Hitchin's equations and Fourier transform on curves

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Abstract

These are informal notes to my talks at the *Lie algebras and moduli spaces* seminar of the Eötvös Loránd University of Budapest in March 2007. First, we review some standard facts and constructions about the moduli space of solutions to Hitchin's equations, then we outline Fourier transformation for Higgs bundles on a curve of genus ≥ 1.

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1 Hitchin's equations

Here we define the self-duality equations reduced to dimension 2 following [4].

Denote $G = U(r)$ and let V be an $U(r)$-vector bundle over $\mathbb{R}^4$. Take a $D_{\tilde{A}}$ G-connection on V and consider its curvature $F(\tilde{A}) = D^2_{\tilde{A}} \in \Omega^2(\mathbb{R}^4, \mathfrak{u}(r))$.

Definition 1. The connection $D_{\tilde{A}}$ is *self-dual* (SD) if

$$*F(\tilde{A}) = F(\tilde{A}), \tag{SD}$$

where $* : \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ is the Hodge star operator.

Remark. We can consider the *anti-self-dual* theory by replacing equation (SD) with

$$*F(\tilde{A}) = -F(\tilde{A}). \tag{ASD}$$

The theory we obtain is similar to the self-dual case.

We expand the equation (SD) in local coordinates. Let x_1, x_2, x_3, x_4 be the canonical coordinates on $\mathbb{R}^4$ and $v_1, \dots, v_r$ a trivialization of V. If $D_{\tilde{A}} = d + \sum_{i=1}^4 A_i dx_i$ in this trivialization for matrices $A_i \in \Omega^0(\mathbb{R}^4, \mathfrak{u}(r))$ then $F(\tilde{A}) =$

$\sum_{i<j} F_{ij} dx_i \wedge dx_j$, with $F_{ij} = \left[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j\right]$ $(i, j = 1, 2, 3, 4)$. With these notations the equation (SD) expands to

$$\begin{aligned} F_{12} &= F_{34} \\ F_{13} &= -F_{24} \\ F_{14} &= F_{23} \end{aligned} \tag{SD}$$

From now on we suppose that the matrices A_1, A_2, A_3, A_4 don't depend on the coordinates x_3, x_4. We define a G-connection $D_A = d + A := d + A_1 dx_1 + A_2 dx_2$ on $V|_{\mathbb{R}^2}$ and denote by F_A its curvature. Furthermore, we consider the endomorphisms $\phi_1 = A_3, \phi_2 = A_4 \in \Omega^0(\mathbb{R}^4, \mathfrak{u}(r))$ called the *real Higgs fields* and $\phi = \phi_1 - i\phi_2 \in \Omega^0(\mathbb{R}^2, \mathfrak{gl}(r, \mathbb{C}))$ the *complex Higgs field*. We can rewrite the equation (SD) in terms of the connection D_A and the Higgs fields:

$$\begin{aligned} F_A &= \frac{1}{2} i[\phi, \phi^*] && \in \mathfrak{u}(r) \\ 0 &= \left[\left(\frac{\partial}{\partial x_1} + A_1\right) + i\left(\frac{\partial}{\partial x_2} + A_2\right), \phi\right] && \in \mathfrak{gl}(r, \mathbb{C}). \end{aligned}$$

The first equation is coordinate dependent (on one side there is a 2-form and on the other side a 0-form). Therefore we introduce the notation $\Phi = \frac{1}{2}\phi dz \in \mathbf{\Omega}^1(\mathbb{R}^2 \mathfrak{gl}(r, \mathbb{C}))$ to write the (SD) equations in coordinate independent form:

$$\begin{aligned} F_A &= -[\Phi, \Phi^*] \\ d_A^{0,1}\Phi &= 0 \end{aligned} \tag{SD'}$$

The coordinate independent form allow us to investigate this equation on complex curves.

2 Higgs-bundles and self-dual connections

2.1 Dolbeault-theory

From now on C is a complex curve over $\mathbb{C}$ (a Riemann-surface) and V is a $GL(r, \mathbb{C})$-bundle over C.

Definition 2. A *Higgs-bundle* on V is a pair $(\bar{\partial}^{\mathcal{E}}, \theta)$ with $\bar{\partial}^{\mathcal{E}} : V \to \Omega^{0,1} \otimes_{\mathcal{C}^\infty} V$ a holomorphic structure on V and $\theta \in \mathbf{\Omega}^1(C, End_{\bar{\partial}^{\mathcal{E}}}(V))$ an $End(V)$-valued $\bar{\partial}^{\mathcal{E}}$-holomorphic 1-form (called *Higgs-field*).

Remark. If (D_A, Φ) satiesfies Hitchin's equations (SD') then $(D_A^{0,1}, \Phi)$ is a Higgs-bundle.

Let $\mathcal{G}^{\mathbb{C}} = \Gamma(C, GL(V))$ be the group of complex gauge-transformations of V, and if there is a Hermitian metric h on V then let $\mathcal{G} = \Gamma(C, U(V))$ be the unitary gauge-transformation group.

Definition 3. The Higgs-bundle $(\bar{\partial}^{\mathcal{E}}, \theta)$ is said to be *stable*, if for any proper θ-invariant subbundle $\mathcal{E}'$ one has the inequality of slopes

$$\frac{\deg(\mathcal{E}')}{\mathrm{rk}(\mathcal{E}')} < \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

The Higgs-bundle is *semistable* if we only have $\leq$ in this inequality. It is *polystable* if it is direct sum of stable Higgs-bundles with same slope.

Theorem 1. (Hitchin)

1. *If (D_A, Φ) is an irreducible solution of Hitchin's equations (SD'), then $(D_A^{0,1}, \Phi)$ is a stable Higgs-bundle.*
2. *If $(\bar{\partial}^{\mathcal{E}}, \theta)$ is a stable Higgs-bundle, then for every D_A with $D_A^{0,1} = \bar{\partial}^{\mathcal{E}}$ there is a $g \in \mathcal{G}^{\mathbb{C}}$ such that $g \cdot D_A$ satisfies Hitchin's equations. Moreover, g is unique up to $\mathcal{G}$ transformations.*
3. *The previous constructions are inverse to each other up to $\mathcal{G}$ and $\mathcal{G}^{\mathbb{C}}$ transformations.*

Remark.

1. If D_A satisfies Hitchin's equations then so does $h \cdot D_A$ for every $h \in \mathcal{G}$. Hence uniqueness in the second statement can only be up to a $\mathcal{G}$ transformation.
2. Since $GL(r, \mathbb{C})/U(r)$ is isomorphic to the space of Hermitian metrics on $\mathbb{C}^r$, the second statement in the theorem can be reformulated as follows. There is a unique metric h (up to $\mathbb{C}^*$) on V such that $\partial^h + \bar{\partial}^{\mathcal{E}} + \theta + \theta^*$ satisfies Hitchin's equations, where $\partial^h + \bar{\partial}^{\mathcal{E}}$ is the Chern-connection corresponding to h and $\bar{\partial}^{\mathcal{E}}$ and θ^* is the adjoint of θ with respect to h. This metric h is called the *Hermitian-Einstein metric*.
3. If instead of $GL(V)$ and $U(V)$ we take $SL(V)$ and $SU(V)$ respectively, then uniqueness of h is strict (not just up to $\mathbb{C}^*$).

Corollary 1. *There is a bijection between the following two sets*

$$M_{Dol} := \left\{ \begin{array}{c} \text{Stable Higgs-} \\ \text{bundles over } C \end{array} \right\} / \mathcal{G}^{\mathbb{C}} \leftrightarrow \left\{ \begin{array}{c} \text{Solutions of Hitchin's} \\ \text{equations on } C \end{array} \right\} / \mathcal{G} =: M_{Hit}.$$

Corollary 2. *The moduli space of stable Higgs-bundles is a hyperKähler manifold.*

Proof. Using the previous corollary it is enough to show that M_{Hit} is a hyperKähler manifold. Gauge-theory implies that the moduli space is a smooth manifold away from points of a smaller dimensional set.

Let $\mathcal{A}$ be the affine space (over $\Omega^{0,1}(C, \mathfrak{gl}(V))$) of $(0,1)$-connections over V. Moreover, let Ω be the affine space (over $\Omega^{1,0}(C, \mathfrak{gl}(V)) \simeq \overline{\Omega^{0,1}(C, \mathfrak{gl}(V))}$) of Higgs-fields. Choose an arbitrary base point $(\bar{\partial}^{\mathcal{E}}, \theta)$ of $\mathcal{A} \times \Omega$; then the tangent space $T_{(\bar{\partial}^{\mathcal{E}}, \theta)}(\mathcal{A} \times \Omega) \simeq T^*\Omega^{0,1}(C, \mathfrak{gl}(V))$ of $\mathcal{A} \times \Omega$ in this point is clearly a flat hyperKähler space. Here, the Kähler metric is the L^2-metric:

$$g\left((\psi_1, \phi_1), (\psi_2, \phi_2)\right) = 2i \int_C \mathrm{tr}(\psi_1^* \wedge \psi_2 + \phi_1^* \wedge \phi_2),$$

and the compex structures are given by:

$$\begin{aligned} I(\psi, \phi) &= (i\psi, i\phi) \\ J(\psi, \phi) &= (i\phi^*, -i\psi^*) \\ K(\psi, \phi) &= (-\phi^*, \psi^*). \end{aligned}$$

The gauge-group $\mathcal{G}$ acts on $\mathcal{A} \times \Omega$ preserving the hyperKähler structure. The real and complex moment maps are given respectively by:

$$\begin{aligned} \mu_{\mathbb{R}} &= F_A + [\Phi, \Phi^*] \\ \mu_{\mathbb{C}} &= D_A^{0,1}\Phi \end{aligned}$$

Thus the space of solutions of Hitchin's equations is the hyperKähler quotient $(\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))/\mathcal{G} = \mathcal{A} \times \Omega /\!/\!/\!/_0 \mathcal{G}$ (for the hyperKähler quotient construction, see the Appendix). □

2.2 The abelian case

In this subsection we describe the Dolbeault moduli space in rank 1. For details, see [2].

Let $\bar{\partial}^{\mathcal{E}}$ be a holomorphic structure on the trivial line bundle V over C. Moreover, let $\theta \in \mathbf{\Omega}^1(C, End_{\bar{\partial}^{\mathcal{E}}}(V)) \simeq \mathbf{\Omega}^1(C)$ be a Higgs-field. By commutativity, the gauge action by $g \in \mathcal{G}^{\mathbb{C}} = \Omega^0(C, GL(V)) = \Omega^0(C, \mathbb{C}^*)$ is given by:

$$\begin{aligned} g \cdot \theta &= g^{-1}\theta g = \theta \\ g \cdot (\bar{\partial}^{\mathcal{E}} + \psi) &= \bar{\partial}^{\mathcal{E}} + (\psi + g^{-1}\bar{\partial} g), \end{aligned}$$

with $\psi \in \Omega^{0,1}(C, \mathbb{C})$. Furthermore, since the rank is 1, every Higgs-bundle is automatically stable.

Proposition 1. *If* $\mathrm{Jac}(C) = H^{0,1}(C)/H^1(C,\mathbb{Z})$ *denotes the Jacobian variety of* C*, then* $M_{Dol}(C,U(1)) = T^*\,\mathrm{Jac}(C)$.

Proof.

$$\left\{ \begin{array}{c} \text{Holomorphic} \\ \text{sturctures on } V \end{array} \right\} \Big/ \mathcal{G}^{\mathbb{C}} \simeq \Omega^{0,1}(C,\mathbb{C})/\mathcal{G}_0^{\mathbb{C}} \times \pi_0(\mathcal{G}^{\mathbb{C}})$$
$$\simeq \left(\Omega^{0,1}(C,\mathbb{C})/\bar{\partial}\Omega^0(C,\mathbb{C})\right)\big/\,\pi_0(\mathcal{G}^{\mathbb{C}}) \qquad (*)$$
$$= H^{0,1}(C,\mathbb{C})\big/\,\pi_0(\mathcal{G}^{\mathbb{C}}).$$

The (*) holds because every $g \in \mathcal{G}_0^{\mathbb{C}}$ can be writen as $g = \exp(f)$ and then $g \cdot \bar{\partial}^{\mathcal{E}} = \bar{\partial}^{\mathcal{E}} + \bar{\partial} f$. Moreover, $\pi_0(\mathcal{G}^{\mathbb{C}}) \simeq H^1(C,\mathbb{Z})$ because of the exact sequence

$$\mathcal{C}^\infty(C,\mathbb{C}) \xrightarrow{\exp} \mathcal{C}^\infty(C,\mathbb{C}^*) \xrightarrow{\pi_1} H^1(C,\mathbb{Z})\ .$$

Moreover, as we already remarked, $\Omega^{1,0}(C) \simeq \overline{\Omega^{0,1}(C)}$ and $\mathcal{G}^{\mathbb{C}}$ acts trivially on $\Omega^{1,0}(C,\mathbb{C})$. Hence

$$\{(\bar{\partial}^{\mathcal{E}},\theta) : \bar{\partial}^{\mathcal{E}}\theta = 0\}\big/\,\mathcal{G}^{\mathbb{C}} = \left(\{\bar{\partial}^{\mathcal{E}}\}/\mathcal{G}^{\mathbb{C}}\right) \times \{\theta : \bar{\partial}^{\mathcal{E}}\theta = 0\} = T^*\,\mathrm{Jac}(C).$$

□

3 Fourier-transform for Higgs-bundles over curves

Here, we are inspired by the article [1], but we proceed slightly differently. In particular, we do not make use of the Abel-map.

Let C be a compact curve over $\mathbb{C}$ of genus $g = g(C) \geq 1$. Since $\mathrm{Jac}(C)$ is topologically a torus, it is paralellizable:

$$T^*\,\mathrm{Jac}(C) = \mathrm{Jac}(C) \times H^1(C,\mathcal{O}_C)^\vee$$
$$\simeq \mathrm{Jac}(C) \times H^0(C,\mathbf{\Omega}^1).$$

Let $(\bar{\partial}^{\mathcal{E}},\theta)$ be a stable degree 0 Higgs-bundle on C and $(\alpha,\beta) \in T^*\,\mathrm{Jac}(C)$ (here $\alpha \in \mathrm{Jac}(C) \simeq Pic^0(C)$, the holomorphic line bundle corresponding to α is denoted by $\mathcal{L}_\alpha$, and $\beta \in H^0(C,\mathbf{\Omega}^1)$). We twist $(\mathcal{E},\theta)$ by (α,β):

$$(\mathcal{E},\theta)_{(\alpha,\beta)} := (\mathcal{E}\otimes\mathcal{L}_\alpha, (\theta - \beta\mathrm{Id}_{\bar{\partial}^{\mathcal{E}}})\otimes\mathrm{Id}_{\mathcal{L}_\alpha}) =: (\mathcal{E}_\alpha,\theta_\beta).$$

Definition 4. Let $\Sigma = \{(x,(\alpha,\beta)) \in C \times T^*\,\mathrm{Jac}(C) : \det(\theta_\beta(x)) = 0\}$ be the *spectral variety.*

Proposition 2. *1.* $\dim \Sigma = 2g$.

2. *Σ contains entire fibers of the projection $T^* \operatorname{Jac}(C) \mapsto H^0(C, \mathbf{\Omega}^1)$, i.e. $(x,(\alpha,\beta)) \in \Sigma \Leftrightarrow (x,(\alpha',\beta)) \in \Sigma$ for any $\beta \in H^0(C,\mathbf{\Omega}^1)$ and $\alpha,\alpha' \in \operatorname{Jac}(C)$.*

3. *If $(\bar\partial^{\mathcal{E}}, \theta)$ is stable of degree 0, then the fibers of Σ for the second projection $\pi_2 : C \times T^* \operatorname{Jac}(C) \to T^* \operatorname{Jac}(C)$ are finite for any $(\alpha,\beta) \in T^* \operatorname{Jac}(C)$ and $\#\pi_2^{-1}(\alpha,\beta) \cap \Sigma = (2g-2)r$ (counted with multiplicity).*

4. *For the first projection $\pi_1 : C \times T^* \operatorname{Jac}(C) \to C$ and for every $x \in C$ we have $\dim_{\mathbb{C}}(\pi_1^{-1}(x) \cap \Sigma) = 2g-1$.*

Proof. 1. Σ is defined by one non-trivial algebraic equation in the $(2g+1)$-dimensional variety $C \times T^* \operatorname{Jac}(C)$.

2. Tensoring by $\operatorname{Id}_{\mathcal{L}_\alpha}$ obviously doesn't change the determinant.

3. The fiber $\Sigma_{(\alpha,\beta)} = \pi_2^{-1}(\alpha,\beta) \cap \Sigma$ is by definition the set of points in C where the determinant $\det(\theta_\beta)$ vanishes. Since θ_β is a map from $\mathcal{E}$ to $\mathcal{E} \otimes \mathbf{\Omega}^1$, it follows that its determinant is a map from $\det(\mathcal{E})$ to

$$\det(\mathcal{E} \otimes \mathbf{\Omega}^1) = \det(\mathcal{E}) \otimes (\mathbf{\Omega}^1)^{\otimes r},$$

or in different terms a section of $(\mathbf{\Omega}^1)^{\otimes r}$. It is not the zero-section, because this would mean that $\mathcal{E}$ has a degree 0 invariant subbundle where θ would agree with β, in contradiction with stability and the degree 0 condition. Therefore, the section $\det(\theta_\beta)$ vanishes in exactly $\deg(\mathbf{\Omega}^1)^{\otimes r} = r(2g-2)$ points (counted with multiplicity).

4. This follows from 1. and 3.

□

Let $\mathcal{P}$ be the pullback of the Poincaré bundle to $C \times T^* \operatorname{Jac}(C)$, i.e. the bundle whose fiber over $C \times \{(\alpha,\beta)\}$ is $\mathcal{L}_\alpha \to C$. Consider $H^0(C,\mathbf{\Omega}^1)$ as the affine part away from infinity in the projective space $\mathbb{P}^g = \mathbb{P}(\mathbb{C} \oplus H^0(C,\mathbf{\Omega}^1))$. Let $(\zeta : \beta)$ be coordinates in $\mathbb{P}^g$, where $\zeta \in \mathbb{C}$ and $\beta \in H^0(C,\mathbf{\Omega}^1)$. Extend every bundle by pull-back to the product $C \times \operatorname{Jac}(C) \times \mathbb{P}^g$. Define the sheaf-map on $C \times \operatorname{Jac}(C) \times \mathbb{P}^g$:

$$\begin{aligned} \Theta : \pi_1^* \mathcal{E} \otimes \mathcal{P} &\longrightarrow \pi_1^* \mathcal{E} \otimes \mathcal{P} \otimes \mathbf{\Omega}^1 \otimes \mathcal{O}_{\mathbb{P}^g}(1) \\ \Theta_{(x,(\alpha,(\zeta:\beta)))} &= (\zeta\theta(x) - \beta \underset{\mathcal{E}}{\operatorname{Id}}) \otimes \underset{\mathcal{L}_\alpha(x)}{\operatorname{Id}}. \end{aligned}$$

Definition 5. The sheaf $M := Coker\Theta$ on $C \times \operatorname{Jac}(C) \times \mathbb{P}^g$ is called the *spectral sheaf.*

Proposition 3. *1. M is a coherent sheaf with support the compactification of Σ in $C \times \operatorname{Jac}(C) \times \mathbb{P}^g$. (For simplicity, we continue to denote this compactification by Σ.)*

2. $M = M_0 \otimes \mathcal{P}$, where M_0 is the restriction of M to $C \times \{0\} \times H^0(C, \mathbf{\Omega}^1)$ (i.e. if $\iota : \{0\} \times H^0(C, \mathbf{\Omega}^1) \mapsto T^* \operatorname{Jac}(C)$ denotes the inclusion, then $M_0 = \iota^* M$).

Proof. 1. This is clear by the definitions.

2. Θ only depends on α through tensoring by $\mathcal{P}$, therefore so does its cokernel.

□

Definition 6. Let $\hat{\mathcal{E}}$ be the coherent sheaf $\hat{\mathcal{E}} = R_0 \pi_{2*} M$, i.e. the sheaf on $\operatorname{Jac}(C) \times \mathbb{P}^g$ satisfying $\hat{\mathcal{E}}_{(\alpha,\beta)} = \oplus_{(x,(\alpha,(\zeta:\beta))) \in \Sigma} M_{(x,(\alpha,(\zeta:\beta)))}$. We call $\hat{\mathcal{E}}$ the *transformed bundle.*

Proposition 4. *1. $\hat{\mathcal{E}}$ is a coherent sheaf on $\operatorname{Jac}(C) \times \mathbb{P}^g$. Moreover, if $(\bar{\partial}^{\mathcal{E}}, \theta)$ is stable of degree 0 then $\hat{\mathcal{E}}$ is a vector bundle of rank $r(2g-2)$.*

2. There is a natural connection $\hat{\nabla}$ and a Hermitian metric $\hat{h}$ on the restriction of $\hat{\mathcal{E}}$ to the affine part $T^* \operatorname{Jac}(C)$ which satisfy the generalized ASD equations.

The generalized ASD equations can be defined on any hyperKähler manifold, see [3].

Proof. 1. $\hat{\mathcal{E}}$ is clearly coherent on $T^* \operatorname{Jac}(C)$; over an element $(0 : \beta)$ of the hyperplane at infinity of $\mathbb{P}^g$, the spectral variety Σ is equal to $\{x : \beta(x) = 0\}$, hence also finite. Finally, if $(\bar{\partial}^{\mathcal{E}}, \theta)$ is stable of degree 0, then Σ is a finite cover of $T^* \operatorname{Jac}(C)$.

2. For the construction of the connection and Hermitian metric, we first have to interpret $\hat{\mathcal{E}}$ as L^2 harmonic 1-forms of a Dirac operator. Then, we define the connection by the projection-of-the-trivial-connection formula, and the Hermitian metric as the L^2-metric of harmonic respresentatives. For details about the construction and the proof that the connection and metric satisfy the generalized Hermitian ASD-equations, see [3].

□

Question 1. *1. Does the connection $\hat{\nabla}$ extend to the hyperplane at infinity?*

2. Is the transform invertible? That is, does every equivalence class of generalized Hermitian ASD-connections on $\mathrm{Jac}(C) \times \mathbb{P}^g$ *arise as the Fourier transform of a Higgs bundle on C?*

4 Appendix: the hyperKähler quotient

Let (X, ω) be a symplectic manifold with an action of a Lie group G preserving the symplectic structure, or (X, ω, g) be a Kähler manifold with a Lie group action preserving both structures. We suppose that there is a moment map $\mu : X \to \mathfrak{g}^*$ which satisfies the relation $d_W \mu(\xi) = \omega(V_\xi, W)$ for any $\xi \in \mathfrak{g}$, $W \in \Gamma(TX)$ and $V_\xi \in \Gamma(TX)$ (the vector field induced by the infinitesimal G action). Suppose that the action of G on $\mu^{-1}(0)$ is free and proper. We can define the Marsden-Weinstein (respectively Kähler) quotient of X by G as $X/\!\!/G := \mu^{-1}(0)/G$. This quotient inherits a symplectic (respectively Kähler) structure from X.

A manifold X is hyperKähler if there are 3 complex structures $I, J, K : TX \to TX$ satisfing the relations $I^2 = J^2 = K^2 = IJK = -1$ and moreover there is a g metric on X such that (X, g, I), (X, g, J), (X, g, K) are Kähler manifolds. Hence the real dimension satisfies $\dim_{\mathbb{R}} X = 4n$ ($n \in \mathbb{N}$).

Suppose that a Lie group G acts freely on (X, g, I, J, K) compatible with all structures and there are three moment maps $\mu_I, \mu_J, \mu_K : X \to \mathfrak{g}^*$, one for each of the associated symplectic structures. We use the notations $\mu_{\mathbb{R}} = \mu_I$ and $\mu_{\mathbb{C}} = \mu_J + i\mu_K$. We have (see [5]):

Theorem 2. (Hitchin-Karlhede-Lindström-Roček) *The quotient*

$$X/\!\!/\!\!/\!\!/G := \mu^{-1}(0)/G$$

is a hyperKähler manifold.

1. If both X and G are finite-dimensional, then $\dim X/\!\!/\!\!/\!\!/G = \dim X - 4 \dim G$.

2. The theorem is true even for infinite dimensional X and G, if the quotient is a finite dimensional manifold.

References

[1] Bonsdorff J. *A Fourier transform for Higgs bundles*, math.AG/0104253

[2] Goldman W., Xia E. *Rank One Higgs Bundles and Representations of Fundamental Groups of Riemann Surfaces*, math.DG/0402429

[3] Frejlich P. , Jardim M. *Nahm transform for Higgs bundles*, math.DG/0702832

[4] Hitchin N. J. *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59-126

[5] Hitchin N. J., Karlhede A., Lindström U., Roček M. *Hyperkähler metrics and supersymmetry* Commun. Math. Phys. 108 (1987) 535.