Asymptotic geometry of non-abelian Hodge theory and Riemann–Hilbert correspondence, rank three $\widetilde{E}_6$ case

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Abstract

We prove the Geometric P=W conjecture in rank three on the three-punctured sphere. For this purpose, we describe the topology at infinity of the related character variety and we use asymptotic abelianization of harmonic bundles away from the ramification divisor to give a complete geometric understanding of the involved maps. A pattern reminiscent of the Stokes phenomenon emerges in the found behavior.

1 Introduction and statement of the main result

The principal objects of this paper are, on the one hand, the moduli space $\mathcal{M}_{\mathrm{Dol}}(\alpha)$ of gauge-equivalence classes of rank 3 Higgs bundles on the complex projective line $\mathbb{C}P^1$ with three logarithmic points (called the Dolbeault

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moduli space) and, on the other hand, the space $\mathcal{M}_{\mathrm{B}}(\mathbf{c})$ of representations of the fundamental group of the thrice-punctured $\mathbb{C}P^1$ in $\mathrm{SL}(3,\mathbb{C})$ up to overall conjugation (called the Betti moduli space or character variety). These two spaces are diffeomorphic to each other via a composition of the non-abelian Hodge and Riemann–Hilbert correspondences. Moreover, they are known to carry natural complex algebraic variety structures of dimension 2, but the diffeomorphism between them is not compatible with their complex structures. We investigate the asymptotic behavior of this diffeomorphism ψ (with specially chosen parameters). Our main result is:

Theorem 1.1. *The Geometric P=W conjecture holds for rank 3 tame harmonic bundles over the three-punctured sphere.*

Along the way, we also give a self-contained proof for the following:

Theorem 1.2 (Proposition 3.1)**.** *The* $\mathrm{GL}(3,\mathbb{C})$ *character variety of the three-punctured sphere admits a smooth compactification by a curve of type* I_1*. In particular, the body of its nerve complex is of homotopy type* S^1*.*

We note that P. Etingof, A. Oblomkov, E. Rains have obtained essentially the same result about the compactifying divisor of type I_1 using representation theoretical techniques [8, Proposition 6.6], including in the cases corresponding to the affine root systems $\widetilde{E}_7$ and $\widetilde{E}_8$ (rather than just $\widetilde{E}_6$ studied here). We now turn to describing the context and precise meaning of Theorem 1.1.

One motivation of this study comes from Hitchin's WKB problem [22], which roughly reads as follows: consider a $\mathbb{C}^\times$ orbit in the Hitchin base and its lift to the Dolbeault space. Consider the family of flat connections corresponding to this lift, and determine the behaviour of the associated transport matrices, as the point of the Hitchin base converges to infinity along a $\mathbb{C}^\times$-orbit. For more about the WKB approximation theory of the Schrödinger operator, see [47]. More recently, [13, Section 13] carried out WKB analysis of Hitchin's equations.

Another source of inspiration is the so-called P=W conjecture. The Betti space is known to be an affine algebraic variety, and as such its cohomology spaces carry a mixed Hodge structure [7]. On the other hand, the Hitchin map endows the cohomology spaces of the Dolbeault space with a perverse Leray filtration. The conjecture, formulated by M. de Cataldo, T. Hausel and L. Migliorini [5], states that the diffeomorphism ψ between the spaces respects these filtrations. Recently this has been an intensely investigated area, with strong ties to other fields such as Cohomological Hall Algebras, the geometry of the affine Springer fiber, and Donaldson–Thomas theory. In [5] it was proved in the rank two case for compact curves. The complete proof for the P=W conjecture came recently from two different sources, using different techniques [17] and [32].

A geometric counterpart of the conjecture was stated in [22, Conjecture 1.1] and [43, Conjecture 11.1]. So far, this Geometric $P = W$ conjecture has received significantly less attention than the original cohomological version. Mauri, Mazzon and Stevenson [31] proved it in genus 1 and type A. Sz. Szabó dealt with the conjecture in case of rank 2 Higgs bundles with irregular singularities over $\mathbb{C}P^1$, belonging to the Painlevé cases [46], and with rank 2 logarithmic Higgs bundles over the five-punctured sphere, by establishing the WKB-analysis of coordinates of the character variety [45]. A. Némethi and Sz. Szabó provided a different proof for these Painlevé cases using low-dimensional topology techniques [37]. As far as the authors know, these are the only cases where the full assertion of the Geometric P=W is proved. The Geometric $P = W$ conjecture asserts the existence of the following homotopy commutative diagram:

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol}} \setminus h^{-1}(B_R(0)) \arrow[r, "\psi"] \arrow[d, "h"'] & \mathcal{M}_B \setminus \psi(h^{-1}(B_R(0))) \arrow[d, "\phi"] \\ \mathcal{H} \setminus B_R(0) \arrow[r] & |\mathcal{D}\partial\mathcal{M}_B| \end{tikzcd} \tag{1}$$

where

$$\psi = \mathrm{RH} \circ \mathrm{NAHC}$$

is the composition of the non-abelian Hodge correspondence

$$\mathrm{NAHC}\colon \mathcal{M}_{\mathrm{Dol}}(\alpha) \to \mathcal{M}_{\mathrm{dR}}(\beta, \tau)$$

and the Riemann Hilbert correspondence

$$\mathrm{RH}\colon \mathcal{M}_{\mathrm{dR}}(\beta, \tau) \to \mathcal{M}_{\mathrm{B}}(\mathbf{c})$$

and h is the Hitchin fibration over the Hitchin base $\mathcal{H}$. The Hitchin base will turn out to be a one-dimensional affine space in our case (see Section 2.3 for more details), and by $\mathcal{H} \setminus B_R(0)$ we denote the neighbourhood of infinity in the base, for $R \gg 1$. Moreover $|\mathcal{D}\partial\mathcal{M}_B|$ denotes the body of the nerve complex of the compactifying divisor of the Betti space, and ϕ is Simpson's natural map from a neighborhood of infinity in $\mathcal{M}_B$ to the body of the nerve complex (see Section 2.1 and 6.2 for more details).

In [22] the conjecture about the existence of a commutative diagram up to homotopy was stated in higher generality. In particular, the homotopy type of the body of the nerve complex is expected to be always that of a sphere. This homotopy sphere assertion was proved by C. Simpson in [43] over $\mathbb{C}P^1$, with an arbitrary finite number of punctures in the rank 2 case. The investigation of the homotopy type of the topological space of the dual boundary complex of the character variety is a basic step to deal with the Geometric $P = W$ conjecture, and numerous results belong to this. A. Komyo [24] proved the assertion for some 2 and 4 dimensional tame cases, and it was generalized by C. Simpson [43], who showed that in the rank 2 case for an arbitrary number n of logarithmic points on $\mathbb{C}P^1$, the homotopy type of the dual boundary complex is that of S^{2n-7}. Another result from M. Mauri, E. Mazzon and M. Stevenson shows that that the dual boundary complex of a log-Calabi-Yau compactification of the $\mathrm{GL}(n,\mathbb{C})$ character variety of a 2-torus is homeomorphic to S^{2n-1}, see [31].

We will adopt the asymptotic abelianization approach used in [46] to establish the Geometric P=W conjecture in rank 2 corresponding to the Painlevé cases. We will adapt this technique here to the rank 3 case. An important technical difference with the rank 2 case is that in [45], [46] we made use of certain local models (called fiducial solutions) only available in rank 2; here we have found a way to get around finding specific model solutions and carry out the analysis solely using abelianization. This simplifies the presentation and clarifies the picture, thus paving the way for potential higher-rank and higher-dimensional generalizations of our viewpoint. As a consequence of the analysis, we find that the asymptotic behavior of ψ depends on a decomposition into sectors of the Hitchin base, namely it is determined by different exponential terms in each sector around infinity. This clearly reminds of the Stokes phenomenon, which is somewhat surprising, because no connection with irregular singularities seems to be present.

This paper is organized as follows. In Section 2, we recall the background material necessary to explain our arguments. In Section 3, we first describe the $\mathrm{GL}(3,\mathbb{C})$ character variety of the three-punctured sphere in general, and prove Theorem 1.2. We then analyze trace coordinates on the character variety, going back to classical work of R. Fricke and F. Klein [12], and which were generalized for $\mathrm{GL}(3,\mathbb{C})$ character varieties by S. Lawton [26], [27]. In Section 4, we use T. Mochizuki's asymptotic abelianization technique [34] to give the large-scale analysis of harmonic bundles. In Section 5 we describe the parallel transport matrices, and apply Riemann Hilbert correspondence to the previous setup. Finally in Section 6, we investigate the asymptotic behaviour of the trace coordinates under the Riemann Hilbert correspondence, and prove Theorem 1.1.

2 Preparatory material

First, let us introduce the material that we will need to establish and prove our result. The structure follows more or less [45] and [10].

2.1 Basic notations, definitions and results

Consider $X = \mathbb{C}P^1$ with the standard Riemannian-metric, and with coordinate charts z and $w = z^{-1}$. For the distinct points $0, 1, \infty \in \mathbb{C}P^1$, let D be the simple effective divisor $D = 0+1+\infty$ (and denote by D the support set of the divisor as well). Consider furthermore a smooth vector bundle E of rank 3 and degree 0 on $\mathbb{C}P^1$. We denote by K and $\mathcal{O}$ the sheaves of holomorphic 1-forms and functions on $\mathbb{C}P^1$, and by $\Omega^{1,0}$ and $\Omega^{0,1}$ the smooth $(1,0)$- and $(0,1)$-forms on $\mathbb{C}P^1$. Then, a 1-form valued $\mathcal{O}$-linear vector bundle morphism θ is called a Higgs field:

$$\theta : E \to E \otimes \Omega^{1,0},$$

Moreover, we consider partial $(0,1)$-connections $\overline{\partial}_E$ on E over $\mathbb{C}P^1$. Together with a Hermitian metric h on E, the basic objects of our investigation will be Higgs bundles $(E, \theta, \overline{\partial}_E)$, which satisfy Hitchin's equations:

$$\begin{cases} \overline{\partial}_E \theta = 0 \\ F_h + [\theta, \theta^{\dagger_h}] = 0 \end{cases}$$

where F_h denotes the curvature form of the Chern connection ∇_h^+, associated with $\overline{\partial}_E$ and h, and $\theta^{\dagger_h}$ denotes the adjoint of the Higgs field with respect to h (i.e. $\theta^{\dagger_h} : E \to E \otimes \Omega^{0,1}$). If the Hitchin's equations are satisfied, then the bundle is called harmonic, and h is called Hermitian-Einstein metric. We also get a holomorphic structure $\overline{\partial}_{\mathrm{Det}E}$ on the complex line bundle $\mathrm{Det}E$, induced by $\overline{\partial}_E$, and a Hermitian metric $h_{\mathrm{Det}E}$ on it, induced by h. Now denote by

$\mathcal{E}$ the holomorphic vector bundle $(E, \overline{\partial}_E)$ on $\mathbb{C}P^1 \setminus D$. The holomorphic vector bundle $\mathcal{E}$ is also defined over D, and we assume that θ has logarithmic singularities at the points of D, that is we are considering logarithmic Higgs bundles.

Let us define the parabolic structure of Higgs bundles, based on [33], [4] and [30]. Fix a weight vector for all $p \in D$: $\underline{\alpha}_P = (\alpha_P^1, \alpha_P^2, \alpha_P^3)$, where α_P^j's lie in a unit interval for all $j = 1, 2, 3$, and $\alpha_P^1 < \alpha_P^2 < \alpha_P^3$. Also consider the filtration on the fiber of $\mathcal{E}$ over each point of D:

$$0 = l_P^3 \subset l_P^2 \subset l_P^1 \subset l_P^0 = \mathcal{E}\big|_P,$$

We always assume that the Higgs field is weakly parabolic, meaning that $\theta : l_P^i \to l_P^i \otimes K(D)$ at each $P \in D$. The Higgs field is called strongly parabolic if $\theta : l_P^i \to l_P^{i+1} \otimes K(D)$ at each $P \in D$, i.e. the residue is nilpotent with respect to the filtration in that the action on the graded pieces $\mathrm{gr}_l^{\cdot}\mathcal{E}$ of $l_P^{\bullet}$ is trivial.

As usual, under stability of a Higgs bundle $(E, \theta, \overline{\partial}_E)$ we mean that for any proper holomorphic subbundle $F \subset E$ which satisfies $\theta : F \to F \otimes K(D)$, the inequality $\mu(F) < \mu(E)$ holds, where $\mu(E) = \frac{\deg E}{\mathrm{rank} E}$ is the slope of the bundle ($\mu(F)$ defined similarly). In the parabolic setting, we speak about α-stability, which depends on the weight vectors $\underline{\alpha}_P$, and means that for all F satisfying the above conditions

$$\frac{\mathrm{pdeg}_{\underline{\alpha}} E}{\mathrm{rank} E} > \frac{\mathrm{pdeg}_{\underline{\alpha}} F}{\mathrm{rank} F},$$

where the parabolic degree of the parabolic bundle (and subbundle) is:

$$\mathrm{pdeg}_{\underline{\alpha}} E = \deg E + \sum_{P \in D} \sum_{j=1}^{3} \alpha_P^j \tag{2}$$

$$\mathrm{pdeg}_{\underline{\alpha}} F = \deg F + \sum_{P \in D} \sum_{j=1}^{3} \alpha_P^j \cdot \dim((F\big|_P \cap l_P^{j-1})/(F\big|_P \cap l_P^j)) \tag{3}$$

The Higgs bundle is called α-polystable, if it is the direct sum of lower rank α-stable Higgs bundles, with the same parabolic slope as $(E, \theta, \overline{\partial}_E)$. By the results of Hitchin [19] and Simpson [41], it is known that a Higgs bundle admits a unique Hermitian-Einstein metric h with $\mathrm{Det} h = h_{\mathrm{Det} E}$ if and only if it is polystable.

Let $\mathrm{SL}(E)$ be the principal bundle of automorphisms of E which induce the identity on $\mathrm{Det}E$. Then the group of complex gauge transformations, denoted by $\mathcal{G}$, is the group of sections of $\mathrm{SL}(E)$. Moreover its Lie algebra consists of the sections of $\mathfrak{sl}(E)$, the vector bundle of traceless endomorphisms of E. The gauge group $\mathcal{G}$ acts on the Higgs bundles via

$$g.(E, \theta, \overline{\partial}_E) = (E, g^{-1}\theta g, g^{-1}\overline{\partial}_E g), \quad \forall g \in \mathcal{G}.$$

Definition 2.1. *The moduli space of harmonic, α-stable, strongly parabolic, meromorphic $SL(3,\mathbb{C})$-Higgs bundles, with at most logarithmic singularities, with given weight vectors $\underline{\alpha}_P$, up to the complex gauge action, is called the Dolbeault moduli space, denoted by $\mathcal{M}_{Dol}(\alpha)$.*

See [25] for a differential geometric construction of this space, and [38] for an algebraic geometric one.

In case of logarithmic Higgs bundles, for θ and h the so called tameness condition is satisfied, that is at each $P \in D$, h admits a lift along any ray to P, which grows at most polynomially in the standard metric. (Here we consider h as an equivariant harmonic map from the universal cover of the Riemann surface to the Hermitian symmetric space $\mathrm{GL}(3,\mathbb{C})/\mathrm{U}(3)$). For such a tame, harmonic bundle $(E, \theta, \overline{\partial}_E, h)$ the connection

$$\nabla = \nabla_h^+ + \theta + \theta^{\dagger_h} \tag{4}$$

is integrable, and $\nabla^{1,0}$ has regular singularities. Fix again for all $P \in D$ some $\underline{\beta}_P = (\beta_P^1, \beta_P^2, \beta_P^3)$ parabolic weight vectors and the $\underline{\tau}_P = (\tau_P^1, \tau_P^2, \tau_P^3)$ eigenvalues of the residue of the connection. The definition of β-stability and parabolic structure of the integrable connection is just the same as for the Higgs field (the parabolic structure of the underlying vector bundle is already given, see also [30]). We again require that (4) is compatible with the filtration, that is $(\mathrm{res}_P\nabla - \tau_P^j \mathrm{id})(l_P^j) \subset l_P^{j+1}$, for all $P \in D$ and $j = 0, 1, 2$. If the eigenvalues τ_P^j are pairwise different that this implies that $\mathrm{res}_P\nabla$ is diagonal with respect to some basis compatible with the filtration. The complex gauge group action on the space of connections is also inherited from the action on the space of Higgs bundles.

Definition 2.2. *The moduli space of β-stable, parabolic, integrable $SL(3,\mathbb{C})$-connections, with regular singularities at the punctures, with given weight vectors $\underline{\beta}_P$ and given residues $\underline{\tau}_P$ at each $P \in D$, up to the complex gauge action, is called the de Rham moduli space, denoted by $\mathcal{M}_{dR}(\beta, \tau)$.*

The third main object of our research is the Betti moduli space, also known as character variety. Under the stability condition, (4) is an irreducible integrable connection. For any choice of base point $x_0 \notin D$, analytic continuation of solutions provides a representation

$$\rho : \pi_1(\mathbb{C}P^1 \setminus D, x_0) \to \mathrm{SL}(3, \mathbb{C})$$

that is well-defined up to simultaneous conjugation by elements of $\mathrm{PGL}(3, \mathbb{C})$ (corresponding to different choices of a basis of solutions at x_0). The eigenvalues, denoted by $\underline{c}_P = (c_P^1, c_P^2, c_P^3)$, of the local monodromy around $P \in D$ are determined by $(\underline{\beta}_P, \underline{\tau}_P)$. (As a matter of fact, the local system admits a filtration and corresponding weights too, but we will not need this extra structure here.)

Definition 2.3. *The moduli space of the above described representations is called the Betti moduli space or character variety, denoted by $\mathcal{M}_B(\boldsymbol{c})$.*

It is known that the Betti space is a smooth, affine algebraic variety for generic parameters. There exists a compactification of the Betti space by a simple normal crossing divisor D_B (see the results of Nagata and Hironaka [36], [18]). In our case D_B is a complex curve. As customary, we define its dual complex $\mathcal{D}D_B$ as the simplicial complex whose vertices are the irreducible components of D_B, and whose edges corresponds to the intersections of the components. We want to apply this to the compactification of the Betti moduli space, therefore the resulting simplicial complex will be called dual boundary complex, denoted by $\mathcal{D}\partial\mathcal{M}_B(\mathbf{c})$.

It is known from Simpson [42]that there is a connection between the above defined parameters. With the eigenvalues of the residues of the Higgs-field being equal to 0, it simplifies to

$$\alpha_P^i = \beta_P^i = \tau_P^i, \text{ and } c_P^i = e^{-2\pi\sqrt{-1}\alpha_P^i}, \forall P \in D, \forall i \in \{1, 2, 3\}.$$

Moreover, the following theorem holds.

Theorem 2.4. *Assume that the parabolic degree of $\mathcal{E}$ is 0.*

1. *[3] The spaces $\mathcal{M}_{Dol}(\alpha)$, $\mathcal{M}_{dR}(\beta, \tau)$ and $\mathcal{M}_B(\boldsymbol{c})$ are $\mathbb{C}$-analytic manifolds, and there exists a diffeomorphism*

$$NAHC : \mathcal{M}_{Dol}(\alpha) \to \mathcal{M}_{dR}(\beta, \tau)$$

called the non-abelian Hodge correspondence.

2. *[20, Theorem 7.1] There exists a complex bianalytic isomorphism*

$$RH : \mathcal{M}_{dR}(\beta,\tau) \to \mathcal{M}_B(\boldsymbol{c}),$$

called the Riemann–Hilbert correspondence.

2.2 Choice of parameters

Now let us choose the parameters introduced in the previous subsection, and explain these choices. For all $P \in D$, set

$$\alpha_P^1 = \beta_P^1 = \tau_P^1 = -\frac{1}{3}$$
$$\alpha_P^2 = \beta_P^2 = \tau_P^2 = 0$$
$$\alpha_P^3 = \beta_P^3 = \tau_P^3 = \frac{1}{3}$$

Consequently

$$c_P^2 = 1, \;\; c_P^3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \varepsilon, \;\; c_P^1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \varepsilon^2,$$

where $i = \sqrt{-1}$ and ε stands for a primitive cubic root of unity.

Lemma 2.5. *With these parameter values*

i) the parabolic degree of $\mathcal{E}$ is zero,

ii) the traces of θ and θ^2 are identically zero.

Proof. i) Since at all $P \in D$, the sum of the parabolic weights equal to 0, it follows from equation (2), that the parabolic degree of $\mathcal{E}$ is 0.

ii) The residues of the Higgs field are traceless at all $P \in D$.

On $\mathbb{C}P^1$ we have for the sheaf of holomorphic 1-forms $K \cong \mathcal{O}(-2)$, and with the divisor $D = 0 + 1 + \infty$, we have $K(D) \cong \mathcal{O}(1)$, via the identification $\frac{dz}{z(z-1)} \leftrightarrow 1$. Then

$$\mathrm{Tr}\theta \in H^0(\mathbb{C}P^1, K(D)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(1)) \cong \mathbb{C}^2$$
$$\mathrm{Tr}\theta^2 \in H^0(\mathbb{C}P^1, K(D)^{\otimes 2}) \cong H^0(\mathbb{C}P^1, \mathcal{O}(2)) \cong \mathbb{C}^3,$$

We have three independent vanishing conditions for both $\mathrm{Tr}\theta$ and $\mathrm{Tr}\theta^2$ at the points $P \in D$, therefore both must be zero globally. (One condition is even redundant in the case of $\mathrm{Tr}\theta$.)

$\square$

2.3 The Hitchin fibration

The characteristic coefficients of a Higgs bundle of rank 3 over $\mathbb{C}P^1$ with logarithmic singularities at D belong to the vector space

$$\begin{gathered}\mathcal{B} = H^0(\mathbb{C}P^1, K(D)) \oplus H^0(\mathbb{C}P^1, K(D)^{\otimes 2}) \oplus H^0(\mathbb{C}P^1, K(D)^{\otimes 3}) \cong \\ \cong H^0(\mathbb{C}P^1, \mathcal{O}(1)) \oplus H^0(\mathbb{C}P^1, \mathcal{O}(2)) \oplus H^0(\mathbb{C}P^1, \mathcal{O}(3)) \cong \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4\end{gathered}$$

According to Lemma 2.5, with the choices made in Section 2.2, the first two components vanish. The third characteristic coefficient of θ is $\mathrm{Det}\theta \in H^0(\mathbb{C}P^1, K(D)^{\otimes 3})$. Let us use the notation $L = K(D)$, with the natural projection from the total space of L, $p_L : \mathrm{Tot}L \to \mathbb{C}P^1$. Define $\zeta \frac{dz}{z(z-1)}$ to be the canonical section of $p_L^* L$ over $p_L^{-1}(\mathbb{C})$, i.e. away from the infinity section. With this, the characteristic polynomial of θ is

$$\mathrm{Det}(\zeta \mathrm{id}_{\mathcal{E}} - \theta) = \zeta^3 \mathrm{id}_{\mathcal{E}}^{\otimes 3} + H_\theta$$

where H_θ lies in the last direct summand of $\mathcal{B}$. The third coefficient $\mathrm{Det}\theta = H_\theta$ has 4 parameters of freedom, but with the three independent vanishing relations at $P \in D$, this reduces to a one-parameter family. One can see [30, Appendix A], that it has the form

$$\mathrm{Det}\theta = H_\theta = (tz(z-1) + p_2(z)) \frac{dz^{\otimes 3}}{z^3(z-1)^3}$$

where $p_2(z) = az^2 + bz + c$ is a quadratic polynomial, and a, b, c, t are the 4 parameters, from which a, b, c are fixed to be 0, and $t \in \mathbb{C}$ is the only free parameter. The 1-dimensional subspace

$$\mathcal{H} = \left\{ t \frac{dz^{\otimes 3}}{z^2(z-1)^2} : t \in \mathbb{C} \right\} \subset \mathcal{B}$$

where $\mathrm{Det}\theta$ may take its values is called the Hitchin base of $\mathcal{M}_{\mathrm{Dol}}(\alpha)$, and the map

$$H : \mathcal{M}_{\mathrm{Dol}}(\alpha) \to \mathcal{H}, \quad (\mathcal{E}, \theta) \mapsto \mathrm{Det}\theta$$

is called the Hitchin fibration. Clearly, for any $\tau \in \mathbb{C}^\times$ and $(\mathcal{E}, \theta)$ we have

$$H((\mathcal{E}, \tau\theta)) = \tau^3 H((\mathcal{E}, \theta)).$$

For every $t \in \mathbb{C}$ the smooth curve called the spectral curve is defined via

$$\Sigma_t = \{(z, \zeta) | \zeta^3 + tz(z-1) = 0\} \subset \mathrm{Tot}L. \tag{5}$$

Notice that Σ_t has maximal ramification over $0, 1$ and a smooth compactification, denoted by $\widetilde{\Sigma}_t$, in Tot$L$ at $z = \infty$ with $(w, \zeta) = (0, 0)$, where it also admits cyclic ramification. One can easily compute from Riemann–Hurwitz formula that its genus is equal to 1.

2.4 Ramification of the spectral curve and the Jacobian

The equation in (5) defining $\widetilde{\Sigma}_t$ has three roots over every point of $\mathbb{C}P^1 \setminus D$:

$$\xi_1 = R^{1/3} e^{i\varphi/3} z^{1/3} (z-1)^{1/3}, \quad \xi_2 = \varepsilon R^{1/3} e^{i\varphi/3} z^{1/3} (z-1)^{1/3},$$
$$\xi_3 = \varepsilon^2 R^{1/3} e^{i\varphi/3} z^{1/3} (z-1)^{1/3},$$

where we recall that ε is a cubic root of unity and we switched to polar coordinates $t = Re^{i\varphi}$. That is, $\xi_{1,2,3}$ is a three-valued holomorphic function on $\mathbb{C}P^1 \setminus D$, and p_L induces a projection map $p_{R,\varphi} : \widetilde{\Sigma}_{R,\varphi} \to \mathbb{C}P^1$. This is indeed a ramified triple cover over $\mathbb{C}P^1$, with ramification points $z = 0, z = 1$ on chart z, and $w = 0$ on chart w, independently of the value of $t > 0$. Thus we introduce the ramification divisor $\Delta = \{0, 1, \infty\} = D$, and denote the lift of the ramification divisor by $\widetilde{\Delta}$ (or $\widetilde{D}$), whose points are the branch points on $\widetilde{\Sigma}_{R,\varphi}$. We summarize properties of the spectral curve.

Proposition 2.6. *$\widetilde{\Sigma}_{R,\varphi}$ is a smooth genus 1 curve, with ramification index at all 3 points of D equal to 3, i.e. all its ramifications are cyclic.*

Let us use the notation for the cubic meromorphic differentials, with double poles at the punctures

$$q_{R,\varphi} := \mathrm{Det}\theta_{R,\varphi} = Re^{i\varphi} \frac{dz^{\otimes 3}}{z^2(z-1)^2} \in H^0(\mathbb{C}P^1, K^{\otimes 3}(2D)). \tag{6}$$

Then the sheets of $\widetilde{\Sigma}_{R,\varphi}$ are just the cubic roots of $q_{R,\varphi}$:

$$Q_{1,R,\varphi} = R^{1/3} e^{i\varphi/3} \frac{dz}{z^{2/3}(z-1)^{2/3}}, \quad Q_{2,R,\varphi} = \varepsilon R^{1/3} e^{i\varphi/3} \frac{dz}{z^{2/3}(z-1)^{2/3}},$$
$$Q_{3,R,\varphi} = \varepsilon^2 R^{1/3} e^{i\varphi/3} \frac{dz}{z^{2/3}(z-1)^{2/3}}, \tag{7}$$

According to [1], the three corresponding eigenspaces determine a line bundle $\mathcal{L}_{R,\varphi} \to \widetilde{\Sigma}_{R,\varphi}$, whose pushforward $(p_{R,\varphi})_*\mathcal{L}_{R,\varphi}$ is isomorphic to $\mathcal{E}$. The degree of $\mathcal{L}_{R,\varphi}$ can be computed via [29, 2.3.3] or [44, Theorem 5.4]:

$$\deg\mathcal{L} = \deg\mathcal{E} + r(1-r)\left(1 - g - \frac{n}{2}\right) = 3,$$

where $r = 3$ is the rank, $n = 3$ is the number of ramification points, and $g = 0$ is the genus of $\mathbb{C}P^1$. Thus an element of the Dolbeault moduli space determines a spectral curve and a line bundle of degree 3 on it, and vica versa. So, the fiber of the Hitchin fibration over the point parameterized by a fixed $t \in \mathcal{H}$ is $\mathrm{Pic}^3(\widetilde{\Sigma}_t)$, that is a 2-torus, namely a torsor over $\mathrm{Jac}(\widetilde{\Sigma}_t)$. In particular, the Hitchin fibration is an elliptic fibration.

Let us discuss this correspondence between the Hitchin fibers and the Jacobian of the spectral curve a bit more in detail. Consider the following period lattice $\Lambda_t \subset H^{0,1}(\widetilde{\Sigma}_t) \cong \mathbb{C}$, provided by the image $\mathrm{Im}(p^{0,1} \circ \iota)$, where

$$\iota : H^1(\widetilde{\Sigma}_t, 2\pi i\mathbb{Z}) \to H^1(\widetilde{\Sigma}_t, \mathbb{C})$$
$$p^{0,1} : H^1(\widetilde{\Sigma}_t, \mathbb{C}) \to H^{0,1}(\widetilde{\Sigma}_t)$$

are the coefficient inclusion on the first cohomology class, and the projection of harmonic forms to their antiholomorphic part respectively. There exists a $\mathbb{C}$-analytic isomorphism $\mathrm{Jac}(\widetilde{\Sigma}_t) \cong H^{0,1}(\widetilde{\Sigma}_t)/\Lambda_t$. Namely, any class in $H^{0,1}(\widetilde{\Sigma}_t)/\Lambda_t$ can be represented by a $\mu \in \Omega^{0,1}(\widetilde{\Sigma}_t)$, because of the abelian Hodge correspondence. Then the connection form $B = \mu - \overline{\mu} = 2i\mathrm{Im}\mu \in \Omega^1(\widetilde{\Sigma}_t)$ defines a flat $U(1)$-connection, and the abelian version of Theorem 2.4 can be expressed as an isomorphism between the Jacobian and the 2-torus:

$$\mathrm{Jac}(\widetilde{\Sigma}_t) \to T^2 = S^1 \times S^1, \quad \mu \mapsto (e^{\int_X B}, e^{\int_Y B}),$$

where X, Y are fixed 1-cycles generating $H_1(\widetilde{\Sigma}_t, \mathbb{Z})$. See [14, Section 4] for more details.

2.5 The Hitchin section

There is a preferred line bundle $\mathcal{L}_0$ over $\widetilde{\Sigma}_t$ giving rise to a section of H analogous to the Hitchin section. Namely, as it is well-known,

$$(p_L)_*\mathcal{O}_{\widetilde{\Sigma}_t} \cong \mathcal{O}_{\mathbb{C}P^1} \oplus K_{\mathbb{C}P^1}(D)^{-1} \oplus K_{\mathbb{C}P^1}(D)^{-2} \cong \mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(-2),$$

the direct summands being generated by $1, \zeta, \zeta^2$ respectively. The preferred choice of spectral sheaf is then $\mathcal{L}_0 = p_L^* L \otimes \mathcal{O}_{\widetilde{\Sigma}_t}$. Notice that $\mathcal{O}_{\widetilde{\Sigma}_t} \cong K_{\widetilde{\Sigma}_t}$, because $\widetilde{\Sigma}_t$ is an elliptic curve. Moreover, from the local form at $z = 0$ of the equation defining $\widetilde{\Sigma}_t$ we get

$$\frac{\mathrm{d}z}{z} = h(w) \frac{\mathrm{d}w}{w}$$

for some local holomorphic function $h(w)$ with $h(0) \neq 0$, i.e.

$$p_L^* K_{\mathbb{C}P^1}(D) \otimes \mathcal{O}_{\widetilde{\Sigma}_t} = K_{\widetilde{\Sigma}_t}(\widetilde{D}).$$

We infer

$$\mathcal{L}_0 = p_L^* L \otimes \mathcal{O}_{\widetilde{\Sigma}_t} \cong K_{\widetilde{\Sigma}_t}(\widetilde{D}) \cong \mathcal{O}_{\widetilde{\Sigma}_t}(\widetilde{D}).$$

By the projection formula we then have

$$\mathcal{E}_0 = (p_L)_* \mathcal{L}_0 \cong K(D) \oplus \mathcal{O} \oplus K(D)^{-1},$$

in particular, the degree of $\mathcal{E}_0$ is equal to zero, as required. Stable Higgs fields

$$\theta_t : K(D) \oplus \mathcal{O} \oplus K(D)^{-1} \to K(D)^2 \oplus K(D) \oplus \mathcal{O} \tag{8}$$

over $\mathcal{E}_0$ are of the form

$$\theta_t = \begin{pmatrix} 0 & 0 & q_t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where q_t is given in (6) (and recall that R, φ are polar coordinates of $t \in \mathbb{C}^*$). Applying a constant (i.e., depending only on t) gauge transformation

$$\begin{pmatrix} t^{-\frac{1}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{\frac{1}{3}} \end{pmatrix}$$

the Higgs field gets transformed into

$$\theta_t = t^{\frac{1}{3}} \begin{pmatrix} 0 & 0 & q_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{9}$$

The strongly parabolic condition (namely, that the residues of θ_t at the points of D are nilpotent) then implies

$$\mathrm{res}_P(\theta_t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, there exists a unique filtration of $\mathcal{E}_0$ over the points of D compatible with this residue, namely

$$l_P^2 = \mathbb{C} \cdot \zeta^2, \quad l_P^1 = \mathbb{C} \cdot \zeta \oplus \mathbb{C} \cdot \zeta^2$$

The parabolic weights corresponding to the generators $1, \zeta, \zeta^2$ are respectively $-\frac{1}{3}, 0, \frac{1}{3}$. See also the results from [11], [16].

3 Description of the Betti moduli space

In this chapter we consider the Betti space (or character variety), parameterizing the irreducible representations of the fundamental group of $\mathbb{C}P^1 \setminus D$ in $\mathrm{GL}(3, \mathbb{C})$, up to the simultaneous conjugation by of $\mathrm{PGL}(3, \mathbb{C})$. We pick $z_0 \in \mathbb{C}P^1 \setminus \{0, 1, \infty\}$ once and for all, and all occurrences of fundamental group will mean with base point z_0. Since the fundamental group of $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ is isomorphic to the free group generated by two elements, this amounts to considering maps

$$\rho : \pi_1 \left(\mathbb{C}P^1 \setminus \{0, 1, \infty\} \right) \cong \langle a, b \rangle \to \mathrm{GL}(3, \mathbb{C})$$

under the constraint that the eigenvalues of $\rho(a), \rho(b)$ and $\rho(ab)$ are fixed: $\{\lambda_1, \lambda_2, \lambda_3\}$, $\{\mu_1, \mu_2, \mu_3\}$ and $\{\nu_1, \nu_2, \nu_3\}$ respectively (previously denoted by the vectors $\underline{c}_P$). Because of the $\mathrm{PGL}(3, \mathbb{C})$ action, we have the freedom to choose $\rho(a)$ to be diagonal with elements $\{\lambda_1, \lambda_2, \lambda_3\}$. Having achieved this, there remains the action of the maximal torus $(\mathbb{C}^\times)^2$ of $\mathrm{PGL}(3, \mathbb{C})$. The action of $(t_1, t_2) \in (\mathbb{C}^\times)^2$ on a matrix $B = [b_{ij}]$ is the standard conjugation

$$(t_1, t_2).B = \begin{pmatrix} b_{11} & t_1 b_{12} & t_1 t_2 b_{13} \\ t_1^{-1} b_{21} & b_{22} & t_2 b_{23} \\ t_1^{-1} t_2^{-1} b_{31} & t_2^{-1} b_{32} & b_{33} \end{pmatrix}.$$

With the notations $\rho(a) = A, \rho(b) = B$, the constraints on the eigenvalues are equivalent to constraints on the traces of A^j, B^j and $(AB)^j$ for $j = 1, 2, 3$. Thus the Betti space can be written as

$$\mathcal{M}_B = \{A, B \in \mathrm{GL}(3,\mathbb{C}) | A = \mathrm{Diag}[\lambda_1, \lambda_2, \lambda_3], B = [b_{i,j}],$$
$$\mathrm{Tr}(B^j) = \sigma_j(\underline{\mu}), \mathrm{Tr}((AB)^j) = \sigma_j(\underline{\nu}), j = 1, 2, 3\} / \left(\mathbb{C}^\times\right)^2$$

where σ_j is the degree j homogeneous symmetric polynomial in 3 variables. Because of the irreducibility of the representations, we are given that both b_{21} and b_{31} can not vanish simultaneously. Possibly passing to a Zariski open subset (that does not alter validity of our arguments), we may assume that they are both nonzero. We may then use the $(\mathbb{C}^\times)^2$ action to remove these coefficients. Since the assumption on $(AB)^3$ is redundant, this gives

$$\mathcal{M}_B = \{B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & b_{22} & b_{23} \\ 1 & b_{32} & b_{33} \end{pmatrix} | \mathrm{Tr}(B^j) = \sigma_j(\underline{\mu}), j = 1, 2, 3,$$
$$\mathrm{Tr}((AB)^j) = \sigma_j(\underline{\nu}), j = 1, 2\}.$$

Now, the conditions $\mathrm{Tr}(B) = b_{11} + b_{22} + b_{33} = \sigma_1(\underline{\mu})$, and $\mathrm{Tr}(AB) = \lambda_1 b_{11} + \lambda_2 b_{22} + \lambda_3 b_{33} = \sigma_1(\underline{\nu})$ can be used to express b_{22} and b_{33} in terms of b_{11}:

$$b_{22} = \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_3} b_{11} + c_1(\underline{\lambda}, \underline{\mu}, \underline{\nu}) =: Q(b_{11})$$
$$b_{33} = \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} b_{11} + c_2(\underline{\lambda}, \underline{\mu}, \underline{\nu}) =: P(b_{11}),$$

where c_1, c_2 are constants depending only on $\underline{\lambda}, \underline{\mu}, \underline{\nu}$, while P and Q are degree 1 polynomials in b_{11}. Switching to the notation

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & b_{22} & b_{23} \\ 1 & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} X & Y & Z \\ 1 & Q(X) & V \\ 1 & W & P(X) \end{pmatrix}$$

implies

$$\mathcal{M}_B = \{(X, Y, Z, V, W) \in \mathbb{C}^5 | \mathrm{Tr}(B^2) = \sigma_2(\underline{\mu}), \mathrm{Tr}(B^3) = \sigma_3(\underline{\mu}), \mathrm{Tr}((AB)^2) = \sigma_2(\underline{\nu})\}.$$

The remaining three conditions read as:

$$\mathrm{Tr}(B^2) = X^2 + 2Y + 2Z + Q^2(X) + P^2(X) + 2VW = \sigma_2(\underline{\mu})$$
$$\mathrm{Tr}((AB)^2) = \lambda_1^2 X^2 + 2\lambda_1\lambda_2 Y + 2\lambda_1\lambda_3 Z + \lambda_2^2 Q^2(X) + \lambda_3^2 P^2(X) + 2\lambda_2\lambda_3 VW = \sigma_2(\underline{\nu})$$
$$\mathrm{Tr}(B^3) = X^3 + P^3(X) + Q^3(X) + 3ZW + 3XZ + 3YV + 3XY$$
$$+3Q(X)VW + 3Q(X)Y + 3P(X)VW + 3P(X)Z = \sigma_3(\underline{\mu})$$

Now, the first two equations allow us to express Y and Z, and eliminate these variables from the third equation. We thus obtain a description of the Betti space as a cubic surface in $\operatorname{Spec}\mathbb{C}[X,V,W]$. Then, we can consider the homogenisation of the resulting equation. This procedure provides us the compatifying curve of $\mathcal{M}_B$ as a homogeneous cubic curve in $\mathbb{C}P^2$, with equation

$$\begin{aligned}\frac{-3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)^2(\lambda_2+\lambda_3)}{\lambda_1(\lambda_2-\lambda_3)^4}X^3+\frac{3(\lambda_1-\lambda_3)^2(\lambda_1\lambda_3-\lambda_2^2)}{\lambda_1(\lambda_2-\lambda_3)^3}X^2V \\ +\frac{3(\lambda_1-\lambda_2)^2(\lambda_3^2-\lambda_1\lambda_2)}{\lambda_1(\lambda_2-\lambda_3)^3}X^2W+\frac{-3(\lambda_1(\lambda_2-\lambda_3)^2+\lambda_2(\lambda_1-\lambda_3)^2+\lambda_3(\lambda_1-\lambda_2)^2)}{\lambda_1(\lambda_2-\lambda_3)^2}XVW \\ +\frac{3\lambda_2(\lambda_1-\lambda_3)}{\lambda_1(\lambda_3-\lambda_2)}VW^2+\frac{3\lambda_3(\lambda_1-\lambda_2)}{\lambda_1(\lambda_2-\lambda_3)}V^2W=0\end{aligned} \tag{10}$$

3.1 Topology of the compactifying divisor

Although we have made special choices for the parameters $\underline{\lambda},\underline{\mu},\underline{\nu}$, here we will see that up to homeomorphism, the compatifying curve is the same as with general choices. Namely, it is the genus 0 curve with nodal singularity, also called fishtail, and denoted by I_1 in Kodaira's list of singular elliptic curves [23].

Proposition 3.1. *The curve C determined by equation (10) in $\mathbb{C}P^2=\{[X:V:W]\}$, is of type I_1.*

Proof. The proof has a similar idea as Proposition 2.3.3 in [15]. Consider the pencil of projective lines passing through the point $P=\left[\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}:\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}:1\right]$ in $\mathbb{C}P^2$, and parameterize them by $[t_0:t_1]\in\mathbb{C}P^1$, via

$$L_{[t_0:t_1]}=\{[X:V:W]\in\mathbb{C}P^2|t_0\left(X-\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}\right)=t_1\left(V-\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}\right)\}.$$

Determine the intersection points of line $L_{[t_0:t_1]}$ with C: if $[t_0:t_1]=[1:0]$, then $X = \frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}$, and substituting this into (10):

$$\frac{-(\lambda_1-\lambda_2)^2(\lambda_2+\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)} + (\lambda_1\lambda_3-\lambda_2^2)V + \frac{(\lambda_1-\lambda_2)^2(\lambda_3^2-\lambda_1\lambda_2)}{(\lambda_3-\lambda_1)^2(\lambda_2-\lambda_3)}W$$
$$+\frac{-(\lambda_1(\lambda_2-\lambda_3)^2+\lambda_2(\lambda_1-\lambda_3)^2+\lambda_3(\lambda_1-\lambda_2)^2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}VW$$
$$+\frac{\lambda_2(\lambda_1-\lambda_3)}{(\lambda_3-\lambda_2)}VW^2 + \frac{\lambda_3(\lambda_1-\lambda_2)}{(\lambda_2-\lambda_3)}V^2W = 0$$

Here, if $W=0$, then $(\lambda_1\lambda_3-\lambda_2^2)V = \frac{(\lambda_1-\lambda_2)^2(\lambda_2+\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}$ has a unique solution for V. If $W \neq 0$, then we can choose $W=1$, and the equation simplifies:

$$V^2 - \frac{2(\lambda_1-\lambda_2)}{\lambda_3-\lambda_1}V + \frac{(\lambda_1-\lambda_2)^2}{(\lambda_3-\lambda_1)^2} = 0,$$

which is a complete square, therefore has a unique solution for V. That is, $L_{[1:0]}$ and C have two intersection points:

$$P = \left[\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1} : \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1} : 1\right], \quad Q = \left[\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1} : \frac{(\lambda_1-\lambda_2)^2(\lambda_2+\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)(\lambda_1\lambda_3-\lambda_2^2)} : 0\right]$$

The other case, if $t_1 \neq 0$ (assume $t_1 = 1$), then $V = t_0\left(X - \frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}\right) + \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}$. If $W \neq 0$ ($W=1$), then substitute the above equation for V into (10)

$$\frac{1}{\lambda_2-\lambda_3}\left(X-\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}\right)^2\left(X\left(\frac{(\lambda_1-\lambda_3)^2(\lambda_1\lambda_3-\lambda_2^2)}{(\lambda_2-\lambda_3)^2}t_0\right.\right.$$
$$\left.+\frac{-(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)^3)}\right) + \lambda_3(\lambda_1-\lambda_2)t_0^2$$
$$+\frac{-\lambda_1^2\lambda_2-2\lambda_1^2\lambda_3+6\lambda_1\lambda_2\lambda_3-2\lambda_2^2\lambda_3-\lambda_2\lambda_3^2}{\lambda_2-\lambda_3}t_0$$
$$\left.+\frac{(\lambda_1-\lambda_2)(\lambda_1^2\lambda_2+2\lambda_1^2\lambda_3-4\lambda_1\lambda_2\lambda_3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2)}{(\lambda_2-\lambda_3)^2}\right) = 0$$

This has solution $X = \frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}$, which provides P. The remaining factor is linear in X, so for fixed t_0, it has a unique solution for X, except if the coefficient of X is 0, i.e.: $t_0' = \frac{(\lambda_1-\lambda_2)^2(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1\lambda_3-\lambda_2^2)}$, this case the only intersection point of C and $L_{[t_0:t_1]}$ is P. In other cases, there are one more intersection point besides P, except when the root of the linear factor provides P again. This happens if the following quadratic equation satisfies for t_0:

$$\lambda_3(\lambda_1-\lambda_2)t_0^2+\frac{(\lambda_1-\lambda_2)(-\lambda_1\lambda_2-3\lambda_1\lambda_3+3\lambda_2\lambda_3+\lambda_3^2)}{\lambda_2-\lambda_3}t_0$$
$$+\frac{(\lambda_1-\lambda_2)(2\lambda_1^2\lambda_2+2\lambda_1^2\lambda_3-\lambda_1\lambda_2^2-6\lambda_1\lambda_2\lambda_3-\lambda_1\lambda_3^2+2\lambda_2^2\lambda_3+2\lambda_2\lambda_3^2)}{(\lambda_2-\lambda_3)^2}=0$$

This has solutions $t_0^+=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_2-\lambda_3}$ and $t_0^-=\frac{\lambda_1\lambda_2+\lambda_1\lambda_3-2\lambda_2\lambda_3}{\lambda_3(\lambda_2-\lambda_3)}$. One can check that for t_0^+, the line $L_{[t_0^+:1]}$ passes through $[0:1:0]$, so it has two intersection points with C. But C has only two common points with the line $W=0$, namely $[0:1:0]$ and Q, thus $L_{[t_0^-:1]}$ intersects C only at P. We deduce that there are exactly two parameters $[t_0':1]$ and $[t_0^+:1]$ for which $L_{[t_0:t_1]}$ has only one intersection point with C, namely P. This means that the map $\mathbb{C}P^1\to C$ sending $[t_0:t_1]$ to the other intersection of $L_{[t_0:t_1]}$ and C (besides P) is one-to-one, except for $[t_0':1]$ and $[t_0^+:1]$. That is, C is homeomorphic with a $\mathbb{C}P^1$ with $[t_0':1]$ and $[t_0^+:1]$ identified. □

3.2 The trace coordinates

We will use the so called trace coordinates on the Betti moduli space, introduced by Lawton [26], [27]. Let ρ be at the SL(3,$\mathbb{C}$) character variety of a rank 2 free group $\langle a,b\rangle$, and consider the character map $\mathcal{M}_B\to\mathbb{C}^9$:

$$\begin{aligned}\rho\mapsto\big(&\mathrm{Tr}(\rho(a)),\mathrm{Tr}(\rho(b)),\mathrm{Tr}(\rho(a)\rho(b)),\mathrm{Tr}(\rho(a)^{-1}),\mathrm{Tr}(\rho(b)^{-1}),\mathrm{Tr}((\rho(a)\rho(b))^{-1}),\\ &\mathrm{Tr}(\rho(a)\rho(b)^{-1}),\mathrm{Tr}(\rho(a)^{-1}\rho(b)),\mathrm{Tr}(\rho(a)\rho(b)\rho(a)^{-1}\rho(b)^{-1})\big)=:(x_1,x_2,...,x_9).\end{aligned} \tag{11}$$

This way the above map gives coordinates on $\mathcal{M}_B$, under the condition $x_9^2-p(\underline{x})x_9+q(\underline{x})=0$, where p and q are two polynomials in the variables $\{x_i\}_{i=1}^9$, see [26, Section 4]. On the three-punctured sphere, where $\gamma_1,\gamma_2,\gamma_3$ are simple loops around the punctures $(0,1,\infty)$, with a common base point, $[\gamma_1]$ and $[\gamma_2]$ generate the fundamental group of the curve, while $[\gamma_3]=([\gamma_1][\gamma_2])^{-1}$. With our special choice of eigenvalues for $\rho([\gamma_1]),\rho([\gamma_2])$, described in Section 2.2, we have the following: $x_1=x_2=x_3=x_4=x_5=x_6=0$, and x_7,x_8,x_9 are the nonzero coordinates. Indeed, the above character map gives a $\mathcal{M}_B(\mathbf{c})\to\mathbb{C}^3$ morphism, and the polynomials simplify to $p(\underline{x})=x_7x_8-3$, and $q(\underline{x})=9-6x_7x_8+x_7^3+x_8^3$. Thus the condition $x_9^2-p(\underline{x})x_9+q(\underline{x})=0$ reads as

$$0=x_9^2-x_7x_8x_9+3x_9+9-6x_7x_8+x_7^3+x_8^3.$$

The homogenisation of this will again provide the equation of the curve at the infinity: $0 = -x_7x_8x_9 + x_7^3 + x_8^3$.

Lemma 3.2. *This curve* $0 = -x_7x_8x_9 + x_7^3 + x_8^3$ *is also of type* I_1*.*

Proof. One can easily see that $(x_7, x_8) = (0, 0)$ is its only singular point. We may set near that point $x_9 = 1$, and the equation becomes $x_7x_8 \approx 0$ up to terms of degree 3. This shows that the singularity is a node. Therefore the singular cubic curve must be a singular elliptic curve, and we can apply Kodaira's classification, which shows that an elliptic curve with one nodal singularity must be an I_1 curve. $\square$

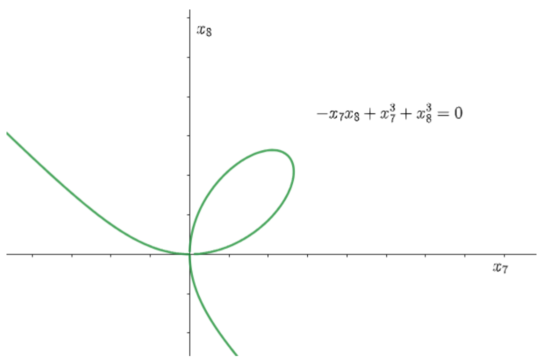

Figure 1: The $-x_7x_8 + x_7^3 + x_8^3 = 0$ nodal (fishtail) curve.

Corollary 3.3. *The fundamental group of the body of the dual boundary complex of the compactifying divisor* D_B *is cyclic, so* $|\mathcal{D}\partial\mathcal{M}_B(\boldsymbol{c})|$ *is of homotopy type* S^1*.*

Notice that we used two different types of coordinates, but the two pictures coincide, as expected: for our special choice of parameters, the compactifying divisor of the Betti space is topologically the same as in the general description. According to Simpson [40], the Riemann Hilbert correspondence provides equivalence between vector bundles with integrable connection and representations of the fundamental group, via the monodromies of the connection around the punctures. Therefore our aim is to apply the trace coordinates to the monodromies of the connection, and investigate their asymptotic behaviour.

4 Asymptotic analysis of the Dolbeault space

Consider a Higgs bundle $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}(\alpha)$ such that $H(\mathcal{E}, \theta) = q$ for some fixed, generic $q \in \mathcal{H}$. For any nonzero complex number t, the Higgs bundle $(\mathcal{E}, \overline{\partial}_E, t\theta)$ is also stable with the same parabolic structure, therefore it is an element of $\mathcal{M}_{\mathrm{Dol}}(\alpha)$ as well. For this Higgs bundle $H(\mathcal{E}, t\theta) = t^3 q$, and the second one of Hitchin's equations reads as

$$F_{h_t} + t^2[\theta, \theta^{\dagger_{h_t}}] = 0,$$

where h_t is the unique Hermitian Einstein metric solving the equation. We consider the 1-parameter family of Higgs bundles $(\mathcal{E}, \overline{\partial}_E, t\theta)$, where $t \to \infty$, or equivalently the family $(\mathcal{E}, \overline{\partial}_E, \theta_t)$, where $t \to \infty$ again on the Hitchin base. Our aim is to analyze the behaviour of h_t under this asymptotics, and describe its limiting behaviour. For this purpose we apply the following theorem of T. Mochizuki [34, Theorem 1.2]:

Theorem 4.1. *For any neighbourhood N of the discriminant D, there exist positive constants C_0 and ϵ_0 only depending on N such that over $\mathbb{C}P^1 \setminus N$ we have*

$$|F_{h_t}|_{h_t} = |t^2||[\theta, \theta^{\dagger_{h_t}}]|_{h_t} \leq C_0 e^{-\epsilon_0 t}.$$

Notice that in [35], T.Mochizuki and the second author developed tools to understand the limiting behaviour of the harmonic bundle at the discriminant locus too.

We now set $N = \overline{B}_\delta(P)$ for $P \in D$ and some $0 < \delta \ll 1$ (see Fig.2). It then follows from the theorem that on simply connected subsets of $\mathbb{C}P^1 \setminus \bigcup_{P \in D} B_\delta(P)$, with respect to some gauge $\nabla_h^+, \theta, \theta^{\dagger_{h_t}}$ can be simultaneously diagonalized, up to exponentially small error terms. Our aim is to describe the matrix of the parallel transport map (4) along a loop around the points $P \in D$, which lies in $\mathbb{C}P^1 \setminus \bigcup_{P \in D} B_\delta(P)$. We will make use of a decomposition of the surface $\mathbb{C}P^1 \setminus D$ into open regions U_1, U_2 where

$$U_1 = \mathbb{C}P^1 \setminus \bigcup_{P \in D} \overline{B}_\delta(P)$$

$$U_2 = \bigcup_{P \in D} B_{2\delta}(P).$$

We pick a loop γ_P in the annulus $U_1 \cap U_2 = B_{2\delta}(P) \setminus \overline{B}_\delta(P)$ winding around P once in positive direction. On the other hand, we pick a path η_P in U_1 from the base point z_0 to the starting point of γ_P see Figure 4.

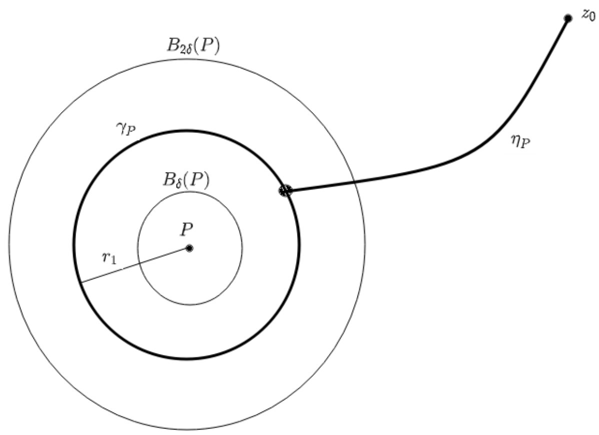

Figure 2: The base point z_0 and path η_P approaching the point $P \in D$, and the setup near P: the loop γ lying in the annulus $B_{2\delta}(P) \setminus \overline{B}_\delta(P)$.

4.1 Away from the ramification divisor

In this section we will work over the set $U_1 = \mathbb{C}P^1 \setminus \bigcup_{P \in D} \overline{B}_\delta(P)$.

From Theorem 4.1, the solutions $(\mathcal{E}, \theta_t, h_t)$ of Hitchin's equations asymptotically decouple into the direct sum of rank 1 (abelian) solutions. Therefore we can construct a model Hermitian metric h_∞ as the orthogonal pushforward of a metric on the line bundle $\mathcal{L}_t \to \widetilde{\Sigma}_t$, which actually solves the decoupled Hitchin's equations (remember that we work away from D), induces a fixed Hermitian metric on Det$E$, and turns out to be the limit of h_t.

First, we study the Hermitian metric $h_{t,\mathrm{Det}E}$ on Det$E$ induced by h_t. We observe that since $\mathrm{Tr}\theta = 0$, i.e. the Higgs field takes values in $\mathfrak{sl}(3, \mathbb{C})$, the metric $h_{t,\mathrm{Det}E}$ turns Det$E$ into a flat unitary parabolic line bundle. Moreover, the parabolic weight of Det$E$ at $P \in D$ is the sum of the parabolic weights of E at P, so it vanishes. Now, Det$\mathcal{E}$ is a line bundle of degree 0 over $\mathbb{C}P^1$, therefore it is isomorphic to $\mathcal{O}_{\mathbb{C}P^1}$. Therefore, we see that with respect to a trivialization of $\mathcal{O}$, the metric $h_{\mathrm{Det}E}$ is some constant. We may rescale the solution for each t so that with respect to a fixed trivialization of $\mathcal{O}$ we have $h_{\mathrm{Det}E} \equiv 1$, see [10, Corollary 3.3].

Now, let us equip $\mathcal{L}_t \to \Sigma_t$ with a parabolic structure. For the lifted points $\widetilde{P} \in \widetilde{D}$ on the spectral curve let us set the parabolic weights to be $\widetilde{\alpha}_{\widetilde{P}} = -1$, for all $\widetilde{P}$. Then $\mathrm{pdeg}\mathcal{L} = 0$, because $\deg\mathcal{L} = 3$. By [42], [2], there exists a metric $h_{\mathcal{L}}$ that

1. is adapted to the above parabolic structure
2. solves the abelian Hitchin's equation, and
3. induces the metric $h_{\mathrm{Det}E}$ on $\mathrm{Det}E$.

Furthermore, $h_{\mathcal{L}}$ is unique up to a constant factor.

Now define h_∞ on U_1 to be the orthogonal pushforward of $h_{\mathcal{L}}$. This means that the eigenspaces of θ are orthogonal with respect to h_∞, and

$$h_\infty((p_L)_* l, (p_L)_* l) = h_{\mathcal{L}}(l, l)$$

for any local section l of $\mathcal{L}$ supported on a single sheet of $\widetilde{\Sigma}_t$. Then h_∞ is the unique solution of the decoupled Hitchin's equations.

As previously discussed, let $\mathcal{L}$ be the line bundle such that $\mathcal{E} = p_*\mathcal{L}$ (now we omit the factor t, because of the isomorphisms $\widetilde{\Sigma}_t \cong \widetilde{\Sigma}_{t'}$ for all $t, t' \neq 0$, compatibly with p). Consider the map

$$\rho : \widetilde{\Sigma}_t \to \widetilde{\Sigma}_t$$

which is a cyclic permutation of $Q_{i,t}$, $i = 1, 2, 3$ (see equation (7)) on the spectral curve. Then there exists a short exact sequence

$$0 \to p^*\mathcal{E} \to \mathcal{L} \oplus \rho^*\mathcal{L} \oplus (\rho^*)^2\mathcal{L} \to \mathcal{O}_{\widetilde{\Delta}} \to 0.$$

Let $U \subset U_1$ be a simply connected open subset such that $p_L^{-1}(U) \cap \widetilde{\Sigma}_t = V_1 \coprod V_2 \coprod V_3$, each V_i being homeomorphic to U. Then we have

$$\mathcal{E}\big|_U = \mathcal{L}\big|_{V_1} \oplus \mathcal{L}\big|_{V_2} \oplus \mathcal{L}\big|_{V_3}$$

so

$$(p\big|_{V_1})^*(\mathcal{E}\big|_U) = \mathcal{L}\big|_{V_1} \oplus \rho^* \left(\mathcal{L}\big|_{V_2}\right) \oplus (\rho^*)^2 \left(\mathcal{L}\big|_{V_3}\right). \tag{12}$$

Then the above definition of h_∞ in detail reads as follows. By definition, the direct summands $\mathcal{L}\big|_{V_1}, \rho^* \left(\mathcal{L}\big|_{V_2}\right), (\rho^*)^2 \left(\mathcal{L}\big|_{V_3}\right)$ are orthogonal to each other, and

$$h_{\mathcal{L}} \otimes \rho^* h_{\mathcal{L}} \otimes (\rho^*)^2 h_{\mathcal{L}} = h_{\mathrm{Det}E} \equiv 1.$$

Furthermore, we construct h_∞, such that its restrictions on the direct summands of (12) are

$$h_{\mathcal{L}}, \quad \rho^* h_{\mathcal{L}}, \quad (\rho^*)^2 h_{\mathcal{L}}.$$

Then the constructed model metric, with respect to a trivialization of $\mathcal{E}$ coming from $\mathcal{L}$, reads as

$$h_\infty = \begin{pmatrix} h_{\mathcal{L}} & 0 & 0 \\ 0 & \rho^* h_{\mathcal{L}} & 0 \\ 0 & 0 & (\rho^*)^2 h_{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$$

where the functions h_1, h_2, h_3 are defined by the second equality.

Moreover, the holomorphic structure on the line bundle $\mathcal{L}$ is provided by the operator $\overline{\partial}_{\mathcal{L}} = \overline{\partial} + \mu$, for some $\mu \in \Omega^{0,1}(\widetilde{\Sigma}_t)$. With respect to the same frame as above, the holomorphic structure of $\mathcal{E}$ reads as

$$\overline{\partial}_E = \overline{\partial} + \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}.$$

The metric $h_{\mathcal{L}}$ is Hermitian-Einstein, thus the associated unitary connection $\nabla^+_{h_{\mathcal{L}}}$ is flat on V_1. Let us denote its connection form by $B_1 = \mu_1 - \overline{\mu}_1$, which is in $\Omega^1(\widetilde{\Sigma})$ (equivalently $p^{0,1} B_1 = \mu_1$). Similarly $B_2 = \mu_2 - \overline{\mu}_2$ and $B_3 = \mu_3 - \overline{\mu}_3$ are the U(1)-connection forms on the direct summands of (12). Then the flat $\mathrm{U}(1)^{\times 3}$-connection form on $\mathcal{E}$, associated with h_∞ is

$$\nabla^+_{h_\infty} = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix}.$$

with respect to some ρ-equivariant smooth unitary frame

$$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) \tag{13}$$

on U. The Higgs field is also diagonal in this frame, with its eigenvalues (7) as diagonal elements. Now we can define the flat model integrable connection:

$$\overline{\partial}_E + \partial^{h_\infty} + \theta_t + \theta_t^{\dagger_{h_\infty}} = \overline{\partial}_E + h_\infty^{-1}\partial h_\infty + \theta_t + h_\infty^{-1}\overline{\theta_t}^T h_\infty =$$
$$= d + \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{pmatrix} + \begin{pmatrix} h_1^{-1}\partial h_1 & 0 & 0 \\ 0 & h_2^{-1}\partial h_2 & 0 \\ 0 & 0 & h_3^{-1}\partial h_3 \end{pmatrix} + \begin{pmatrix} Q_{1,t}+\overline{Q_{1,t}} & 0 & 0 \\ 0 & Q_{2,t}+\overline{Q_{2,t}} & 0 \\ 0 & 0 & Q_{3,t}+\overline{Q_{3,t}} \end{pmatrix} =$$
$$= d + \begin{pmatrix} B_1+\partial(\ln(h_1))+2\mathrm{Re}Q_t & 0 & 0 \\ 0 & B_2+\partial(\ln(h_2))+2\mathrm{Re}(\varepsilon Q_t) & 0 \\ 0 & 0 & B_3+\partial(\ln(h_3))+2\mathrm{Re}(\varepsilon^2 Q_t) \end{pmatrix} \tag{14}$$

with the simplifying notation $Q_{1,t} = Q_t$. Notice that in the formula

$$B_i + \partial(\ln(h_1)) + 2\mathrm{Re}(\varepsilon^{i-1}Q_t)$$

the first term is its imaginary part and the remaining two are its real part.

The situation greatly simplifies over the Hitchin section. Indeed, when $\mathcal{L} = \mathcal{L}_0 = \mathcal{O}_{\widetilde{\Sigma}_t}(\widetilde{D})$, then the parabolic weights of

$$\mathcal{L}_0(-\widetilde{D}) \cong \mathcal{O}_{\widetilde{\Sigma}_t}$$

induced by the parabolic structure of $\mathcal{L}_0$ all vanish. Therefore, this is just the trivial line bundle over $\widetilde{\Sigma}_t$ with trivial parabolic structure, so the solution of the Hermite–Einstein equations written in a global holomorphic trivialization $\mathbf{1}$ of $\mathcal{L}_0(-\widetilde{D})$ is $h_{\mathcal{L}_0(-\widetilde{D})} \equiv 1$. Moreover, we also have $\mu \equiv 0$ for the $(0,1)$-partial connection defining $\mathcal{L}_0(-\widetilde{D})$. This then implies $B_i \equiv 0$ for all $1 \leq i \leq 3$. In conclusion, over the Hitchin section, with respect to the trivialization

$$\mathbf{1}\big|_{V_1}, \mathbf{1}\big|_{V_2}, \mathbf{1}\big|_{V_3}$$

of $\mathcal{E}_0$ over a domain U as above, the limiting connection simplifies to

$$\nabla_t^{\lim} = d + 2\mathrm{Re}\begin{pmatrix} Q_t & 0 & 0 \\ 0 & \varepsilon Q_t & 0 \\ 0 & 0 & \varepsilon^2 Q_t \end{pmatrix}.$$

Now if h_t is the metric solving the Hitchin's equations for $(\mathcal{E}, \theta_t)$, and ∇_t is the associated flat integrable connection form, as in eq. (4), then we can refer again to the result of T. Mochizuki, and conclude the asymptotic equality $\nabla_t \approx \nabla_t^{\lim}$, precisely by [34, Corollary 2.13].

Theorem 4.2. *Over any simply connected open subset U of U_1, there exists a gauge transformation g_t and positive constants C_1 and ϵ_1 depending only on $\delta > 0$ used to define U_1, such that*

$$|g_t \cdot \nabla_t - \nabla_t^{lim}|_{h_t} \leq C_1 e^{-\epsilon_1 t}.$$

4.2 Near the ramification divisor

In this section we describe asymptotic behaviour of solutions of Hitchin's equations on U_2. Obviously, it is sufficient to deal with $B_{2\delta}(P)$ for a given $P \in D$, see Figure 4, where $0 < \delta \ll 1$. Let us consider the cyclic triple cover $\widetilde{B}_{2\delta}(P) \to B_{2\delta}(P)$ defined by $\widetilde{\Sigma}_1$, i.e.

$$\zeta^3 = z(z-1),$$

where we use the trivialization

$$\frac{\mathrm{d}z}{z(z-1)}$$

of $L = K(D)$. Over $B_{2\delta}(P) \subset \mathbb{C}P^1$, $\mathcal{E}$ can be decomposed as

$$\mathcal{E}\Big|_{B_{2\delta}(P)} \cong \mathcal{O}_{B_{2\delta}(P)}\langle 1\rangle \oplus \mathcal{O}_{B_{2\delta}(P)}\langle \zeta\rangle \oplus \mathcal{O}_{B_{2\delta}(P)}\langle \zeta^2\rangle$$

(see (8)), and with respect to a suitable frame the Higgs field is a multiple of a companion matrix (see (9)):

$$\theta_t = t^{\frac{1}{3}} \begin{pmatrix} 0 & 0 & \frac{dz^{\otimes 3}}{z^2(z-1)^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $0 < \delta \ll 1$, let us use the approximations up to higher order terms on γ_0, γ_1, near the points $0, 1 \in \mathbb{C}P^1$:

$$\theta_t\Big|_{\gamma_0} \approx t^{\frac{1}{3}} \begin{pmatrix} 0 & 0 & \frac{\mathrm{d}z^{\otimes 3}}{z^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \theta_t\Big|_{\gamma_1} \approx t^{\frac{1}{3}} \begin{pmatrix} 0 & 0 & \frac{\mathrm{d}z^{\otimes 3}}{(z-1)^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We focus on the first one, i.e. the neighbourhood of 0, but a similar analysis holds near 1 too. Our next aim is to introduce a smooth approximately unitary frame

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \tag{15}$$

defined over $\widetilde{B}_{2\delta}(P)$ in which the Higgs field is diagonal. These vectors can be determined uniquely (up to phase) as the unit length eigenvectors of $\theta_t\Big|_{\gamma_0}$. Therefore, let us switch to local polar coordinates on the circle $\gamma_0(t)$ of radius

$r_0 = \frac{3\delta}{2}$ centered at 0, with angle coordinate $\vartheta \in [0, 2\pi]$. Then, independently of the value of t, the frame (15) is:

$$\mathbf{e}_1 = \frac{1}{\sqrt{|z|^{4/3} + |z|^{2/3} + 1}} \begin{pmatrix} z^{2/3}\left(\frac{\mathrm{d}z}{z}\right)^{\otimes 2} \\ z^{1/3}\frac{\mathrm{d}z}{z} \\ 1 \end{pmatrix}$$
$$\mathbf{e}_2 = \frac{1}{\sqrt{|z|^{4/3} + |z|^{2/3} + 1}} \begin{pmatrix} \varepsilon z^{2/3}\left(\frac{\mathrm{d}z}{z}\right)^{\otimes 2} \\ \varepsilon^2 z^{1/3}\frac{\mathrm{d}z}{z} \\ 1 \end{pmatrix}$$
$$\mathbf{e}_3 = \frac{1}{\sqrt{|z|^{4/3} + |z|^{2/3} + 1}} \begin{pmatrix} \varepsilon^2 z^{2/3}\left(\frac{\mathrm{d}z}{z}\right)^{\otimes 2} \\ \varepsilon z^{1/3}\frac{\mathrm{d}z}{z} \\ 1 \end{pmatrix}$$

By Theorem **??** applied to the complement of $B_\delta(0)$, the limit of the metrics h_t is the decoupled metric h_∞ over γ_0. Therefore, in the smooth unitary frame (15) both θ_t and $\theta_t^{\dagger_{h_t}}$ are asymptotically diagonal, up to exponentially small terms. Thus the approximation for $\theta_t + \theta_t^{\dagger_{h_t}}$ over γ_0 (in polar coordinates $t = Re^{i\varphi}$) is

$$\theta_t + \theta_t^{\dagger_{h_t}} \approx \begin{pmatrix} 2R^{1/3}\mathrm{Re}(e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z) & 0 & 0 \\ 0 & 2\varepsilon^2 R^{1/3}\mathrm{Re}(e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z) & 0 \\ 0 & 0 & 2\varepsilon R^{1/3}\mathrm{Re}(e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z) \end{pmatrix}$$

The model integrable connection is $\nabla_t^{mod} = d + D_t$, where D_t is a diagonal matrix, with diagonal elements:

$$\begin{aligned} &B_1 + \partial(\ln(h_1)) + 2R^{1/3}\mathrm{Re}(e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z) \\ &B_2 + \partial(\ln(h_2)) + 2R^{1/3}\mathrm{Re}(\varepsilon^2 e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z) \\ &B_3 + \partial(\ln(h_3)) + 2R^{1/3}\mathrm{Re}(\varepsilon e^{i\varphi/3}z^{-2/3}\,\mathrm{d}z). \end{aligned} \tag{16}$$

(Similarly with $(z-1)$ instead of z over γ_1). As a consequence of the previously cited theorems, asymptotically $\nabla_t^{mod} \approx \nabla_t$, as $t \to \infty$, similarly as in the previous subsection. From now on, "$\approx$" will mean "up to exponentially small terms", or more precisely:

$$A \approx B \iff AB^{-1} = \mathrm{id} + O(e^{-R^{1/3}})$$

5 Parallel transport and the Riemann Hilbert correspondence

In the sequel, our aim is to apply the non-abelian Hodge correspondence and the Riemann Hilbert correspondence to a Higgs bundle in a Hitchin fiber close to infinity. Therefore the next step is to determine the monodromy on a positively oriented loop around the punctures. Let us fix some base point $z_0 \in \mathbb{C}P^1 \setminus D$, and consider the concatenations of paths $\eta_P * \gamma_P * \eta_P^{-1}$, where η_P^{-1} is the path η_P with opposite orientation (see Fig.2). This is indeed homotopic to a loop around the puncture $P \in D$ separating it from the other two punctures. We need to determine along these paths the parallel transport maps of the flat connections whose approximations were given in the previous section, and determine the monodromies by integrating the connection forms on the paths, and taking their exponential. We will do it separately on the paths γ_P and η_P, and the mondoromy on the concatenation will be the triple product of the parallel transport maps corresponding to the above decomposition into paths.

5.1 Monodromy of the diagonalizing frame

Firstly consider the loop γ_0, and the connection form with diagonal elements (16). In order to apply the Riemann–Hilbert correspondence, we need to find a fundamental solution of the local system. For this purpose, we first need to integrate the connection form on γ_0, that is:

$$\begin{pmatrix} 2R^{1/3}\mathrm{Re}(e^{i\varphi/3}\int_{\gamma_0} z^{-2/3}\,\mathrm{d}z) & 0 & 0 \\ 0 & 2R^{1/3}\mathrm{Re}(\varepsilon^2 e^{i\varphi/3}\int_{\gamma_0} z^{-2/3}\,\mathrm{d}z) & 0 \\ 0 & 0 & 2R^{1/3}\mathrm{Re}(\varepsilon e^{i\varphi/3}\int_{\gamma_0} z^{-2/3}\,\mathrm{d}z) \end{pmatrix} + U_0.$$

The integrals of the terms $\partial(\ln(h_i))$, $i = 1, 2, 3$ vanish because we integrate them on a closed loop, and U_0 is the diagonal matrix with diagonal entries $\int_{\gamma_0} B_i$. The integrals appearing here can be determined explicitly, because γ_0 is just the boundary of a disc of radius r_1 around $0 \in \mathbb{C}P^1$:

$$\int_{\gamma_0} z^{-2/3}\,\mathrm{d}z = \int_0^{2\pi} \frac{r_1 i e^{i\vartheta}\,\mathrm{d}\vartheta}{r_1^{2/3} e^{2i\vartheta/3}} = r_1^{1/3}(\varepsilon - 1)$$

Now, to get the parallel transport map, we take the exponential of this matrix, and we obtain:

$$N_{0,R,\varphi} = \begin{pmatrix} \exp(2(Rr_1)^{1/3}\mathrm{Re}(e^{i\varphi/3}(\varepsilon-1))) & 0 & 0 \\ 0 & \exp(2(Rr_1)^{1/3}\mathrm{Re}(\varepsilon^2 e^{i\varphi/3}(\varepsilon-1))) & 0 \\ 0 & 0 & \exp(2(Rr_1)^{1/3}\mathrm{Re}(\varepsilon e^{i\varphi/3}(\varepsilon-1))) \end{pmatrix}.$$

However this is not yet the monodromy of the flat connection on γ_0, because when we transport the frame (15) around the loop in positive direction, the effect on the frame is a cyclic permutation. Indeed, one can easily check that

$$\begin{aligned} \mathbf{e}_1(\vartheta = 0) &= \mathbf{e}_2(\vartheta = 2\pi) \\ \mathbf{e}_2(\vartheta = 0) &= \mathbf{e}_3(\vartheta = 2\pi) \\ \mathbf{e}_3(\vartheta = 0) &= \mathbf{e}_1(\vartheta = 2\pi) \end{aligned}$$

Thus the monodromy of the diagonalizing frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the permutation matrix

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and the monodromy of the connection on γ_0 can be obtained by multiplying the $N_{0,R,\varphi}e^{U_0}$ with T, i.e. $M_{0,R,\varphi} = TN_{0,R,\varphi}e^{U_0}$. Notice that the monodromy $M_{1,R,\varphi}$ on γ_1 near the puncture $1 \in \mathbb{C}P^1$ is just the same, because the integrals coincide:

$$\int_{\gamma_0} z^{-2/3}dz = \int_{\gamma_1} (z-1)^{-2/3}dz$$

5.2 Parallel transport to the punctures and the whole monodromy

Similarly as above, apply the Riemann–Hilbert correspondence to the parallel transport map (14) on η_0, and we obtain the matrix:

$$\begin{pmatrix} \frac{h_1(\eta_0(0))}{h_1(\eta_0(1))}\exp(\int_{\eta_0} B_1 + 2\mathrm{Re}\int_{\eta_0} Q_t) & 0 & 0 \\ 0 & \frac{h_2(\eta_0(0))}{h_2(\eta_0(1))}\exp(\int_{\eta_0} B_2 + 2\mathrm{Re}(\varepsilon\int_{\eta_0} Q_t)) & 0 \\ 0 & 0 & \frac{h_3(\eta_0(0))}{h_3(\eta_0(1))}\exp(\int_{\eta_0} B_3 + 2\mathrm{Re}(\varepsilon^2\int_{\eta_0} Q_t)) \end{pmatrix}$$

and similarly for η_1.

Lemma 5.1. *The factors* $\frac{h_i(\eta_0(0))}{h_i(\eta_0(1))}$, $i = 1, 2, 3$ *can be omitted from the above matrix.*

Proof. Since η_0 lies in a region, where the conditions of Theorem 4.2 satisfy, we can apply the theorem. Thus there exists a gauge transformation, more precisely a torus action, with which on $L_{0,t}$, we can simultaneously get rid of the factors $\frac{h_i(\eta_0(0))}{h_i(\eta_0(1))}$, $i = 1, 2, 3$. $\square$

Again, a similar result holds for η_1. The monodromies on η_0 and η_1 are respectively

$$L_{0,t} = \begin{pmatrix} \exp(\int_{\eta_0} B_1 + 2\mathrm{Re}\int_{\eta_0} Q_t) & 0 & 0 \\ 0 & \exp(\int_{\eta_0} B_2 + 2\mathrm{Re}(\varepsilon \int_{\eta_0} Q_t)) & 0 \\ 0 & 0 & \exp(\int_{\eta_0} B_3 + 2\mathrm{Re}(\varepsilon^2 \int_{\eta_0} Q_t)) \end{pmatrix}$$
$$L_{1,t} = \begin{pmatrix} \exp(\int_{\eta_1} B_1 + 2\mathrm{Re}\int_{\eta_1} Q_t) & 0 & 0 \\ 0 & \exp(\int_{\eta_1} B_2 + 2\mathrm{Re}(\varepsilon \int_{\eta_1} Q_t)) & 0 \\ 0 & 0 & \exp(\int_{\eta_1} B_3 + 2\mathrm{Re}(\varepsilon^2 \int_{\eta_1} Q_t)) \end{pmatrix}$$

Now the monodromy on $\eta_0 * \gamma_0 * \eta_0^{-1}$ will be provided as the product of $L_{0,t}, M_{0,R,\varphi}$ and $L_{0,t}^{-1}$. However, there is one more thing we need to consider: over η_0 we had smooth unitary frame (13), but over γ_0 we had another smooth unitary frame (15), so we need to determine the matrix of the change of bases at $\eta_0(1)$. Let us denote this base change by $J_{0,t}$.

Lemma 5.2. *For suitable choice of the frame* (13)*, the matrix of the change of bases to* (15) *at* $\eta_0(1)$ *is given by*

$$J_{0,t} = \begin{pmatrix} \exp(2\,\mathrm{Re}\int_{\eta_0} Q_t) & 0 & 0 \\ 0 & \exp(2\,\mathrm{Re}(\varepsilon \int_{\eta_0} Q_t)) & 0 \\ 0 & 0 & \exp(2\,\mathrm{Re}(\varepsilon^2 \int_{\eta_0} Q_t)) \end{pmatrix}$$

Proof. The proof goes very similar as in [45, Proposition 11]. The parallel transport map $L_{0,t}$ carries the frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ from $\eta_0(0)$ to another frame at $\eta_0(1)$. At $\eta_0(0)$ it has unit length, but then at $\eta_0(1)$ the vectors $L_{0,t}\mathbf{f}_1, L_{0,t}\mathbf{f}_2, L_{0,t}\mathbf{f}_3$ will respectively be of length

$$|L_{0,t}\mathbf{f}_1(\eta_0(1))| = e^{2\,\mathrm{Re}\int_{\eta_0} Q_t}$$
$$|L_{0,t}\mathbf{f}_2(\eta_0(1))| = e^{2\,\mathrm{Re}(\varepsilon\int_{\eta_0} Q_t)}$$
$$|L_{0,t}\mathbf{f}_3(\eta_0(1))| = e^{2\,\mathrm{Re}(\varepsilon^2\int_{\eta_0} Q_t)}.$$

The other frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ at $\eta_0(1)$ has unit length, and both frames diagonalize the Higgs field at this point. It is known that any two diagonalizing frames of a given semi-simple endomorphism of a finite-dimensional $\mathbb{C}$-vector space differ from each other by a diagonal automorphism with nonzero diagonal elements. The norms of these elements are the above determined values $|L_{0,t}\mathbf{f}_i(\eta_0(1))|$, $i = 1, 2, 3$, so the matrix of the basis change is

$$J_{0,t} = \begin{pmatrix} a_0 \exp(2\,\mathrm{Re} \int_{\eta_0} Q_t) & 0 & 0 \\ 0 & b_0 \exp(2\,\mathrm{Re}(\varepsilon \int_{\eta_0} Q_t)) & 0 \\ 0 & 0 & c_0 \exp(2\,\mathrm{Re}(\varepsilon^2 \int_{\eta_0} Q_t)) \end{pmatrix}$$

for some $a_0, b_0, c_0 \in \mathrm{U}(1)$ unit length complex numbers. Now the point $\eta_0(1)$ lies in a region where Theorem **??** holds, thus we can apply a gauge transformation on $J_{0,t}$ as well, and get rid of the factors a_0, b_0, c_0. □

Again, we have the analogue for $J_{1,t}$ with the same argument. Now we can describe the monodromies around the punctures on the two loops $\eta_0 * \gamma_0 * \eta_0^{-1}$ and $\eta_1 * \gamma_1 * \eta_1^{-1}$, denoted by A and B respectively, depending of course on the parameter t, or R and φ in polar coordinates:

$$\begin{aligned} A_t = A_{R,\varphi} &= L_{0,t}^{-1} J_{0,t}^{-1} T N_{0,R,\varphi} e^{U_0} J_{0,t} L_{0,t} \\ B_t = B_{R,\varphi} &= L_{1,t}^{-1} J_{1,t}^{-1} T N_{1,R,\varphi} e^{U_1} J_{1,t} L_{1,t} \end{aligned} \tag{17}$$

After we execute the matrix multiplications, we get that both matrices have the following form similar to a companion matrix:

$$A_t = A_{R,\varphi} = \begin{pmatrix} 0 & A_1(t) & 0 \\ 0 & 0 & A_2(t) \\ A_3(t) & 0 & 0 \end{pmatrix}, \quad B_t = B_{R,\varphi} = \begin{pmatrix} 0 & B_1(t) & 0 \\ 0 & 0 & B_2(t) \\ B_3(t) & 0 & 0 \end{pmatrix}.$$

Proposition 5.3. *For the nonzero elements of A_t, as $t \to \infty$ we have*

$$\begin{aligned} A_1(t) &\approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\pi_a) + \int_{\eta_0} (B_2 - B_1) + \int_{\gamma_0} B_2 \right) \\ A_2(t) &\approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\varepsilon\pi_a) + \int_{\eta_0} (B_3 - B_2) + \int_{\gamma_0} B_3 \right) \\ A_3(t) &\approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\varepsilon^2\pi_a) + \int_{\eta_0} (B_1 - B_3) + \int_{\gamma_0} B_1 \right) \end{aligned}$$

where $\pi_a = 4\int_{\eta_0}(\varepsilon-1)\frac{dz}{z^{2/3}(z-1)^{2/3}}$. *Similarly for* B_t *we have*

$$B_1(t) \approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\pi_b) + \int_{\eta_1}(B_2 - B_1) + \int_{\gamma_1} B_2\right)$$
$$B_2(t) \approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\varepsilon\pi_b) + \int_{\eta_1}(B_3 - B_2) + \int_{\gamma_1} B_3\right)$$
$$B_3(t) \approx \exp\left(R^{1/3}\,\mathrm{Re}(e^{i\varphi/3}\varepsilon^2\pi_b) + \int_{\eta_1}(B_1 - B_3) + \int_{\gamma_1} B_1\right)$$

where $\pi_b = 4\int_{\eta_1}(\varepsilon-1)\frac{dz}{z^{2/3}(z-1)^{2/3}}$.

Proof. From the matrix multiplication (17) we get

$$A_1(t) = \exp\left(-\int_{\eta_0} B_1 - 4\mathrm{Re}\left(\int_{\eta_0} Q_t\right) + \int_{\eta_0} B_2 + 4\mathrm{Re}\left(\varepsilon\int_{\eta_0} Q_t\right)\right)\cdot$$
$$\cdot\exp\left(2R^{1/3}r_1^{1/3}\mathrm{Re}(\varepsilon^2 e^{i\varphi/3}(\varepsilon-1)) + \int_{\gamma_0} B_2\right)$$

Now using the notation π_a and $Q_t = R^{1/3}e^{i\varphi/3}\frac{dz}{z^{2/3}(z-1)^{2/3}}$ from (7), this formula becomes

$$A_1(t) = \exp\left(R^{1/3}\left[\mathrm{Re}(e^{i\varphi/3}\pi_a) + 2r_1^{1/3}\mathrm{Re}(\varepsilon^2 e^{i\varphi/3}(\varepsilon-1))\right] + \int_{\eta_0}(B_2 - B_1) + \int_{\gamma_0} B_2\right)$$

Similarly

$$A_2(t) = \exp\left(R^{1/3}\left[\mathrm{Re}(e^{i\varphi/3}\varepsilon\pi_a) + 2r_1^{1/3}\mathrm{Re}(\varepsilon e^{i\varphi/3}(\varepsilon-1))\right] + \int_{\eta_0}(B_3 - B_2) + \int_{\gamma_0} B_3\right)$$
$$A_3(t) = \exp\left(R^{1/3}\left[\mathrm{Re}(e^{i\varphi/3}\varepsilon^2\pi_a) + 2r_1^{1/3}\mathrm{Re}(e^{i\varphi/3}(\varepsilon-1))\right] + \int_{\eta_0}(B_1 - B_3) + \int_{\gamma_0} B_1\right)$$

Now the values $\int_{\eta_0} B_{i+1} - B_i$, and $\int_{\gamma_0} B_i$ $i = 1,2,3$ (understood mod3), and the values $\mathrm{Re}(e^{i\varphi/3}\varepsilon^j\pi_a)$, $j = 0,1,2$ are fixed, and similarly the terms of B_t with η_1 and π_b . Although the r_1 parameter is one, we can choose. Recall that is the radius of the circles around the punctures, which actually describe γ_0 and γ_1. For the local analysis we already settled $0 < r_1 \ll 1$, but now we can choose it to be far smaller than the absolute values of all the above values. Then the terms $2r_1^{1/3}\mathrm{Re}(\varepsilon^i e^{i\varphi/3}(\varepsilon-1))$, $i = 0,1,2$ will have approximately no contribution in the above formulas, and we receive the desired approximations for the elements of A_t and B_t. $\square$

6 Asymptotics of the trace coordinates and proof of the main theorem

Finally, in this section we will compute the trace coordinates introduced in Section 3.2 to the approximations for the monodromy matrices determined in Section 5.2. Curiously, we will find a structure that is reminiscent to the Stokes phenomenon, namely that depending on sectors in the Hitchin base, different exponential terms of the coordinates dominate.

From the shape of the above determined matrices A_t and B_t, it trivially follows that six out of the nine coordinates from (11) are zero.

Proposition 6.1. *The only nonzero trace coordinates are $Tr(A_tB_t^{-1})$, $Tr(A_t^{-1}B_t)$ and $Tr(A_tB_tA_t^{-1}B_t^{-1})$.*

Introduce the notations

$$X_{R,\varphi} = \mathrm{Tr}(A_{R,\varphi}B_{R,\varphi}^{-1}) = \mathrm{Tr}(A_tB_t^{-1})$$
$$Y_{R,\varphi} = \mathrm{Tr}(A_{R,\varphi}^{-1}B_{R,\varphi}) = \mathrm{Tr}(A_t^{-1}B_t)$$
$$Z_{R,\varphi} = \mathrm{Tr}(A_{R,\varphi}B_{R,\varphi}A_{R,\varphi}^{-1}B_{R,\varphi}^{-1}) = \mathrm{Tr}(A_tB_tA_t^{-1}B_t^{-1})$$

After performing the matrix multiplications, we can give the approximations for the absolute values of this three coordinates:

$$\begin{gathered}
|X_{R,\varphi}| \approx \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}(\pi_a-\pi_b))) + \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}\varepsilon(\pi_a-\pi_b)))+ \\
+\exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}\varepsilon^2(\pi_a-\pi_b))) \\
|Y_{R,\varphi}| \approx \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}(\pi_b-\pi_a))) + \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}\varepsilon(\pi_b-\pi_a)))+ \\
+\exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}\varepsilon^2(\pi_b-\pi_a))) \\
|Z_{R,\varphi}| \approx \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}(1-\varepsilon)(\pi_a-\pi_b)))+ \\
+\exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}(1-\varepsilon)\varepsilon(\pi_a-\pi_b))) + \exp(R^{1/3}\mathrm{Re}(e^{i\varphi/3}(1-\varepsilon)\varepsilon^2(\pi_a-\pi_b)))
\end{gathered} \tag{18}$$

Notice here that the terms $\int_{\eta_0} B_{i+1} - B_i$, and $\int_{\gamma_0} B_i$ $i = 1,2,3$ are purely imaginary, thus they have no contribution to the absolute values of the coordinates.

6.1 Asymptotics of the trace coordinates

Remember that the polar coordinates $(R,\varphi) \in \mathbb{R}_+ \times S^1$ parameterize the Hitchin base (except for its origin), and the Dolbeault moduli space admits the Hitchin fibration over the Hitchin base. Now consider $1 \ll R$ fixed, and φ going through $[0, 2\pi]$, parameterizing the circle S^1_R of radius R in the base. We consider any smooth lift of this loop to the Dolbeault space. (A natural choice is to use the Hitchin section, but we will not exploit this choice anywhere.) The question we would like to answer is: where does the Riemann–Hilbert correspondence map such a loop, as $R \to \infty$? The Geometric P=W conjecture follows from the property that it maps into the generator of the fundamental group of the body of the dual boundary complex of the Betti moduli space.

Proposition 6.2. *Let $\sigma \colon S^1 \to \mathcal{M}_{Dol}(\alpha)$ be a lift of the loop $Re^i\varphi$, i.e. $h \circ \sigma = I_{S^1}$. Assume that $\dim_{\mathbb{R}} \mathcal{M}_{Dol}(\alpha) = 4$ (which holds for the case at hand). Assume that $\phi \circ \psi \circ \sigma$ induces an isomorphism*

$$\mathbb{Z} \cong \pi_1(S^1) \to \pi_1(|\mathcal{D}\partial\mathcal{M}_B|) \cong \mathbb{Z}$$

for $R \gg 1$. Then the diagram (1) *is commutative up to homotopy.*

Proof. The argument of [37, Section 5] applies verbatim. It only depends on the condition that the moduli space is a 4-manifold, not on the rank nor the type of singularities of the underlying Higgs bundles. □

For fixed $0 \ll R$, and φ going through $[0, 2\pi]$ the coordinates $X_{R,\varphi}$, $Y_{R,\varphi}$, $Z_{R,\varphi}$ describe the image of the above mentioned loop under the Riemann Hilbert map. To prove the statement of the Geometric P=W conjecture, we investigate the magnitude of the $X_{R,\varphi}$, $Y_{R,\varphi}$, $Z_{R,\varphi}$ coordinates with respect to each other.

Introduce $x = e^{i\varphi/3}(\pi_a - \pi_b) = a + b\sqrt{-1}$ for simplicity. Then

$$\begin{gathered}\mathrm{Re}(x) = a, \quad \mathrm{Re}(\varepsilon x) = -\frac{1}{2}a - \frac{\sqrt{3}}{2}b, \quad \mathrm{Re}(\varepsilon^2 x) = -\frac{1}{2}a + \frac{\sqrt{3}}{2}b \\ \mathrm{Re}(-x) = -a, \quad \mathrm{Re}(-\varepsilon x) = \frac{1}{2}a + \frac{\sqrt{3}}{2}b, \quad \mathrm{Re}(-\varepsilon^2 x) = -\frac{1}{2}a - \frac{\sqrt{3}}{2}b \\ \mathrm{Re}((1-\varepsilon)x) = \frac{3}{2}a + \frac{\sqrt{3}}{2}b, \;\; \mathrm{Re}((1-\varepsilon)\varepsilon x) = -\sqrt{3}b, \;\; \mathrm{Re}((1-\varepsilon)\varepsilon^2 x) = -\frac{3}{2}a + \frac{\sqrt{3}}{2}b\end{gathered} \tag{19}$$

Consider $|X_{R,\varphi}|$ from (18). It has three terms, all exponential. The question, which of these three terms dominates as $R \to \infty$, depends on which of three exponents $\{\mathrm{Re}(x), \mathrm{Re}(\varepsilon x), \mathrm{Re}(\varepsilon^2 x)\}$ is the largest. Similarly for $|Y_{R,\varphi}|$ and $|Z_{R,\varphi}|$, the dominating term depends on, which is the largest from $\{\mathrm{Re}(-x), \mathrm{Re}(-\varepsilon x), \mathrm{Re}(-\varepsilon^2 x)\}$ and from $\{\mathrm{Re}((1-\varepsilon)x), \mathrm{Re}((1-\varepsilon)\varepsilon x), \mathrm{Re}((1-\varepsilon)\varepsilon^2 x)\}$ respectively. Therefore we can decompose the complex plane (parameterized by x) into 12 sectors, defined as follows (see also Figure 3):

$$S_j = \{x \in \mathbb{C} | \frac{\pi(j-1)}{6} < \arg(x) < \frac{\pi j}{6}\}, \quad j = 1, 2, ..., 12$$

Since we want to analyze the asymptotic behaviour of the coordinates, we introduce the concept of dominance: we say, that function $F(R)$ dominates the function $G(R)$, if and only if $\frac{|G(R)|}{|F(R)|} \to 0$, as $R \to \infty$. Now we can sum up in the following table, in which sector which term dominates for the coordinates.

Sector	Dominating term of each coordinate
S_1	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(x)) = \exp(R^{1/3}a)$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon x)) = \exp(R^{1/3}(\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)x)) = \exp(R^{1/3}(\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_2	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(x)) = \exp(R^{1/3}a)$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon x)) = \exp(R^{1/3}(\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)x)) = \exp(R^{1/3}(\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_3	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon x)) = \exp(R^{1/3}(\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)x)) = \exp(R^{1/3}(\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_4	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon x)) = \exp(R^{1/3}(\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_5	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-x)) = \exp(R^{1/3}(-a))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_6	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-x)) = \exp(R^{1/3}(-a))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_7	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon x)) = \exp(R^{1/3}(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-x)) = \exp(R^{1/3}(-a))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon^2 x)) = \exp(R^{1/3}(-\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$
S_8	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon x)) = \exp(R^{1/3}(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-x)) = \exp(R^{1/3}(-a))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon x)) = \exp(R^{1/3}(-\sqrt{3}b))$
S_9	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon x)) = \exp(R^{1/3}(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon^2 x)) = \exp(R^{1/3}(\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon x)) = \exp(R^{1/3}(-\sqrt{3}b))$
S_{10}	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(\varepsilon x)) = \exp(R^{1/3}(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon^2 x)) = \exp(R^{1/3}(\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon x)) = \exp(R^{1/3}(-\sqrt{3}b))$
S_{11}	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(x)) = \exp(R^{1/3}(a))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon^2 x)) = \exp(R^{1/3}(\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)\varepsilon x)) = \exp(R^{1/3}(-\sqrt{3}b))$
S_{12}	$\lvert X_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(x)) = \exp(R^{1/3}(a))$ $\lvert Y_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}(-\varepsilon^2 x)) = \exp(R^{1/3}(\frac{1}{2}a - \frac{\sqrt{3}}{2}b))$ $\lvert Z_{R,\varphi}\rvert \approx \exp(R^{1/3}\mathrm{Re}((1-\varepsilon)x)) = \exp(R^{1/3}(\frac{3}{2}a + \frac{\sqrt{3}}{2}b))$

Lemma 6.3. *i) $|Z_{R,\varphi}|$ dominates $|X_{R,\varphi}|$ and $|Y_{R,\varphi}|$ in all sectors.*

ii) On sectors S_i, $i = 1,4,5,8,9,12$ $|X_{R,\varphi}|$ dominates $|Y_{R,\varphi}|$, while on sectors S_i, $i = 2,3,6,7,10,11$ $|Y_{R,\varphi}|$ dominates $|X_{R,\varphi}|$.

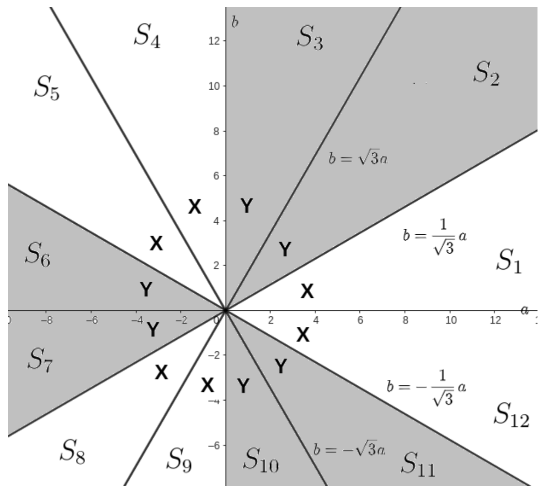

Figure 3: The (a,b) complex plane, divided into 12 sectors, marked in every sector whether $|X_{R,\varphi}|$ dominates $|Y_{R,\varphi}|$ or the other way.

Actually lines $b = \pm\sqrt{3}a$, $b = \pm\frac{1}{\sqrt{3}}a$ and the two axis divide the plane into the above defined S_i, $i = 1,...,12$ sectors, which are open subsets of the plane, as on Figure 3. Define R_i to be the ray between sectors S_i and S_{i+1} (where $i+1$ understood modulo 12).

Proof. The comparison of the exponents from the table in each sector shows straightforward this result.

On S_1: $\frac{3}{2}a + \frac{\sqrt{3}}{2}b > a > \frac{1}{2}a + \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.

On S_2: $\frac{3}{2}a + \frac{\sqrt{3}}{2}b > \frac{1}{2}a + \frac{\sqrt{3}}{2}b > a$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.

On S_3: $\frac{3}{2}a + \frac{\sqrt{3}}{2}b > \frac{1}{2}a + \frac{\sqrt{3}}{2}b > -\frac{1}{2}a + \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.

On S_4: $-\frac{3}{2}a + \frac{\sqrt{3}}{2}b > -\frac{1}{2}a + \frac{\sqrt{3}}{2}b > \frac{1}{2}a + \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.

On S_5: $-\frac{3}{2}a + \frac{\sqrt{3}}{2}b > -\frac{1}{2}a + \frac{\sqrt{3}}{2}b > -a$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.

On S_6: $-\frac{3}{2}a + \frac{\sqrt{3}}{2}b > -a > -\frac{1}{2}a + \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.

On S_7: $-\frac{3}{2}a + \frac{\sqrt{3}}{2}b > -a > -\frac{1}{2}a - \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.

On S_8: $-\sqrt{3}b > -\frac{1}{2}a - \frac{\sqrt{3}}{2}b > -a$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.
On S_9: $-\sqrt{3}b > -\frac{1}{2}a - \frac{\sqrt{3}}{2}b > \frac{1}{2}a - \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.
On S_{10}: $-\sqrt{3}b > \frac{1}{2}a - \frac{\sqrt{3}}{2}b > -\frac{1}{2}a - \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.
On S_{11}: $-\sqrt{3}b > \frac{1}{2}a - \frac{\sqrt{3}}{2}b > a$, that is $|Z_{R,\varphi}| > |Y_{R,\varphi}| > |X_{R,\varphi}|$.
On S_{12}: $\frac{3}{2}a + \frac{\sqrt{3}}{2}b > a > \frac{1}{2}a - \frac{\sqrt{3}}{2}b$, that is $|Z_{R,\varphi}| > |X_{R,\varphi}| > |Y_{R,\varphi}|$.

□

6.2 Proof of the main theorem

As previously denoted, $\mathcal{D}\partial\mathcal{M}_B(\mathbf{c})$ stands for the dual boundary complex of the compactifying divisor on the Betti side. By the results of Section 3, the divisor is of type I_1, i.e. a so called fishtail curve, and its dual simplicial complex has one vertex with a graph loop. Denote by V the vertex, and by E the loop edge of the graph of the complex (V corresponds to the one irreducible component of the I_1 curve). Let ϕ be Simpson's natural map from the punctured neighbourhood of the compactifying divisor of the Betti space, to the body of its dual boundary complex (as denoted in the commutative diagram in the introduction). See the definition "evaluation map" in [31].

Consider any section $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ of the h Hitchin map over some fixed R. Then its image under the non-abelian Hodge correspondence, the Riemann Hilbert correspondence (their composition denoted by ψ earlier) and ϕ

$$\phi \circ \mathrm{RH} \circ \mathrm{NAHC}(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) = \phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$$

is a loop, and we need to shows, that it is a generator of the fundamental group of $|\mathcal{D}\partial\mathcal{M}_B|$. The coordinates $X_{R,\varphi}, Y_{R,\varphi}, Z_{R,\varphi}$ describe the Betti space, we need to see what happens with them, as $0 \ll R$ is fixed, and φ goes through the interval $[0, 2\pi]$. Lemma 6.2 shows, that $|Z_{R,\varphi}|$ dominates the other two coordinates, so as $R \to \infty$, the image of the loop tends to the point $[0 : 0 : 1]$ on the open sectors, which is actually the nodal point of the curve C determined by the equation

$$X^3 + Y^3 - XYZ = 0,$$

and this point corresponds to the E edge of $|\mathcal{D}\partial\mathcal{M}_B(\mathbf{c})|$. The remaining question, whether it maps into the generator of its fundamental group. Consider $X_{R,\varphi}$ and $Y_{R,\varphi}$, as the function of φ, see formulas (18). In the exponent of this formulas, $e^{i\varphi/3}$ appears, so the exponent suffers a rotation of total degree 120° on the (a, b)-coordinate plane (see Figure 3). Here each of the 12 sectors are actually a 2-dimensional open cone with angle 30°, so under the rotation, exactly two times changes whether the exponent is in some white or grey sector, that is $X_{R,\varphi}$ dominates or $Y_{R,\varphi}$. Therefore there are two critical angles $\varphi_1 = \varphi_{\text{crit}}$ and $\varphi_2 = \varphi_{\text{crit}} + \pi$, where the dominance of $|X_{R,\varphi}|$ or $|Y_{R,\varphi}|$ changes. That is, we can decompose S^1 into two closed arcs

$$S^1 = I_1 \cup I_2,$$

such that, let's say,

- for $\varphi \in \mathrm{Int} I_1$: $|X_{R,\varphi}| > |Y_{R,\varphi}|$
- for $\varphi \in \mathrm{Int} I_2$: $|Y_{R,\varphi}| > |X_{R,\varphi}|$.

Now consider a blow up of the compactifying curve at its nodal point. Then the dual boundary graph changes, as on the below picture:

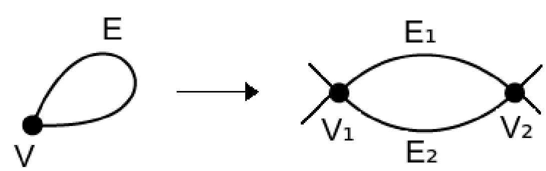

Figure 4: The transformation of the graph of the dual boundary complex under the blow up. The vertex V transforms into V_1, and V_2 corresponds to the appearing exceptional divisor.

The curve C locally around the nodal point is of two component, which are the $X-$ and $Y-$ axis in the $[X : Y : Z]$ coordinate system locally. After the blow up at its nodal point, C transforms into C_1 curve, and exceptional curve C_2 appears, such that C_1 and C_2 have two intersection points. These two intersection points correspond to E_1 and E_2 in the dual picture.

That is

- for $\varphi \in \mathrm{Int} I_1$: $\phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in E_1$
- for $\varphi \in \mathrm{Int} I_2$: $\phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in E_2$.

Lemma 6.4. *Consider some tiny open neighbourhoods of the critical angles $U_1 = (\varphi_1 - \epsilon, \varphi_1 + \epsilon)$ and $U_2 = (\varphi_2 - \epsilon, \varphi_2 + \epsilon)$. Then one of U_1 and U_2 is mapped onto C_1, and the other one is mapped onto C_2 under ψ.*

Proof. There is two different behaviour of the asymptotics of the coordinates at two different type of critical angles. One of them is on R_3, R_7 and R_{11}. Here

$$|Z_{R,\varphi}| = |Y_{R,\varphi}| = |X_{R,\varphi}|.$$

Therefore the image of this type of critical angles are away from the nodal point $[0:0:1]$, after the blow up it is inner point of C_1. On the other hand on R_1, R_5 and R_9:

$$|Z_{R,\varphi}| > |Y_{R,\varphi}| = |X_{R,\varphi}|.$$

Therefore the image of this type of critical angles are in the nodal point $[0:0:1]$, and the image of tiny open interval around it have intersection with both component: one coming from $X-$ direction, one coming from $Y-$ direction, so the image of the critical angle must be inner point of C_2 after the blow up (also follows from $|Y_{R,\varphi}| = |X_{R,\varphi}|$).

Since we have of rotation of total angle $120°$, and the arrangement of the two type of critical angles is $120°$-periodic, one of φ_1 and φ_2 is of the first type, and the other one is of the second type. $\square$

As a consequence of the above lemma

- for $\varphi = \varphi_1$: $\phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in V_1$
- for $\varphi = \varphi_2$: $\phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in V_2$

It also follows, that some tiny open neighbourhoods of the critical angles are mapped around the vertices of the dual graph, such way, that their images connect the images of $\mathrm{Int}I_1$ and $\mathrm{Int}I_2$ on E_1 and E_2. So as φ ranges through the interval $[0, 2\pi]$, the images of the tiny open intervals U_1, U_2 connect the images of $\mathrm{Int}I_1$ and $\mathrm{Int}I_2$ at V_1 and V_2. Thus the elements $\phi \circ \psi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ describe a path in the dual graph, which is exactly the generator of the fundamental group of $|\mathcal{D}\partial\mathcal{M}_B(\mathbf{c})|$.

Now recall the commutative diagram, which is the subject of the conjecture.

$$\begin{array}{ccc}
\mathcal{M}_{\mathrm{Dol}} \setminus h^{-1}(B_R(0)) & \xrightarrow{\psi} & \mathcal{M}_B \setminus \psi(h^{-1}(B_R(0))) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathcal{H} \setminus B_R(0) & \xrightarrow{\qquad\qquad} & |\mathcal{D}\partial\mathcal{M}_B|
\end{array}$$

Here we constructed the map

$$\mathcal{H} \setminus B_R(0) \to |\mathcal{D}\partial\mathcal{M}_B|,$$

such as applying $\phi \circ \psi$ for an arbitrary section $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ of h. Denote this section by:

$$\sigma : (R, \varphi) \mapsto (\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$$

Therefore we saw, that $\phi \circ \psi \circ \sigma : S^1 \to S^1$ is homotopic to the identity. But from the above analysis also follows, that $\phi \circ \psi$ is constant on the preimage $h^{-1}(R, \varphi)$ (which is a smooth elliptic curve of genus 1), thus $\phi \circ \psi$ is homotopic to h on S^1, which shows the commutativity of the above diagram, and completes the proof.

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