Geometry of Hodge moduli spaces in the Painlevé cases

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Outline

Hodge theory, Riemann–Hilbert

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Dolbeault

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DATA OF WILD NON-ABELIAN HODGE THEORY (NAHT)

Hitchin '87, Simpson '90, Sabbah '99, Biquard–Boalch '04: fix

- C: smooth projective curve over $\mathbb{C}$
- $r \geqslant 2$ rank (i.e., $G = \mathrm{Gl}_r(\mathbb{C})$)
- $p_1, \ldots, p_n \in C$ irregular singularities (with local charts z_j), and for each p_j:
- a parabolic subalgebra $\mathfrak{p}_j \subset \mathfrak{gl}_r$ with associated Levi $\mathfrak{l}_j$
- parabolic weights $\{\alpha_j^i\}_i$
- an unramified irregular type $Q_j \in \mathfrak{t} \otimes \mathbb{C}((z_j))/\mathbb{C}[[z_j]]$ with centralizer $\mathfrak{h}_j$
- an adjoint orbit $\mathcal{O}_j$ in $\mathfrak{l}_j \cap \mathfrak{h}_j$.

Hitchin's equations and wild NAHT

Fix a smooth vector bundle $V \to C$ of rank r.
The space of solutions of Hitchin's equations

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for

- a Hermitian metric h on V with behaviour $h \approx \mathrm{diag}(|z|^{2\alpha_j^1}, \dots, |z|^{2\alpha_j^r})$ near p_j,
- a unitary connection D in (V, h),
- and a field $\theta : V \to V \otimes \Omega_C^{1,0}$ having prescribed irregular part and residue in $\mathcal{O}_j$ at p_j

$\rightsquigarrow$ hyper-Kähler moduli space $\mathcal{M}_{\mathrm{Hod}}$.

de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\mathrm{Hod}}$ have a geometric meaning:

- de Rham: $\mathcal{M}_{\mathrm{dR}}$ parametrizing S-equivalence classes of certain semistable parabolic connections with irregular singularities;
- Dolbeault: $\mathcal{M}_{\mathrm{Dol}}$ parametrizing S-equivalence classes of certain semistable parabolic Higgs bundles $(\mathcal{E}, \theta)$ with higher-order poles, where $\mathcal{E}$ is the holomorphic vector bundle structure on V defined by $D^{0,1}$.

By non-abelian Hodge theory, $\mathcal{M}_{\mathrm{dR}}$ and $\mathcal{M}_{\mathrm{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\mathrm{Hod}}$).

Semistability, S-equivalence

The parabolic degree and slope of $\mathcal{E}$ are defined respectively as

$$\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{j=0}^{n} \sum_{k=1}^{r} \alpha_k^j$$

and

$$\text{par-slope}(\mathcal{E}) = \frac{\text{par-deg}(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

$(\mathcal{E}, \theta)$ is said to be parabolically semistable if for all non-trivial proper subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\text{Im}(\theta|_{\mathcal{F}}) \subset K(D) \otimes \mathcal{F}$, one has

$$\text{par-slope}(\mathcal{F}) \leqslant \text{par-slope}(\mathcal{E}).$$

S-equivalence: the Jordan–Hölder graded objects are isomorphic.

Irregular Riemann–Hilbert correspondence

- Birkhoff, Mebkhout, Kashiwara, Deligne, Malgrange, Jimbo–Miwa–Ueno...: equivalence between the categories of irregular connections and Stokes-filtered local systems.
- Boalch '07: algebraic geometric construction of wild character varieties $\mathcal{M}_{\mathrm{B}}$ parametrizing Stokes data. Generically, $\mathcal{M}_{\mathrm{B}}$ is a holomorphic symplectic manifold.
- Irregular Riemann–Hilbert correspondence (RH): bi-analytic map
$$\mathrm{RH} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}.$$

Conclusion: $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are all diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}$).

Painlevé cases

From now on, we set $C = \mathbb{C}P^1$ and we assume $r = 2$ and $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Hod}} = 4$. There exists a finite list

$$PI, PII, PIII(D6), PIII(D7), PIII(D8), PIV, PV_{\mathrm{deg}}, PV, PVI$$

of irregular types with this property (depending on some parameters), called Painlevé cases. From now on, we let

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}$$

and we write PX to refer to one of the above Painlevé cases. We therefore have smooth non-compact Kähler surfaces

$$\mathcal{M}^X_{\mathrm{dR}}, \quad \mathcal{M}^X_{\mathrm{Dol}}, \quad \mathcal{M}^X_{\mathrm{B}}$$

diffeomorphic to each other (and to $\mathcal{M}^X_{\mathrm{Hod}}$) for any fixed X.

Singularity type of Painlevé cases

X	$D = \sum n_i p_i$
VI	$p_1 + p_2 + p_3 + p_4$
V	$2p_1 + p_2 + p_3$
$III(D6) = V_{\deg}$	$2p_1 + 2p_2; \frac{3}{2}p_1 + p_2 + p_3$
$III(D7)$	$\frac{3}{2}p_1 + 2p_2$
$III(D8)$	$\frac{3}{2}p_1 + \frac{3}{2}p_2$
IV	$3p_1 + p_2$
II	$4p_1; \frac{5}{2}p_1 + p_2$
I	$\frac{7}{2}p_1$

Half-integer coefficients refer to ramified irregular type.

Example: PIV

$n = 2$, singularities:

- at $z = 0$: logarithmic (i.e., $n_0 = 1$), with full flag parabolic filtration
$$0 \subset \ell \subset \mathbb{C}^2$$
such that ℓ is preserved by $\mathrm{res}_0(\theta)$, and the eigenvalues of $\mathrm{res}_0(\theta)$ on ℓ and on $\mathbb{C}^2/\ell$ are equal to μ_+, μ_-;
- at $z = \infty$: Poincaré rank 2 (i.e., $n_\infty = 3$), with trivial parabolic filtration and local form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} z dz + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} dz + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \frac{dz}{z} + \text{lower order terms}$$

If one fixes values of the parameters $a \in \mathbb{C}^\times, \mu_\pm, b, c \in \mathbb{C}$, one gets a smooth complex analytic surface $\mathcal{M}_{\mathrm{Dol}}^{IV}(\mu_\pm, a, b, c)$.

Hitchin fibration

Hitchin '87: for the Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ over a compact curve C there exists a proper surjective map to an affine space

$$\begin{aligned} h\colon \mathcal{M}_{\mathrm{Dol}} &\to \mathcal{B} = H^0(C, K_C) \oplus \cdots \oplus H^0(C, K_C^r) \\ (\mathcal{E}, \theta) &\mapsto (s_1, \ldots, s_r) \end{aligned}$$

where

$$\det(\zeta - \theta) = \zeta^r + s_1 \zeta^{r-1} + \cdots + s_r.$$

Irregular version of the Hitchin fibration in the Painlevé cases:

$$\begin{aligned} h\colon \mathcal{M}_{\mathrm{Dol}}^X &\to H^0(\mathbb{C}P^1, K(D)) \oplus H^0(\mathbb{C}P^1, K^2(2D)) \cong \mathbb{C}^8 \\ (\mathcal{E}, \theta) &\mapsto (\operatorname{tr}\theta, \det\theta), \end{aligned}$$

with image a 1-dimensional affine subspace.

Compactifying divisors of $\mathcal{M}_B^X$ and their dual complexes

Let M be an abstract algebraic variety M over $\mathbb{C}$ and $\overline{M}$ a compactification (Nagata '62). Hironaka '64: if M is smooth then $\overline{M}$ may be chosen smooth with $D = \overline{M} \setminus M$ a simple normal crossing (SNC) divisor.
Given a smooth compactification $\tilde{M}$ of M by a SNC divisor D, one can define a simplicial complex $\mathcal{N}(M)$ (called nerve complex or dual complex) as follows:

- the vertices (0-dimensional simplices) are in bijection with the irreducible components D_i of D,
- edges (1-dimensional simplices) between a pair (ii') of vertices are in bijection with irreducible components of $D_i \cap D_{i'}$,
- etc.

Homotopy type of dual complexes

Danilov '75: The homotopy type of $\mathcal{N}(M)$ is independent of the choice of smooth compactification $\widetilde{M}$.
We denote by $|\mathcal{N}(M)|$ the (homotopy type of the) associated topological space.
Simpson '15: for T a sufficiently small neighbourhood of D in $\widetilde{M}$, there exists a surjective map

$$\phi\colon T \to |\mathcal{N}(M)|,$$

defined up to homotopy.
Let $\widetilde{\mathcal{M}}_{\mathrm{B}}^{X}$ be a smooth compactification of $\mathcal{M}_{\mathrm{B}}^{X}$ by a SNC divisor D^X and denote by $\mathcal{N}^X$ the nerve complex of D^X.

Geometric $P = W$ in the Painlevé cases

Theorem (A. Némethi–Sz '20)

For all X and for some large compact set $K \subset \mathcal{M}_{\mathrm{B}}^X$ there exists a homotopy commutative square

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}^X \setminus K & \xrightarrow{\psi} & \mathcal{M}_{\mathrm{B}}^X \setminus K \\ \downarrow h & & \downarrow \phi \\ D^\times & \dashrightarrow & |\mathcal{N}^X|, \end{array}$$

the bottom map being a homotopy equivalence. Here, $D^\times = \mathbb{C} - B_R(0) \subset \mathbb{C}$, h is the irregular Hitchin map and $\psi = RH \circ NAHT$.

Background: $P = W$ conjecture

- 2012: M. de Cataldo, T. Hausel, L. Migliorini proved that the perverse Leray filtration on $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{C})$ gets identified to the weight filtration on $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ for moduli spaces of Higgs bundles of rank 2 and their corresponding character varieties on projective curves of arbitrary genus, and conjectured a similar correspondence in any rank. This became known as the $P = W$ conjecture.
- 2015: L. Katzarkov, A. Noll, P. Pandit and C. Simpson conjectured in higher generality a homotopy commutativity property as in the above theorem. Conjecturally, it implies the "highest degree part" of the $P = W$ conjecture.

Background: Geometric $P = W$ conjecture

An obvious necessary condition for the homotopy commutativity property is that $|\mathcal{N}^X|$ have the homotopy type of a sphere.

- 2015: A. Komyo proved that for the complex 4-dimensional moduli space of logarithmic Higgs bundles of rank 2 over $\mathbb{C}P^1$ with 5 logarithmic points, $|\mathcal{N}^X|$ is homotopy equivalent to S^3.
- 2015: C. Simpson generalized Komyo's homotopy equivalence assertion to logarithmic Higgs bundles of rank 2 over $\mathbb{C}P^1$ with an arbitrary number of logarithmic points, and called the homotopy commutativity assertion "Geometric P = W conjecture".
- 2019: Sz. proved the Geometric P = W conjecture in the case $X = VI$ with nilpotent residues.

Extension of h to a fibration

Recall the irregular Hitchin map

$$h : \mathcal{M}_{\mathrm{Dol}}^{X} \to \mathbb{C}.$$

Theorem (Ivanics–Stipsicz–Szabó '19)

For generic parabolic weights, there exists an embedding

$$\mathcal{M}_{\mathrm{Dol}}^{X} \hookrightarrow E(1) = \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P}^2$$

and an elliptic fibration

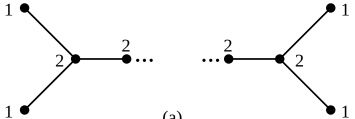
$$\tilde{h} : E(1) \to \mathbb{C}P^1$$

extending h.

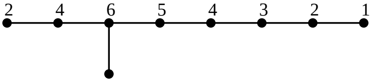
Denote by F_{∞}^{X} the non-reduced curve $E(1) \setminus \mathcal{M}_{\mathrm{Dol}}^{X} = \tilde{h}^{-1}(\infty)$.

Types of singular fibers

Kodaira's classification '63: the degenerations of elliptic curves corresponding to the affine root systems: (a) $I^*_{n-4} = D^{(1)}_n$, (b) $E^{(1)}_8$, (c) $E^{(1)}_7$, (d) $E^{(1)}_6$. The label of a node records multiplicity of the corresponding divisor component.



(a)



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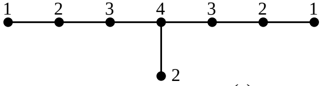

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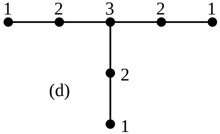


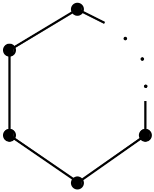
(d)

Table of fibers at infinity

X	F_{∞}^{X}
VI	$I_0^* = D_4^{(1)}$
V	$I_1^* = D_5^{(1)}$
V_{deg}	$I_2^* = D_6^{(1)}$
$III(D6)$	$I_2^* = D_6^{(1)}$
$III(D7)$	$I_3^* = D_7^{(1)}$
$III(D8)$	$I_4^* = D_8^{(1)}$
IV	$E_6^{(1)}$
II	$E_7^{(1)}$
I	$E_8^{(1)}$

Singular fibers I_n and II

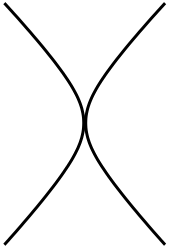
The I_n fiber ($n \geqslant 2$) is a collection of n rational curves of self-intersection -2, all with multiplicity one, intersecting each other transversally in a circular manner, as shown on the figure. The case $n = 1$ corresponds to a single nodal $\mathbb{C}P^1$.



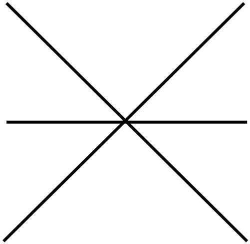
The II fiber is an S^2 with a single cuspidal singular point.

SINGULAR FIBERS *III* AND *IV*

Figure (a): type *III*, two rational curves of self-intersection -2, having a tangency of order 2 (tacnode singularity) at a point.
Figure (b): type *IV*, three rational curves of self-intersection -2 intersecting each other pairwise transversally in a common point.



(a)



(b)

Refined version of the Extension Theorem

For sake of concreteness, let $X = IV$ and assume $\mu_+ = \mu_-$. In addition to $F_\infty^{IV} = E_6^{(1)}$, there are other singular fibers of h.

Theorem (Ivanics–Stipsicz–Szabó '19)

The set of further singular fibers of h is one of the following:

1. *a type IV curve*
2. *a type II curve and a type I_2 curve*
3. *a type I_3 curve and a type I_1 curve*
4. *a type III curve and a type I_1 curve*
5. *a type I_2 curve and two type I_1 curves.*

There exist explicit conditions in terms of the parameters a, b, c that determine which one of the above possibilities holds.

Spectral data

Consider the rational ruled surface

$$p \colon \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus K(D)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2)) \to \mathbb{C}P^1,$$

endowed with the canonical section ζ of $p^* K(D)$.
The spectral sheaf of an irregular Higgs bundle $(\mathcal{E}, \theta)$ is the rank 1 pure sheaf M defined by

$$0 \to p^*\mathcal{E} \otimes K^\vee(-D) \xrightarrow{\zeta - p^*\theta} p^*\mathcal{E} \to M \to 0.$$

The support of M is called the spectral curve of $(\mathcal{E}, \theta)$.
Beauville–Narasimhan–Ramanan correspondence:

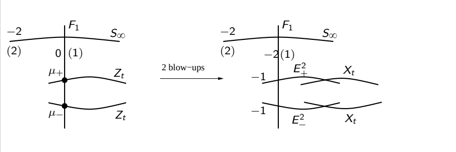
$$p_* M \cong \mathcal{E}, \quad p_*(\zeta \wedge) = \theta.$$

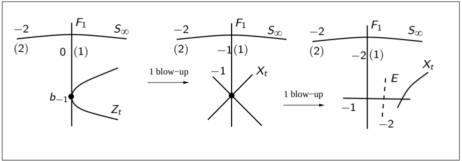
Sketch of proof of the Extension Theorem

1. Identify the pencil of spectral curves of the Higgs fields having the desired local behaviour.
2. Apply a sequence of blow-ups to eliminate the base locus.
3. Determine the singular fibers of the fibration.
4. For each singular spectral curve Z_t, determine the set of spectral sheaves M supported on Z_t that give rise to $\vec{\alpha}$-stable Higgs bundles.
5. For each spectral sheaf, find the family of parabolic structures compatible with the corresponding Higgs field.

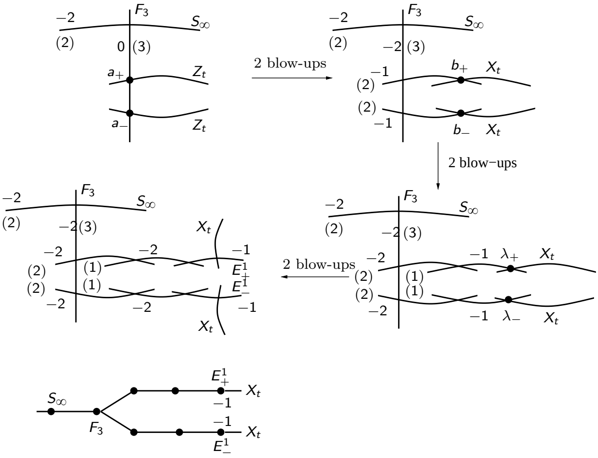
Elimination of base locus over 0

Z_t (and X_t) refer to the spectral curve in the pencil (respectively, the fibration). The non-positive number next to a curve refers to its self-intersection number, and the number in parentheses refers to its multiplicity in F_∞^{IV} (0 if not indicated).





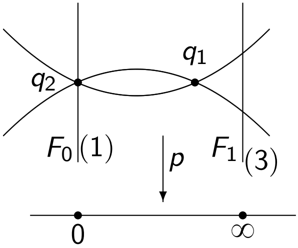
Elimination of base locus over ∞



Spectral curve corresponding to I_3 fiber

Assume for instance we have found a singular curve X_t of type I_3 in the fibration.

In the pencil, this situation corresponds to a singular spectral curve Z_t consisting of two rational curves transversally intersecting in 2 points: a point q_1 in generic position and $q_2 \in F_0 = p^{-1}(0)$.



Stratification by locus of freeness

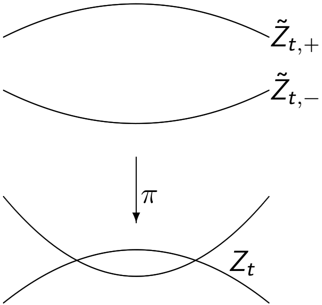
For a rank 1 torsion-free sheaf M on Z_t, either

- M is invertible,
- M is locally free at q_1, but not locally free at q_2,
- M is locally free at q_2, but not locally free at q_1,
- M is locally free neither at q_1 nor at q_2.

Stability rules out the last possibility.

Normalization

Let $\pi\colon \tilde{Z}_t \to Z_t$ stand for the normalization map, and $\tilde{Z}_{t,\pm}$ for the connected components of $\tilde{Z}_t$.



Bidegree

Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be the invertible sheaves associated to M on $\tilde{Z}_{t,\pm}$ respectively:

$$\mathcal{L}_\pm = (\pi^* M \otimes \mathcal{O}_{\tilde{Z}_{t,\pm}})/\mathcal{T}or^{\mathcal{O}_{\tilde{Z}_{t,\pm}}}(\pi^* M).$$

We define the bidegree of M as

$$(\delta_+, \delta_-) = (\deg(\mathcal{L}_+), \deg(\mathcal{L}_-)).$$

From now on, we focus on the case $\deg(\mathcal{E}) = 1, \text{par-deg}(\mathcal{E}) = 0$, and generic parabolic weights $\alpha^j_\pm$.

Sheaf counting

For invertible sheaves M, stability is equivalent to

$$\delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 1 \quad \text{or} \quad \delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 2,$$

with $\delta_- = 1 - \delta_+$. In both cases one gets a family of isomorphism classes of sheaves parametrized by $\mathbb{C}^\times$.
For non-invertible sheaves M of both types, stability gives

$$\delta_+ = -\lceil \alpha_+^1 + \alpha_+^2 \rceil + 1, \quad \delta_- = -\delta_+,$$

and there exists up to isomorphism a unique such M.

Counting compatible parabolic structures

The non-invertible sheaf M that is not locally free at q_2 is locally of the form $\pi_* \tilde{M}$ for an invertible sheaf $\tilde{M}$ on $\tilde{Z}_t$. The residue of the associated Higgs field at 0 is

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}.$$

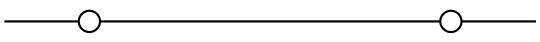
There exists a family parametrized by $\mathbb{C}P^1$ of lines $\ell \subset \mathcal{E}|_{q_2}$ preserved by θ.
For all other sheaves, there exists a unique compatible parabolic structure for $(\mathcal{E}, \theta)$.

Singular fiber of h

In all, we have found that the parabolic Higgs bundles with given spectral sheaf Z_t form a family parametrized by

$$\mathbb{C}^{\times} \coprod \mathbb{C}^{\times} \coprod \{pt\} \coprod \mathbb{C}P^1.$$

These strata form a degenerate elliptic fiber of type I_3.

$-\lceil \alpha_+^1 + \alpha_+^2 \rceil + 2$ 

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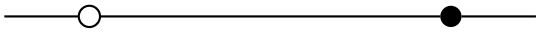
$-\lceil \alpha_+^1 + \alpha_+^2 \rceil + 2$ 

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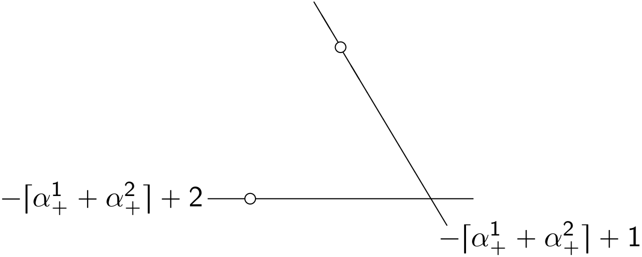

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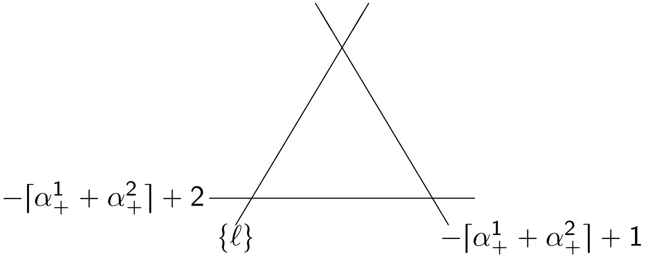

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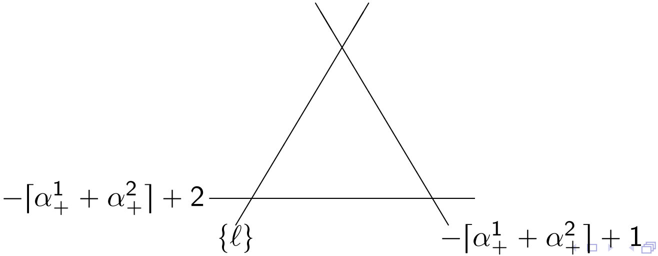
These strata form a degenerate elliptic fiber of type I_3.
The space of parabolic weights is $\mathbb{R}$, the non-generic weights are $\mathbb{Z} \subset \mathbb{R}$. What happens when $\vec{\alpha}$ "crosses a wall" $n \in \mathbb{Z}$? Before:



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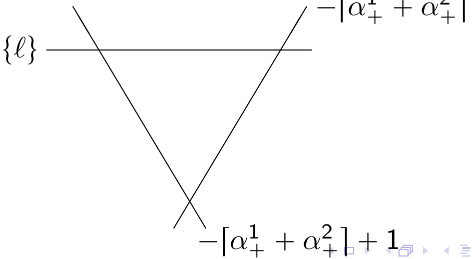


Betti spaces and affine cubic surfaces

P. Boalch (2007): General construction of wild character varieties using quasi-Hamiltonian reduction.
Fricke–Klein 1926, … , van der Put–Saito '09: for each X there exists a (not homogeneous) quadric $Q^X \in \mathbb{C}[x_1, x_2, x_3]$ such that

$$\mathcal{M}_{\mathrm{B}}^X = (f^X) \subset \mathbb{C}^3$$

where

$$f^X(x_1, x_2, x_3) = x_1 x_2 x_3 + Q^X(x_1, x_2, x_3).$$

For instance, for some $s_1 \in \mathbb{C}$, $s_2 \in \mathbb{C}^\times$ we have

$$Q^{IV}(x_1, x_2, x_3) = x_1^2 - (s_2^2 + s_1 s_2)x_1 - s_2^2 x_2 - s_2^2 x_3 + s_2^2 + s_1 s_2^3.$$

Stokes matrices for PIV

We sketch the argument of van der Put–Saito in the case $X = IV$. The monodromy matrix along a loop enclosing ∞ is of the form

$$T_\infty = \begin{pmatrix} e^c & 0 \\ 0 & e^{-c} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}$$

for some $a_i \in \mathbb{C}$. The first factor is the formal monodromy and the subsequent ones are Stokes matrices.

The multiplicative group $\mathbb{C}^\times$ acts on the quadruple (a_1, a_2, a_3, a_4) with weights $(1, -1, 1, -1)$.

The ring of invariants $\mathbb{C}[a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}, a_4^{\pm 1}]^{\mathbb{C}^\times}$ is spanned by $(a_{12}, a_{14}, a_{23}, a_{34})$ with $a_{ij} = a_i a_j$.

GIT-quotient construction of $\mathcal{M}_{\mathrm{B}}^{IV}$

There is the obvious relation $a_{12}a_{34} - a_{14}a_{23} = 0$. Using it, we may eliminate say a_{23}.
There is a further relation

$$\mathrm{tr}(T_\infty) = 2\cosh(\mu).$$

This gives the polynomial relation f^{IV} after some algebraic manipulations.

Compactifications of Betti spaces

Let

$$F^X \in \mathbb{C}[x_0, x_1, x_2, x_3]_3$$

be the homogenization of f^X and set

$$\overline{\mathcal{M}}_{\mathrm{B}}^X = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]/(F^X)).$$

Projective cubic surface (possibly) with singularities at

$$P_1 = [0 : 1 : 0 : 0], \quad P_2 = [0 : 0 : 1 : 0], \quad P_3 = [0 : 0 : 0 : 1].$$

Let

$$\widetilde{\mathcal{M}}_{\mathrm{B}}^X \to \overline{\mathcal{M}}_{\mathrm{B}}^X$$

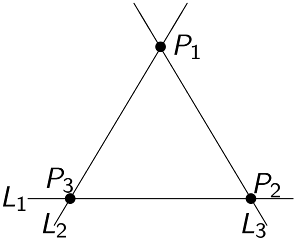
denote the minimal resolution of singularities.

Compactifying divisors

The divisor at infinity of $\overline{\mathcal{M}}_B^X$ is

$$D = L_1 \cup L_2 \cup L_3 = (x_1 x_2 x_3) \subset \mathbb{C}P^2_\infty = \{[0 : x_1 : x_2 : x_3]\}$$

where $L_i = (x_i)$ are lines pairwise intersecting each other in P_1, P_2, P_3.

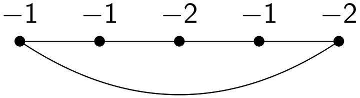


Singularities of projectivization

X	Singularities of $\overline{\mathcal{M}}_B^X$
VI	$\varnothing$
V	A_1
V_{deg}	A_2
$III(D6)$	A_2
$III(D7)$	A_3
$III(D8)$	A_4
IV	$A_1 + A_1$
II	$A_1 + A_1 + A_1$
I	$A_2 + A_1 + A_1$

Graph of divisor in smooth compactification

We resolve the singular points of $\overline{\mathcal{M}}_{\mathrm{B}}^{X}$ to get the smooth compactification $\widetilde{\mathcal{M}}_{\mathrm{B}}^{X}$.
For $X = IV$, the nerve complex $\mathcal{N}^{IV}$ of the compactifying divisor D^{IV} in $\widetilde{\mathcal{M}}_{\mathrm{B}}^{IV}$ is:



The labels of the vertices give the self-intersection number of the corresponding component.
The neighbourhood of D^X in $\mathcal{M}_{\mathrm{B}}^{X}$ has the homotopy type of an oriented plumbed 3-manifold $Y_{\mathrm{B}}^{X} = Y_\Gamma$ over some graph Γ, with base curves of genus 0.

Plumbed 3-manifolds

A class of smooth 3-manifolds Y_Γ obtained from a decorated graph $(\Gamma, \vec{g}, \vec{e})$ by "plumbing":

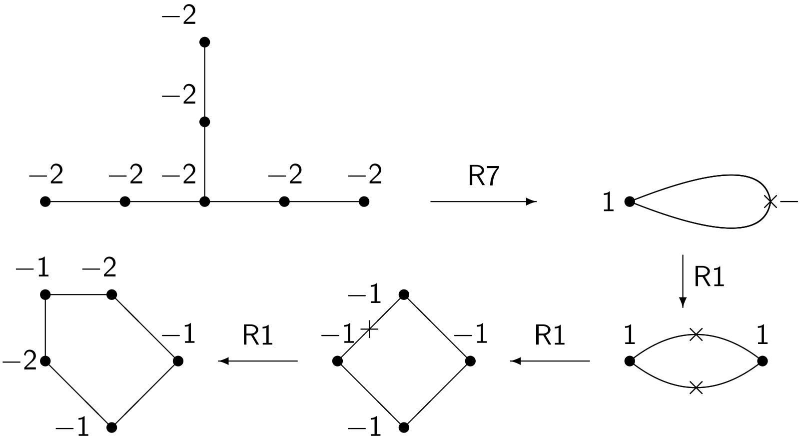
- for each node i of Γ, take a genus g_i oriented surface Σ_i, take the total space Y_i of an S^1-bundle on Σ_i of specified Euler class e_i,
- for each edge (ii') of Γ, remove an open disc D_i (respectively $D_{i'}$) from Σ_i (respectively $\Sigma_{i'}$), then in trivializations on the discs the S^1-bundles are given by $D_i \times S^1$ and $D_{i'} \times S^1$; now we glue the S^1-bundles on the corresponding curves with boundary $\Sigma_i \backslash D_i$ and $\Sigma_{i'} \backslash D_{i'}$ switching base and fiber circles: $\partial D_i \mapsto S^1, S^1 \mapsto \partial D_{i'}$ via some diffeomorphisms:

$$Y_i \cup_{\partial D_i \times S^1 \cong \partial D_{i'} \times S^1} Y_{i'}$$

Plumbing calculus

Different decorated graphs may give rise to diffeomorphic 3-manifolds.
W. D. Neumann '81: complete set of moves on decorated graphs that leave invariant the diffeomorphism type of the corresponding plumbed 3-manifold.
For example: blow-up $R1$, Seifert graph exchange $R7$.

Diffeomorphism ψ using plumbing calculus



Reformulation of Geometric P = W

Aim: show the existence of a commutative square up to homotopy

$$\begin{array}{ccc} Y^X_{\mathrm{Dol}} & \xrightarrow{\psi} & Y^X_{\mathrm{B}} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi} \\ S^1_R & \longrightarrow & |\mathcal{N}^X| \end{array},$$

the bottom row being a homotopy equivalence, where

- Y^X_{Dol} is the plumbed 3-manifold homotopy equivalent to a neighbourhood of F^X_∞ in $\mathcal{M}^X_{\mathrm{Dol}}$,
- Y^X_{B} is the plumbed 3-manifold homotopy equivalent to a neighbourhood of D^X in $\mathcal{M}^X_{\mathrm{B}}$.

The map ϕ

For simplicity, we assume $\overline{\mathcal{M}}_{\mathrm{B}}^{X}$ is smooth. Let T_i be an small open tubular neighbourhood of L_i in $\overline{\mathcal{M}}_{\mathrm{B}}^{X}$ and set

$$T = T_1 \cup T_2 \cup T_3 \supset D^X.$$

Let $\{\phi_i\}$ be a partition of unity subordinate to the cover of T by $\{T_i\}$. Define the map

$$\begin{aligned} \phi : T &\to \mathbb{R}^3 \\ x &\mapsto \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix}. \end{aligned}$$

Then, $\mathrm{Im}(\phi)$ is the boundary of the standard 2-simplex in $\mathbb{R}^3$.

FIRST HOMOLOGY OF PLUMBED 3-MANIFOLDS

For any plumbed 3-manifold Y_Γ we have

$$H_1(Y_\Gamma, \mathbb{Z}) = \mathrm{coker}(I) \oplus \mathbb{Z}^{2g} \oplus \mathbb{Z}^{b_1(|\Gamma|)},$$

where

- I is the intersection form associated to the decorated graph Γ,
- $g = \sum_i g_i$ is the sum of genera of the curves (equal to 0 in our cases),
- $b_1(|\Gamma|)$ is the first Betti number of the graph Γ.

Elzein–Némethi '02: the above decomposition gives the weight of the generators in the Mixed Hodge Structure.

Purity

In our cases of interest, we get

$$b_1(Y^X_{\mathrm{Dol}}) = 1 + 0 + 0$$
$$b_1(Y^X_{\mathrm{B}}) = 0 + 0 + 1,$$

consequently we have

$$H_1(Y^X_{\mathrm{Dol}}, \mathbb{C}) = \mathrm{Gr}^W_{-2} H_1(Y^X_{\mathrm{Dol}}, \mathbb{C})$$
$$H_1(Y^X_{\mathrm{B}}, \mathbb{C}) = \mathrm{Gr}^W_{0} H_1(Y^X_{\mathrm{B}}, \mathbb{C}).$$

Proof of homotopy commutativity

We have for both $Y = Y^X_{\text{Dol}}$ and $Y = Y^X_{\text{B}}$ an isomorphism

$$[Y, S^1] = H^1(Y, \mathbb{Z}) = \mathbb{Z}.$$

It is sufficient to prove that h_* and ϕ_* are epimorphisms on H_1. It is known that both h and ϕ are torus bundles over S^1, whence surjectivity follows easily.