WKB-analysis of Hitchin system in rank 2

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Basic notation

Fix the data

- $\Sigma = \mathbb{C}P^1$: complex projective line (more generally, a compact Riemann surface),
- $r = 2$ rank (i.e., $G = \mathrm{Sl}_2(\mathbb{C})$),
- $t_0, \dots, t_n \in \mathbb{C}P^1$ logarithmic singularities (with local charts z_j),
- $D = t_0 + \cdots + t_n$ parabolic divisor,
- $\alpha_j^- = \frac{1}{4} < \alpha_j^+ = \frac{3}{4}$ Dolbeault parabolic weights

Hitchin's equations

Let (V, h) be a fixed rank 2 smooth Hermitian vector bundle on a compact Riemann surface Σ. Two-dimensional reduction of the Yang–Mills equations are Hitchin's equations (1987)

$$D^{0,1}\theta = 0$$

$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection D and a field $\theta : V \to V \otimes \Omega^{1,0}_C$.

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$$D^{0,1}\theta = 0 \in \Omega^2(\Sigma, \mathfrak{gl}(V)) \quad \text{complex equation}$$
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for a unitary connection D and a field $\theta : V \to V \otimes \Omega^{1,0}_C$.
The space of solutions of Hitchin's equations is preserved by unitary gauge transformations. Taking solutions up to unitary gauge equivalence $\rightsquigarrow$ $\mathcal{M}_{\text{Hod}}$ non-abelian Hodge moduli space, a hyper-Kähler manifold.

Higgs bundles, stability

Set $\bar{\partial}_{\mathcal{E}} = D^{0,1}$ and denote by $\mathcal{E}$ the holomorphic vector bundle structure on V defined by $\bar{\partial}_{\mathcal{E}}$. The couple $(\mathcal{E}, \theta)$ is called a Higgs bundle if

$$\theta \colon \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}} K_{\Sigma}$$

is holomorphic ($\Leftrightarrow$ complex Hitchin equation). For a holomorphic bundle introduce its slope

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

A Higgs bundle is (semi-)stable if for any proper holomorphic subbundle $0 \subset \mathcal{F} \subset \mathcal{E}$ preserved by θ we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \qquad (\text{respectively, } \mu(\mathcal{F}) \leq \mu(\mathcal{E})).$$

A Higgs bundle is poly-stable if it is a direct sum of stable Higgs bundles of fixed slope μ.

Hitchin–Kobayashi correspondence

Theorem (Hitchin, 1987)

Let $(\mathcal{E}, \theta)$ be a Higgs bundle of rank 2 and degree 0 on Σ. Then, there exists a Hermitian metric h on V satisfying the real Hitchin equation if and only if $(\mathcal{E}, \theta)$ is poly-stable.

Corollary

In one of the complex structures I of the hyper-Kähler family, $\mathcal{M}_{\mathrm{Hod}}$ parameterizes complex gauge equivalence classes of poly-stable Higgs bundles $(\mathcal{E}, \theta)$.

The Kähler manifold $(\mathcal{M}_{\mathrm{Hod}}, I)$ is called the Dolbeault moduli space, denoted by $\mathcal{M}_{\mathrm{Dol}}$.

Hitchin map

Hitchin, 1987: for $\mathcal{M}_{\text{Dol}}$ a Dolbeault moduli space there exists a surjective proper algebraic map of quasi-projective varieties

$$H \colon \mathcal{M}_{\text{Dol}} \to Y = \mathbb{C}^N,$$

with $\dim_{\mathbb{C}} \mathcal{M}_{\text{Dol}} = 2N$. For $(\mathcal{E}, \theta)$ we set

$$\det(\zeta - \theta) = \zeta^r + s_1 \zeta^{r-1} + \cdots + s_r$$

for the characteristic polynomial of θ. Then,

$$H(\mathcal{E}, \theta) = (s_1, \ldots, s_r).$$

The generic fiber of H is an abelian variety, H is a completely integrable system.

Flat connections, holomorphic connections

Define

$$\nabla = D + \theta + \theta^{\dagger_h}.$$

Hitchin's equations $\Rightarrow F_\nabla = 0$, i.e. ∇ is a flat connection. Then $\nabla^{0,1} = D^{0,1} + \theta^{\dagger_h}$ specifies a holomorphic vector bundle structure on V (in general different from $\mathcal{E}$!), and $\nabla^{1,0} = D^{1,0} + \theta$ is a holomorphic connection on $(V, \nabla^{0,1})$. There are similar notions of (poly-)stability as for Higgs bundles.

Theorem (Donaldson, 1987)

For a given holomorphic connection $\nabla^{1,0}$ on a holomorphic bundle of rank 2, there exists a Hermitian metric h satisfying Hitchin's equations if and only if $\nabla^{1,0}$ is poly-stable.

de Rham Kähler structure

Corollary

In another complex structure J of the hyper-Kähler family, $\mathcal{M}_{\text{Hod}}$ parameterizes S-equivalence classes of stable holomorphic connections.

The Kähler manifold $(\mathcal{M}_{\text{Hod}}, J)$ is called the de Rham moduli space, denoted by $\mathcal{M}_{\text{dR}}$.

Non-abelian Hodge theory

Corollary

$\mathcal{M}_{\mathrm{dR}}$ and $\mathcal{M}_{\mathrm{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\mathrm{Hod}}$):

$$\mathrm{NAHT}\colon \mathcal{M}_{\mathrm{Dol}} \xrightarrow{\sim} \mathcal{M}_{\mathrm{dR}}.$$

Tame harmonic bundles

Let Σ be a noncompact curve of the form

$$\Sigma = \overline{\Sigma} \setminus \{p_1, \dots, p_n\},$$

with $\overline{\Sigma}$ compact. Tame harmonic bundles (C. Simpson, 1990): at the parabolic divisor we require

- θ has first order poles at p_j,
- the eigenvalues of $\mathrm{res}_{p_j}(\theta)$ are equal to 0 (strongly parabolic),
- and with respect to a compatible trivialization,

$$h \approx \mathrm{diag}(|z_j|^{2\alpha_j^-}, |z_j|^{2\alpha_j^+}) = \mathrm{diag}(|z_j|^{\frac{1}{2}}, |z_j|^{\frac{3}{2}})$$

Konno, 1993: solutions up to unitary gauge equivalence form $\mathcal{M}_{\mathrm{Hod}}$ non-abelian Hodge moduli space, a hyper-Kähler manifold.

Character variety, Riemann–Hilbert correspondence

Character variety $\mathcal{M}_{\mathrm{B}}$: moduli space parameterising (filtered) local systems ρ on $\mathbb{C}P^1 \setminus D$, with eigenvalues on a simple positive loop around p_j given by

$$c_j^{\pm} = \exp(-2\pi\sqrt{-1}\alpha_j^{\pm}) = \pm\sqrt{-1}.$$

Regular-singular Riemann–Hilbert correspondence: bi-analytic map

$$\mathrm{RH}\colon \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}},$$

obtained by unique analytical continuation of local parallel sections of ∇.

Conclusion: $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are all diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}$), via

$$\mathrm{RH} \circ \mathrm{NAHT}\colon \mathcal{M}_{\mathrm{Dol}} \longrightarrow \mathcal{M}_{\mathrm{B}}.$$

Hitchin WKB-problem, P=W conjecture

Hitchin WKB-problem

Katzarkov–Noll–Pandit–Simpson, 2013: understand RH $\circ$ NAHT "close to infinity", i.e. for $(\mathcal{E}, \theta)$ with $|H(\mathcal{E}, \theta)| \gg 1$.

Related to:

the Geometric P=W conjecture, a Geometric version of de Cataldo–Hausel–Migliorini's cohomological P=W conjecture, 2012.

"Standard" WKB-method

Analogous problem (...,Voros, 1983,...): asymptotic analysis of solutions ψ of Schrödinger's equation

$$-\hbar^2 \frac{\mathrm{d}^2\psi(q)}{\mathrm{d}q^2} + V(q)\psi(q) = 0$$

as $\hbar \to 0$. This involves expressions of the form

$$\exp\left(\sqrt{-1}\hbar^{-1}\int_\gamma u\,\mathrm{d}q\right)$$

for paths $\gamma\colon [0,1] \to \Sigma$ avoiding the ramification points and

$$u(q,\hbar) = \zeta + O(\hbar^2),$$

with $\zeta^2 + V(q) = 0$.

Hitchin base and map

From now on we let $n = 5$, based on arXiv:2103.00932. Let $(\mathcal{E}, \theta)$ be a strongly parabolic Higgs bundle of rank 2 with 5 logarithmic points $t_0, \dots, t_4$, and set $D = t_0 + \cdots + t_4$.
We then have $\mathrm{tr}(\theta) \equiv 0$. Hitchin base:

$$\mathcal{B} = \{q : \quad q(t_j) = 0 \text{ for all } 0 \leq j \leq 4\} \subset H^0(\mathbb{C}P^1, K^{\otimes 2}(2D)) \cong \mathbb{C}^7,$$

so $\dim_{\mathbb{C}}(\mathcal{B}) = 2$. Hitchin map:

$$\begin{aligned} H \colon \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha}) &\to \mathcal{B} \\ (\mathcal{E}, \theta) &\mapsto -\det(\theta). \end{aligned}$$

Spectral curve

For $q \in S^3_1 \subset \mathcal{B}$ we write $\zeta_\pm(Rq, z)$ for the roots of

$$\zeta^2 - Rq = 0,$$

specifically

$$\zeta_\pm(Rq, z) = \pm\sqrt{Rq(z, 1)}.$$

Denote the corresponding meromorphic 1-forms by

$$Z_\pm(Rq, z) = \pm\sqrt{Rq(z, 1)}\frac{\mathrm{d}z}{\prod_{j=0}^4 (z - t_j)}.$$

We denote by

$$X_{Rq} = \{([z : w], \pm\sqrt{Rq(z, w)})\} \to \mathbb{C}P^1$$

the Riemann surface of the bivalued function $\zeta_\pm(Rq, z)$.

Ramification divisor and Hopf fibration

We set

$$\Delta_q = \{z \in \mathbb{C} : \quad q(z) = 0\}$$

for the ramification divisor of X_{Rq} (independent of R). We then have

$$\Delta_q = \{t_0, t_1, t_2, t_3, t_4, t(q)\}$$

for some $t(q) \in \mathbb{C}P^1$. Namely,

$$q(z) = \frac{(az - b)\mathrm{d}z^{\otimes 2}}{\prod_{j=0}^{4}(z - t_j)}.$$

for some $(a, b) \in \mathbb{C}^2$ with $at(q) - b = 0$. The map

$$t \colon S^3_1 \to \mathbb{C}P^1$$
$$q \mapsto t(q)$$

is the Hopf fibration.

Idea of solution

We fix a generic element $q \in S_1^3$ and consider $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha})$ such that

$$H(\mathcal{E}, \theta) = q.$$

For $R > 0$ we have

$$H(\mathcal{E}, \sqrt{R}\theta) = Rq.$$

It is then possible to express the $R \to \infty$ asymptotic behaviour of RH $\circ$ NAHT in function of $\int_\gamma Z_\pm(q, z)$ over various paths γ in $\mathbb{C}P^1$, up to factors belonging to $U(1)$.
We then choose a suitable smooth section

$$\sigma \colon S_1^3 \to \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha})$$

of H to get rid of the $U(1)$ factors.

Asymptotic abelianization

Let $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$ denote the Hermite–Einstein metric and integrable connection associated to $(\mathcal{E}, \sqrt{R}\theta)$. Introduce

$$\nabla^{\text{model}}_{\sqrt{R}} = \nabla_{h_{q,\infty}} + \begin{pmatrix} 2\Re Z_+(Rq, z) & 0 \\ 0 & 2\Re Z_-(Rq, z) \end{pmatrix}.$$

where $h_{q,\infty}$ is some explicit abelian solution of Hitchin's equation and $\nabla_{h_{q,\infty}}$ the corresponding unitary connection.

Theorem (T. Mochizuki '16)

Over any simply connected compact set $K \subset \mathbb{C} \setminus \Delta_q$ there exists a gauge transformation $g_{\sqrt{R}}$ such that

$$g_{\sqrt{R}} \cdot \nabla_{\sqrt{R}} - \nabla^{\text{model}}_{\sqrt{R}} \to 0$$

(measured with respect to $h_{\sqrt{R}}$) as $R \to \infty$, uniformly over K.

Fiducial solution, Painlevé 3

R. Mazzeo, J. Swoboda, H. Weiss, F. Witt '16 (near the ramification point $t(q)$), L. Fredrickson, R. Mazzeo, J. Swoboda, H. Weiss '20 (near parabolic points D): local models for the $R \gg 0$ behaviour of $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$, called fiducial solutions.

Near $t(q)$: let $\ell_{\sqrt{R}}$ be the solution of the Painlevé 3-type equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\right)\ell_{\sqrt{R}} = 8R\tilde{r}\sinh(2\ell_{\sqrt{R}})$$

satisfying the boundary behaviours

$$\ell_{\sqrt{R}}(\tilde{r}) \approx -\frac{1}{2}\log(\tilde{r}), \quad \tilde{r} \to 0+$$

$$\ell_{\sqrt{R}}(\tilde{r}) \approx \frac{1}{\pi}K_0\left(\frac{8}{3}\sqrt{R\tilde{r}^3}\right) \approx \frac{\sqrt{3}}{2\pi\sqrt{2}\sqrt[4]{R\tilde{r}^3}}e^{-\frac{8}{3}\sqrt{R\tilde{r}^3}}, \quad \tilde{r} \to \infty,$$

with K_0 the modified Bessel function of order 0.

Fiducial solution, approximate solution

Then, for a co-ordinate $\tilde{z}$ on the disc $|\tilde{z}| < 1$ introduce a unitary connection and Higgs field:

$$A^{\mathrm{fid}}_{\sqrt{R}} = \left(\frac{1}{8} + \frac{1}{4} \tilde{r} \partial_{\tilde{r}} \ell_{\sqrt{R}} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 2\sqrt{-1} \mathrm{d}\tilde{\varphi}$$

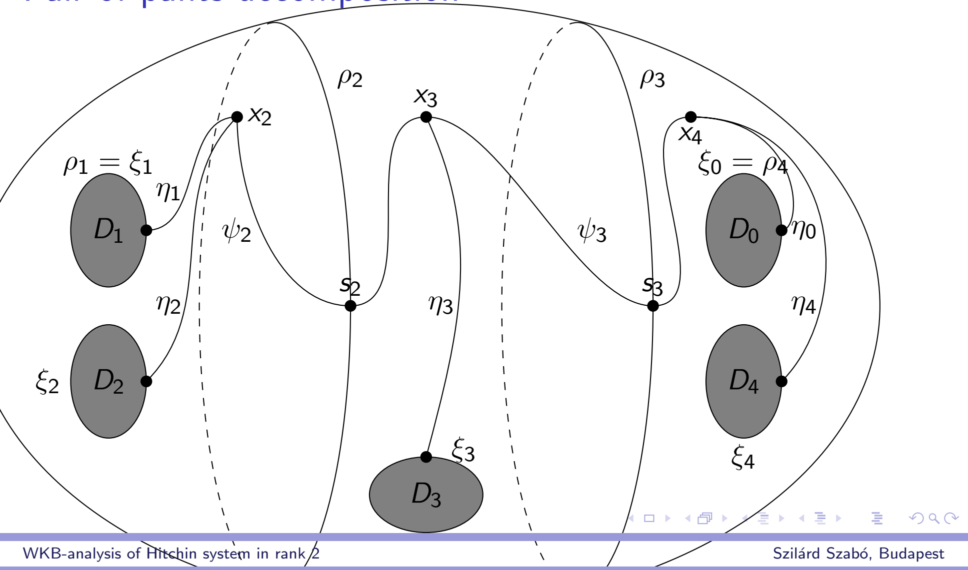
$$\theta^{\mathrm{fid}}_{\sqrt{R}} = \begin{pmatrix} 0 & \tilde{r}^{1/2} e^{\ell_{\sqrt{R}}(\tilde{r})} \\ \tilde{z}\tilde{r}^{-1/2} e^{-\ell_{\sqrt{R}}(\tilde{r})} & 0 \end{pmatrix} \mathrm{d}\tilde{z}.$$

Gluing construction of the fiducial solution and Mochizuki's abelian form $\rightsquigarrow$ approximate solution $h^{\mathrm{appr}}_{\sqrt{R}}$.

Theorem (MSWW '16, FMSW '20)

Assume that all the zeroes of q are simple. Then, there exists a small perturbation of the Hermitian metric $h^{\mathrm{appr}}_{\sqrt{R}}$ that satisfies Hitchin's equation for $(\mathcal{E}, \sqrt{R}\theta)$.

Pair-of-pants decomposition



Simpson's Fenchel–Nielsen co-ordinates

Simpson '16: (an open subset of) $\mathcal{M}_{\mathrm{B}}$ carries complex length co-ordinates

$$t_i = \operatorname{tr} \mathrm{RH}(\nabla)[\rho_i] \in \mathbb{C} \quad (i \in \{2, 3\}),$$

and complex twist co-ordinates

$$[p_i : q_i] \in \mathbb{C}P^1 \quad (i \in \{2, 3\}),$$

subject to the condition

$$p_i^2 + t_i p_i q_i + q_i^2 \neq 0.$$

Parallel transport map

For any loop γ in $\mathbb{C}P^1 \setminus \Delta_q$ let us write

$$\mathrm{RH}(\nabla_{\sqrt{R}})[\gamma] = \begin{pmatrix} a(\gamma, q, R) & b(\gamma, q, R) \\ c(\gamma, q, R) & d(\gamma, q, R) \end{pmatrix}.$$

For $0 \leq j \leq 2$ introduce

$$\pi_j(q) = \int_{x_2}^{t_j} Z_+(q, z) \in \mathbb{C},$$

$$\tau_j(q) = \frac{a t_j - b}{\prod_{0 \leq k \leq 4, k \neq j} (t_j - t_k)} \in \mathbb{C}.$$

Asymptotics of parallel transport around parabolic points

Proposition

Fix any $q \in S_1^3$ and consider the loop $\gamma = \xi_j$, a positively oriented circle around p_j, of radius $r_0 > 0$ in the Euclidean metric.

1. The behaviour of the diagonal entries of $\mathrm{RH}(\nabla_{\sqrt{R}})[\xi_j]$ as $R \to \infty$ is given by the limits

$$a(\xi_j, q, R) \to 0$$
$$d(\xi_j, q, R) \to 0;$$

2. the behaviour of the off-diagonal entries is given by

$$b(\xi_j, q, R) e^{8\Re\sqrt{\tau_j}\sqrt{R r_0}} \to \sqrt{-1}$$
$$c(\xi_j, q, R) e^{-8\Re\sqrt{\tau_j}\sqrt{R r_0}} \to \sqrt{-1}.$$

Asymptotics of complex length co-ordinates t_2

Theorem

Fix $q \in S^3_1$ and consider the loop $\gamma = \rho_2$. In case $\Re(\pi_1 - \pi_2) \neq 0$ there exists a complex 1-parameter family of sections σ of the Hitchin map H such that as $R \to \infty$ we have the limit

$$t_2(q, R) = \exp\left(4\sqrt{R}|\Re(\pi_1 - \pi_2)|\right) + o(1).$$

In case $\Re(\pi_1 - \pi_2) = 0$ the limit of $t_2(q, R)$ as $R \to \infty$ exists and is finite.

Corollary

Fix $q \in S^3_1$ and assume $\pi_1(q) \neq \pi_2(q)$. Then there exists a unique $\varphi_2 \in [0, 2\pi)$ such that $t_2(e^{\sqrt{-1}\varphi_2} q, R)$ is bounded as $R \to \infty$.

Asymptotics of complex length co-ordinates

Corollary

Let $q \in S_1^3$ *satisfy*

$$\pi_4(q) - \pi_0(q) \neq 0 \neq \pi_1(q) - \pi_2(q).$$

Then there exists a section σ *of* H *such that we have limits*

$$\lim_{R \to \infty} t_2(q, R) \exp\left(-4\sqrt{R}|\Re(\pi_1(q) - \pi_2(q))|\right) = 1$$

and

$$\lim_{R \to \infty} t_3(q, R) \exp\left(-4\sqrt{R}|\Re(\pi_4(q) - \pi_0(q))|\right) = 1.$$

Limit of complex twist co-ordinate $[p_2 : q_2]$

Theorem

Fix $q \in S^3_1$ such that $\Re(\pi_2 - \pi_1) \neq 0$. Then, the complex twist co-ordinate $[p_2 : q_2]$ associated to Rq converges to $[0 : 1]$ as $R \to \infty$ if the conditions

$$\int_{\psi_2} \Re Z_+ < 2\Re(2\sqrt{s_2\tau_2} - \sqrt{s_2\tau_1} - \sqrt{s_3\tau_3})$$

$$|\Re(\pi_1 - \pi_2)| = 2\sqrt{s_2}\Re(\sqrt{\tau_1} - \sqrt{\tau_2})$$

hold for one choice of a square root Z_+ of Q.
On the other hand, $[p_2 : q_2] \to [1 : 0]$ if

$$|\Re(\pi_1 - \pi_2)| \neq 2\sqrt{s_2}\Re(\sqrt{\tau_1} - \sqrt{\tau_2}).$$

Asymptotics of complex twist co-ordinate $[p_2 : q_2]$

Specifically, in the first situation we have

$$\frac{p_2}{q_2} \approx \exp 4\sqrt{R}\Re\left(\int_{\psi_2} Z_+(q) - 2(2\sqrt{s_2\tau_2}(q) - \sqrt{s_2\tau_1}(q) - \sqrt{s_3\tau_3}(q))\right).$$

This behaviour follows from some miraculous cancellations.
Conclusion:

- the behaviour $\frac{p_2}{q_2} \to \infty$ is generic,
- the challenge is to find $q \in S^3_1$ such that $\frac{p_2}{q_2} \to 0$.

Geometry of period integrals

Define the open subset

$$U_2(s_2) \subset S^3_1$$

by the conditions

$$0 \neq \pi_1(q) - \pi_2(q) \neq \pm 2\sqrt{s_2}(\sqrt{\tau_1}(q) - \sqrt{\tau_2}(q)).$$

For every $q \in U_2$ there exists a unique $\varphi^* \in [0, 2\pi)$ such that

$$\Re(\pi_1(e^{\sqrt{-1}\varphi^*}q) - \pi_2(e^{\sqrt{-1}\varphi^*}q)) = 2\sqrt{s_2}\Re(\sqrt{\tau_1}(e^{\sqrt{-1}\varphi^*}q) - \sqrt{\tau_2}(e^{\sqrt{-1}\varphi^*}q$$

This provides a smooth section of the Hopf fibration

$$S_2 \colon t(U_2) \to S^3_1$$
$$[a : b] \mapsto e^{\sqrt{-1}\varphi^*(q)} q$$

Finding small $[p_2 : q_2]$

We make the choices

$$t_0 = -\frac{1}{k}, \quad t_1 = 0, \quad t_2 = 1, \quad t_3 = -1, \quad t_4 = \frac{1}{k}$$

for some $0 < k < 1$.

Proposition

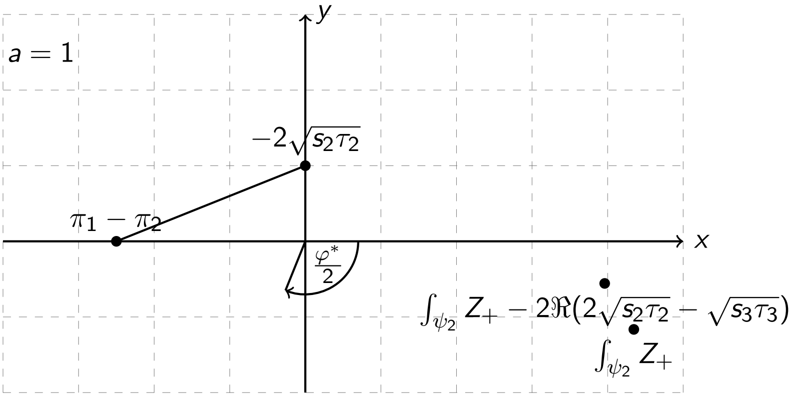
Let $q = S_2(t_1)$. Then q belongs to $U_2(s_2)$ for every $s_2 > 0$, and we have $\Re(\pi_1(q) - \pi_2(q)) \neq 0$. Moreover, there exist distinct points $x_2, x_3 \in \mathbb{C}P^1 \setminus D$ and

$$\rho = \rho(q, t_0, \ldots, t_4, x_2, x_3) > 0$$

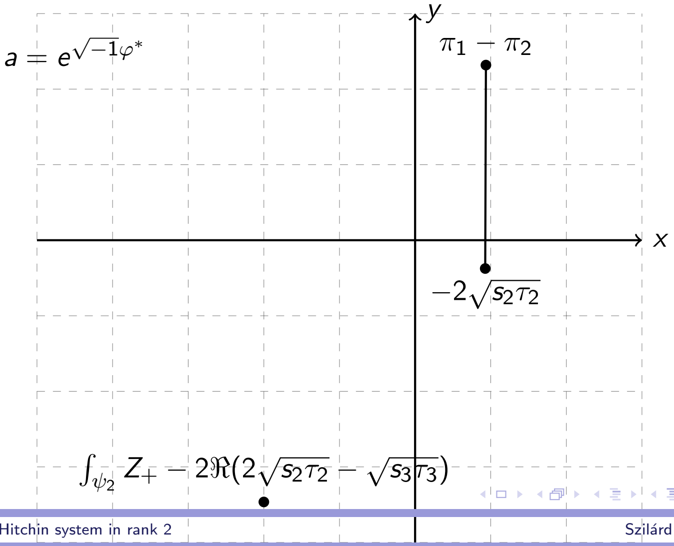
such that for every $0 < s_2, s_3 < \rho$ we have $[p_2 : q_2] \to [0 : 1]$ as $R \to \infty$.

Idea of proof to find small $[p_2 : q_2]$

Rotating triangles. Before:



Idea of proof to find small $[p_2 : q_2]$; after:



Finding small $[p_2 : q_2]$ and $[p_3 : q_3]$ simultaneously

Proposition

There exist $0 < s_2, s_3, s_4 < \rho''$ such that $S_2(t_1) = S_3(t_1)$. For the choice $q = S_2(t_1)$, we have $[p_2 : q_2] \to [0 : 1]$ and $[p_3 : q_3] \to [0 : 1]$ as $R \to \infty$.

Further directions

- Use the above approach to study the (Geometric) $P = W$ conjecture, along the lines of the Painlevé 6 case, Sz. (Adv. Math., 2021).
- Generalization to higher rank.
- Generalization to irregular singularities.