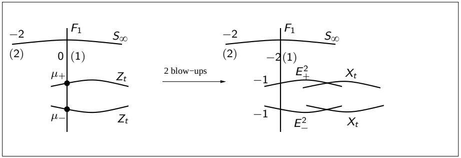
$$p_* M \cong \mathcal{E}, \quad p_*(\zeta \wedge) = \theta.$$

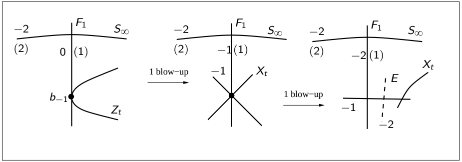
Sketch of proof of the Extension Theorem

1. Identify the pencil of spectral curves of the Higgs fields having the desired local behaviour.
2. Apply a sequence of blow-ups to eliminate the base locus.
3. Determine the singular fibers of the fibration.
4. For each singular spectral curve Z_t, determine the set of spectral sheaves M supported on Z_t that give rise to $\vec{\alpha}$-stable Higgs bundles.
5. For each spectral sheaf, find the family of parabolic structures compatible with the corresponding Higgs field.

Elimination of base locus over 0

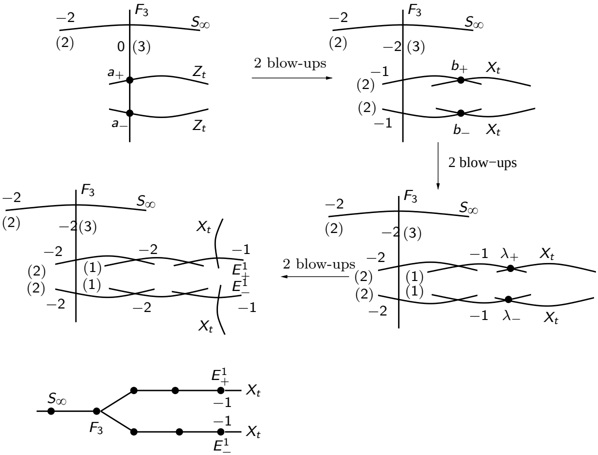
Z_t (and X_t) refer to the spectral curve in the pencil (respectively, the fibration). The non-positive number next to a curve refers to its self-intersection number, and the number in parentheses refers to its multiplicity in F_∞^{IV} (0 unless indicated). We set $b_{-1} = \mu_+$.





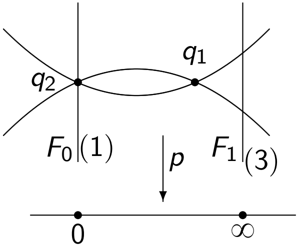
Elimination of base locus over ∞

Recall local form of PIV at ∞.



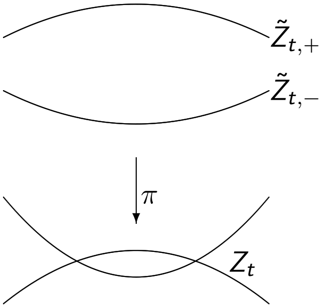
Spectral curve corresponding to I_3 fiber

Assume for instance that we have found a singular curve X_t of type I_3 in the fibration.
In the pencil, this situation corresponds to a singular spectral curve Z_t consisting of two rational curves transversally intersecting in 2 points: a point q_1 in generic position and $q_2 \in F_0 = p^{-1}(0)$.



Normalization

Let $\pi\colon \tilde{Z}_t \to Z_t$ stand for the normalization map, and $\tilde{Z}_{t,\pm}$ for the connected components of $\tilde{Z}_t$.



Structure of torsion-free sheaves near ordinary double points

At regular points of the support, M is torsion-free if and only if it is locally free.
Let M be a rank 1 torsion-free sheaf on Z_t. Then, locally near q_i exactly one of the following possibilities holds:

- M is locally free;
- there exists a rank 1 locally free sheaf $\tilde{M}$ on $\tilde{Z}_t$ such that $M = \pi_* \tilde{M}$.

Stratification by locus of freeness

For a rank 1 torsion-free sheaf M on Z_t, either

- M is locally free at q_1, but not locally free at q_2,

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- M is invertible (i.e., everywhere locally free),
- M is locally free neither at q_1 nor at q_2.

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It turns out that stability rules out the last possibility.

Bidegree

Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be the invertible sheaves associated to M on $\tilde{Z}_{t,\pm}$ respectively:

$$\mathcal{L}_\pm = (\pi^* M \otimes \mathcal{O}_{\tilde{Z}_{t,\pm}})/\mathcal{T}or^{\mathcal{O}_{\tilde{Z}_{t,\pm}}}(\pi^* M).$$

We define the bidegree of M as

$$(\delta_+, \delta_-) = (\deg(\mathcal{L}_+), \deg(\mathcal{L}_-)) \in \mathbb{Z}^2.$$

From now on, we focus on the case $\deg(\mathcal{E}) = 1, \text{par-deg}(\mathcal{E}) = 0$, and generic parabolic weights $\alpha^j_\pm$.

Sheaf counting

For invertible sheaves M, stability is equivalent to

$$\delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 1 \quad \text{or} \quad \delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 2,$$

with $\delta_- = 1 - \delta_+$. In both cases one gets a family of isomorphism classes of sheaves parametrized by $\mathbb{C}^\times$.
For non-invertible sheaves M of both types, stability gives

$$\delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 1, \quad \delta_- = -\delta_+,$$

and there exists up to isomorphism a unique such M.

Counting compatible parabolic structures

The non-invertible sheaf M that is not locally free at q_2 is locally of the form $\pi_* \tilde{M}$ for an invertible sheaf $\tilde{M}$ on $\tilde{Z}_t$. The residue of the associated Higgs field at 0 is

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}.$$

There exists a family parametrized by $\mathbb{C}P^1$ of lines $\ell \subset \mathcal{E}|_{q_2}$ preserved by θ.
For all other sheaves, there exists a unique compatible parabolic structure for $(\mathcal{E}, \theta)$.

Singular fiber of h

In all, we have found that the parabolic Higgs bundles with given spectral sheaf Z_t form a family parametrized by

$$\mathbb{C}^\times \qquad .$$

These strata form a degenerate elliptic fiber of type I_3.

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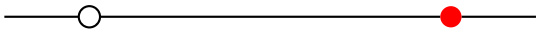
$-\lceil \alpha_+^1 + \alpha_+^2 \rceil + 2$ 

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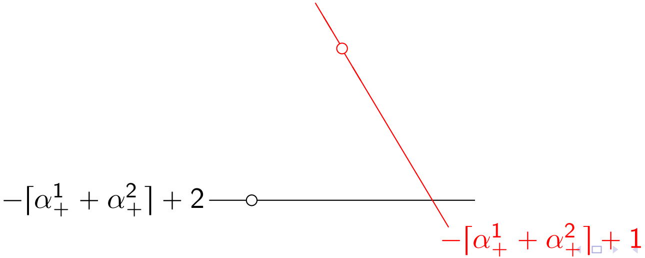

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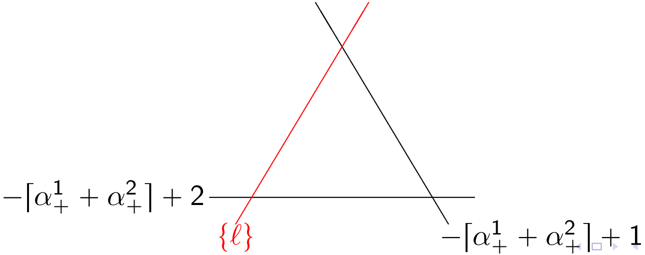

Wall-crossing

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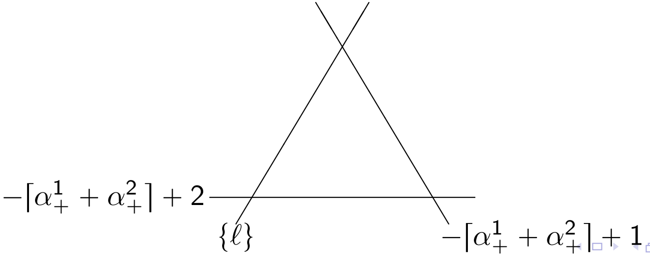
The space of parabolic weights is $\mathbb{R}$, the non-generic weights are $\mathbb{Z} \subset \mathbb{R}$. What happens when $\vec{\alpha}$ "crosses a wall" $n \in \mathbb{Z}$?



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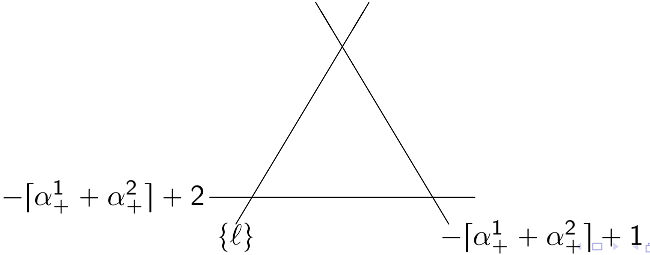

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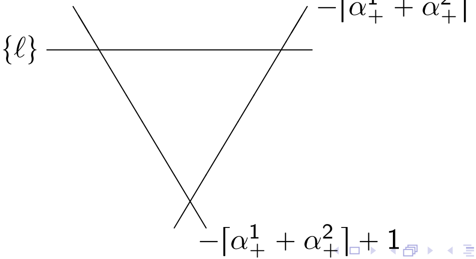


Pencils and fibrations with fiber I_0^*

Persson, Miranda (1990): Classification of configurations of Kodaira singular fibers that appear in a rational elliptic fibration. Next to an I_0^* fiber they have found 19 possibilities:

$$
\begin{array}{lllll}
I_0^*, & I_4 + 2I_1, & IV + II, & IV + 2I_1; & \\
I_3 + II + I_1, & I_3 + 3I_1; & & & \\
2III, & III + I_2 + I_1, & III + II + I_1, & III + 3I_1; & \\
3I_2, & 2I_2 + 2I_1, & I_2 + 2II, & I_2 + II + 2I_1, & I_2 + 4I_1; \\
3II, & 2II + 2I_1, & II + 4I_1, & 6I_1. &
\end{array}
$$

Realization of configurations of singular fibers

Theorem (Ivanics–Stipsicz–Szabó '19 SIGMA)

Given any configuration of singular fibers on the complement of an I_0^* fiber in a rational elliptic surface, there exists a choice of parameters for the Painlevé VI moduli spaces such that the Hitchin fibration h has the given set of singular fibers.

Double sections

We are looking for curves in $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2))$ defined by

$$\zeta^2 - \sigma = 0$$

for some $\sigma \in H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(4))$. Denoting by $[u : v]$ homogeneous co-ordinates of $\mathbb{C}P^1$, σ may be viewed as a homogeneous polynomial of degree 4 in u, v. For instance,

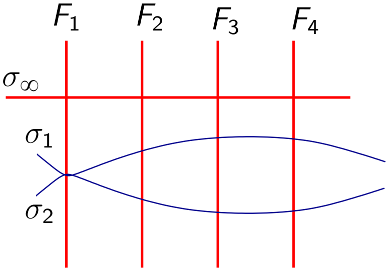
- for $\sigma(u, v) = u^4$ we get the union of two sections of $\mathcal{O}_{\mathbb{C}P^1}(2)$ which are tangent to each other over the point $[0 : 1] \in \mathbb{C}P^1$;
- for $\sigma(u, v) = u^3 v$ we get a cuspidal curve which is a double section — the cusp point is over $[0 : 1]$ and the fiber over $[1 : 0]$ is tangent to the cuspidal curve.

Example: $I_0^* + IV + II$

In order to get the configuration of singular fibers $I_0^* + IV + II$, we take the elliptic pencil generated by the double sections u^4 (two sections tangent over $[0:1]$) and u^3v (cuspidal double section). These curves lead to singular fibers IV and II respectively in the associated fibration.

Then we choose the points $p_1, \dots, p_4$ appropriately to have a singular fiber of type I_0^* too in the fibration. It turns out that the appropriate choice is $p_4 = [0:1]$ and $p_i = [u_i : 1]$ where u_i are the solutions of $u^3 = 1$ (cubic roots of unity).

Pencil with $I_0^* + IV$



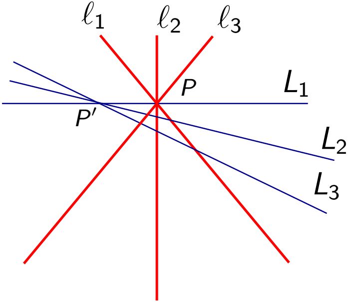


Figure: Left: the pencil in $\mathbb{F}_2$; right: corresponding pencil in $\mathbb{C}P^2$, both giving fibrations with an I_0^* fiber and a type IV fiber.