SOLUTION SHEET OF DIFFERENTIAL GEOMETRY MID-TERM, MARCH 29, 2022

(1) Compute the length of the conical helix

$$\gamma(t) = \begin{pmatrix} at\cos t \\ at\sin t \\ bt \end{pmatrix}$$

over $t \in [0, 2\pi]$, where $a, b > 0$ are fixed parameters.

Solution: We find

$$\dot\gamma(t) = \begin{pmatrix} a\cos t - at\sin t \\ a\sin t + at\cos t \\ b \end{pmatrix}$$

and

$$|\dot\gamma(t)| = \sqrt{a^2 + b^2 + a^2t^2}.$$

It follows that

$$l(\gamma) = \int_0^{2\pi} \sqrt{a^2 + b^2 + a^2t^2}\mathrm{d}t = \sqrt{a^2+b^2}\int_0^{2\pi} \sqrt{1 + \frac{a^2}{a^2+b^2}t^2}\mathrm{d}t.$$

We make the substitution

$$\sinh(u) = \frac{at}{\sqrt{a^2+b^2}},$$

so

$$\cosh(u)\mathrm{d}u = \frac{a}{\sqrt{a^2+b^2}}\mathrm{d}t.$$

We then find

$$l(\gamma) = \frac{a^2+b^2}{a}\int_0^{\mathrm{arsh}\left(\frac{2a\pi}{\sqrt{a^2+b^2}}\right)} \cosh^2(u)\mathrm{d}u.$$

Using the hyperbolic trigonometric formula

$$\cosh^2(u) = \frac{1}{2}\cosh(2u) + \frac{1}{2},$$

this becomes

$$l(\gamma) = \frac{a^2+b^2}{a}\left[\frac{\sinh(2u)}{4} + \frac{u}{2}\right]_{u=0}^{\mathrm{arsh}\left(\frac{2a\pi}{\sqrt{a^2+b^2}}\right)}.$$

Making use of the hyperbolic trigonometric formulas

$$\sinh(2u) = 2\sinh(u)\cosh(u)$$

and

$$\cosh(\operatorname{arsh}(A)) = \sqrt{1+A^2},$$

we get

$$l(\gamma) = \frac{a^2+b^2}{2a} \cdot \frac{2a\pi}{\sqrt{a^2+b^2}} \cdot \sqrt{1+\frac{4a^2\pi^2}{a^2+b^2}} + \frac{a^2+b^2}{2a} \cdot \operatorname{arsh}\left(\frac{2a\pi}{\sqrt{a^2+b^2}}\right).$$

(2) Using the Gram–Schmidt procedure, determine the normal (i.e., second Frenet) vector of the curve γ given in the previous part, assuming that $b = a$.

Solution: Using the assumption the speed simplifies to

$$|\dot\gamma(t)| = \sqrt{2a^2+a^2t^2} = a\sqrt{2+t^2}$$

and thus we find

$$\vec{t}(t) = \frac{1}{\sqrt{2+t^2}} \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 1 \end{pmatrix}.$$

Moreover, we find

$$\ddot\gamma(t) = \begin{pmatrix} -2a\sin t - at\cos t \\ 2a\cos t - at\sin t \\ 0 \end{pmatrix}.$$

A simple computation then gives

$$\langle \ddot\gamma(t), \vec{t}(t)\rangle = \frac{at}{\sqrt{2+t^2}},$$

and so

$$\begin{aligned} &\ddot\gamma(t) - \langle \ddot\gamma(t), \vec{t}(t)\rangle \vec{t}(t) = \\ &= \frac{1}{2+t^2} \begin{pmatrix} (2+t^2)(-2a\sin t - at\cos t) - at(\cos t - t\sin t) \\ (2+t^2)(2a\cos t - at\sin t) - at(\sin t + t\cos t) \\ -at \end{pmatrix} \\ &= \frac{1}{2+t^2} \begin{pmatrix} -4a\sin t - 3at\cos t - at^2\sin t - at^3\cos t \\ 4a\cos t - 3at\sin t + at^2\cos t - at^3\sin t \\ -at \end{pmatrix} \end{aligned}$$

The squared length of this vector is

$$\frac{a^2}{(2+t^2)^2}\left(16+18t^2+7t^4+t^6\right)$$

By Gram–Schmidt we therefore get

$$\vec{n}(t) = \frac{1}{\sqrt{16+18t^2+7t^4+t^6}} \begin{pmatrix} -4\sin t - 3t\cos t - t^2\sin t - t^3\cos t \\ 4\cos t - 3t\sin t + t^2\cos t - t^3\sin t \\ -t \end{pmatrix}.$$

(3) Still assuming $a = b$, compute the differential of the first Frenet vector of γ and find κ.

Solution: We find

$$\dot{\vec{t}}(t) = -\frac{t}{(2+t^2)^{\frac{3}{2}}}\begin{pmatrix}\cos t - t\sin t\\ \sin t + t\cos t\\ 1\end{pmatrix} + \frac{1}{\sqrt{2+t^2}}\begin{pmatrix}-2\sin t - t\cos t\\ 2\cos t - t\sin t\\ 0\end{pmatrix}$$

$$= \frac{1}{(2+t^2)^{\frac{3}{2}}}\begin{pmatrix}-t(\cos t - t\sin t) + (2+t^2)(-2\sin t - t\cos t)\\ -t(\sin t + t\cos t) + (2+t^2)(2\cos t - t\sin t)\\ -t\end{pmatrix}$$

$$= \frac{1}{(2+t^2)^{\frac{3}{2}}}\begin{pmatrix}-4\sin t - 3t\cos t - t^2\sin t - t^3\cos t\\ 4\cos t - 3t\sin t + t^2\cos t - t^3\sin t\\ -t\end{pmatrix},$$

so

$$\frac{\dot{\vec{t}}(t)}{|\dot{\gamma}(t)|} = \frac{1}{a(2+t^2)^2}\begin{pmatrix}-4\sin t - 3t\cos t - t^2\sin t - t^3\cos t\\ 4\cos t - 3t\sin t + t^2\cos t - t^3\sin t\\ -t\end{pmatrix}$$

Using Frenet's first formula and the form of $\vec{n}$, we finally obtain

$$\kappa(t) = \frac{\sqrt{16 + 18t^2 + 7t^4 + t^6}}{a(2+t^2)^2}.$$