Solution sheet 2: symmetries of the plane and of the space

Introduction to Geometry, Szilárd Szabó, special thanks to Zoltán Kovács

November the 13th, 2017

Exercise 1. Let ABC be any negatively oriented triangle and AB_1C, BCA_1 and be AC_1B be postively oriented regular triangles. Denote by O the center of AC_1B. Show that then A_1OB_1 is an isosceles triangle with base angles equal to $\pi/6$.

Solution: Consider the rotations $F_{B_1,\pi/3}$ and $F_{A_1,\pi/3}$. Their composition carries B to

$$F_{A_1,\pi/3}(F_{B_1,\pi/3}(B)) = F_{A_1,\pi/3}(C) = A.$$

Denote by ℓ the line joining A_1 and B_1 and by ℓ_A the line passing through A_1 forming angle $\pi/6$ with ℓ (so that ℓ be carried to ℓ_A by a negative rotation by angle $\pi/6$ about A_1). Similarly, denote by ℓ_B the line passing through B_1 forming angle $\pi/6$ with ℓ (so that ℓ be carried to ℓ_B by a positive rotation by angle $\pi/6$ about B_1). Furthermore, denote by R, R_A, R_B the reflections with respect to ℓ, ℓ_A, ℓ_B respectively. Then, we have

$$F_{A_1,\pi/3}F_{B_1,\pi/3} = (R_BR)(RR_A) = R_BR_A.$$

Let O' be the intersection point of ℓ_A and ℓ_B, then we see that the triangle $A_1O'B_1$ is isosceles with base angles equal to $\pi/6$. In particular,

$$F_{A_1,\pi/3}F_{B_1,\pi/3}$$

is rotation by positive angle $2\pi/3$ about O'. We have seen above that this rotation maps B to A. On the other hand, the only rotation by positive angle $2\pi/3$ mapping B to A has center O. We infer that $O = O'$, hence by the above we get that $A_1O'B_1$ is isosceles with base angles equal to $\pi/6$.

Exercise 2. Let ℓ_1, ℓ_2, ℓ_3 be three lines in the plane and R_1, R_2, R_3 respectively the reflections with respect to them. Show that $(R_1R_2R_3)^2$ is a translation.

Solution: First, assume that ℓ_1, ℓ_2, ℓ_3 are all parallel to each other. Then, the composition R_1R_2 is a translation. It is then possible to find a line ℓ_1' such that

$$R_1R_2 = R_1'R_3,$$

where R_1' stands for the reflection with respect to ℓ_1'. We find

$$(R_1R_2R_3)^2 = (R_1'R_3R_3)^2 = (R_1')^2 = \mathrm{Id}.$$

Assume now that the lines are not all parallel. Then, ℓ_2 is not parallel to at least one of ℓ_1, ℓ_3. Without restricting the generality, we suppose that ℓ_2 is not parallel to ℓ_1, i.e. they intersect each other in some point O. Then, the composition R_1R_2 is a rotation $F_{O,\varphi}$ about O. Then, we may find lines ℓ_1', ℓ_2' passing through O and forming the same angle $\varphi/2$ with each other as ℓ_1, ℓ_2 such that in addition ℓ_2' is perpendicular to ℓ_3. Then, $F_{O,\varphi}$ may also be expressed as

$$F_{O,\varphi} = R_1'R_2',$$

where R_1', R_2' stand for the reflections with respect to ℓ_1', ℓ_2' respectively. We then find:

$$R_1R_2R_3 = F_{O,\varphi}R_3 = R_1'R_2'R_3.$$

Let us denote by O' the intersection of ℓ_2' and ℓ_3. Then, $R_2'R_3$ is the half-turn $H_{O'}$ with respect to O'. In particular, we find that $R_1R_2R_3$ is a composition of a reflection with a half-turn, i.e. a glide-reflection. But it is clear that the square of a glide-reflection is a translation.

Exercise 3. Let $ABCDE$ be vertices of a positively oriented regular pentagon. Show that D is the center of the positively oriented regular decagon (i.e., 10-gon) with vertices A and B.

Solution: Let us set $\alpha = ADB\sphericalangle$. It is sufficient to show that

$$\alpha = \frac{2\pi}{10},$$

because then the triangle ADB is isosceles with the same central angle at D as the regular decagon, however this property uniquely characterizes the center of the regular decagon.

Let us denote by O the center of the regular pentagon. Consider the rotation F by angle $\frac{2\pi}{5}$ in the negative direction about O. We will use a description of F as a product of two reflections. Introduce the following lines given by two incident points:

$$\ell_1 = AD$$
$$\ell_2 = DO$$
$$\ell_3 = AO.$$

Denote by R_j the reflection with respect to ℓ_j. Then, we have

$$F = R_3R_2.$$

On the other hand, let F' be rotation centered at D by angle α in the positive direction. We then have

$$F' = R_2R_1.$$

We know from the theory that FF' is a rotation by angle

$$\alpha - \frac{2\pi}{5} \pmod{2\pi}.$$

Now, we see that

$$FF' = (R_3R_2)(R_2R_1) = R_3R_1.$$

In particular, we have that

$$FF'(A) = A, \quad FF'(D) = C,$$

so the center of the rotation obtained by composition is A and its angle is $-\alpha$. We infer that

$$\alpha - \frac{2\pi}{5} \equiv -\alpha \pmod{2\pi},$$

i.e.

$$2\alpha = \frac{2\pi}{5} + k2\pi$$

for some integer k. As $\alpha \in (0, \pi/2)$, we see that

$$\alpha = \frac{2\pi}{10},$$

and we are done.

Exercise 4. Let $n = 2k + 1$ be any odd number bigger than 3. Let $P_0, \ldots, P_{n-1}$ denote consecutive vertices of a regular n-gon, increasingly labelled in the positive direction. Let N denote the intersection point of the diagonals P_0P_2 and P_1P_{k+1}. Show that the length of the segment P_0N is equal to the side of the n-gon.

Solution: Since $P_0P_{k+1}P_1$ is an equilateral triangle, it is sufficient to prove that P_1P_0N is similar to it. As these two triangles share the angle at P_1, we merely need to show that they share another angle too. We will show that $P_1P_0N\sphericalangle = P_0P_{k+1}P_1\sphericalangle$. We achieve this by showing that both of these angles are equal to $P_0P_2P_1\sphericalangle$. Indeed, $P_1P_0N\sphericalangle = P_1P_0P_2\sphericalangle$ (because N is on the diagonal P_0P_2) and $P_1P_0P_2\sphericalangle = P_0P_2P_1\sphericalangle$ (because the triangle $P_0P_1P_2$ is equilateral), show that $P_1P_0N\sphericalangle = P_0P_2P_1\sphericalangle$. On the other hand, we have $P_0P_2P_1\sphericalangle = P_0P_{k+1}P_1\sphericalangle$ because P_0, P_1, P_2, P_{k+1} lie on a circle (the circumscribed circle of the regular n-gon). This finishes the proof.