Billiard Models and Energy Transfer

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Dedicated to the memory of R. L. Dobrushin on the occasion of his 80'th anniversary

For two (or more) interacting classical particles the existing few results (for diffusion or energy transfer, for instance) assume that the mass of one of them - as compared to the other mass - becomes negligible in the limit (cf. [ChD 09]). Here two models are presented for energy transfer in systems with two or more identical hard disks. The first one, a stochastic paradigm for two Lorentz disks, suggests that the joint diffusive limit of two disks is the mixture of independent pairs of Wiener processes ([P-GySz 09]. In the second one it is shown that in a quasi-1D mechanical chain of localized hard disks - in the scaling of [GG 08] - the limit for the energies of the disks is a n. n. interacting Markov process ([SzT 09]. This latter result should open the way toward a rigorous derivation of Fourier law of heat conduction for a deterministic particle system.

Keywords: billiard models, Lorentz disk, energy transfer, joint diffusion, derivation of master equation

1. Introduction

In the last decade the theory of hyperbolic billiards got enriched with highly efficient methods, for instance by LS Young (towers, [Y 98]), by N Chernov and D Dolgopyat (standard pairs, growth properties, averaging, Young coupling á la Chernov-Dolgopyat, [ChD 09]), by P Bálint-IP Tóth (multidim towers, [BT 08]), by I Melbourne, M Nicol, A Torok (probability theory of Young towers, citeMN09, MT04), by D Szász, T Varjú (local limit laws, [SzV 04]), etc. An superb reference is the survey at IMC06, [ChD 07] on what had happened until and was happening around 2006. Consequently, it became possible to achieve essential steps toward the main goal of statistical physics: the derivation of macroscopic behavior from microscopic laws, for example, in the study of diffusion or Fourier's law of heat conduction. For two (or more) interacting particles, however, so far there has been no results except when the mass of one of them - as compared to the other mass - becomes negligible in the limit (cf. [ChD 09]). Here we present results in both of the aforementioned questions of physics which treat or are motivated by the dynamics of Lorentz disks of equal masses: 2D models of elastic disks moving among periodic scatterers. In both of them energy transfer is at the heart of the problem.

2. Diffusive limit of two Lorentz disks

For simplicity consider two moving disks of radii ρ among a periodic configuration of disjoint circular scatterers. Assume the scatterer configuration has finite horizon and ρ is so small that the disks can move unboundedly. For understanding the joint limit of the two disks in the diffusive scaling, we present a stochastic model.

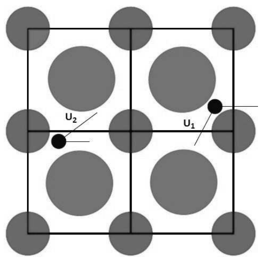

Fig. 1. Four cells of the infinite scatterer configuration (red). Dark circles - moving disks. Grey ones - scatterers.

Definition 2.1. (Continuous time random walk with internal states with general state space.) Assume we are given a rate $\lambda > 0$ and a family $\{P_x(u,.)|x \in \mathbb{Z}^d \setminus \{0\}\}$ of substochastic kernels over $\tilde{H}$ such that $Q = \sum_{x\in\mathbb{Z}^d\setminus\{0\}} P_x$ is a stochastic kernel over $\tilde{H}$. A continuous time pure jump Markov process $\{\xi_t = (x_t, u_t)\}$ - where $(x_t, u_t) \in \mathbb{Z}^d \times \tilde{H}$ - is called a (generalized) Random Walk with Internal States (RWwIS) if

$$\mathbb{P}(\xi_{t+dt} = \xi_t) = 1 - \lambda dt + o(dt)$$

and for every $(x_t, u) \in \mathbb{Z}^d \times \tilde{H}$ and $\forall u \in \tilde{H}, \forall A \subset \tilde{H}\ \ x_{t+dt} - x_t \neq 0$

$$\mathbb{P}(\xi_{t+dt} = (x_{t+dt}, u'), u' \in A|\xi_t = (x_t, u)) = \lambda P_{x_{t+dt}-x_t}(u, A)dt + o(dt).$$

In our stochastic model the motions of the two particles are independent RWwIS's unless the two particles occupy the same lattice point. Let $\xi_t^i = (\eta_t^i, \varepsilon_t^i),\ i = 1, 2$ be two RWwIS with the same kernels but different rates λ_1, λ_2 which only change their values in the moments of collisions. Whenever $\eta_t^1 \neq \eta_t^2$, the joint generator of the two Markov processes is the product of the two individual generators (modeling two independent Lorentz processes).

Next we define the collision interaction. Whenever $\eta_t^1 = \eta_t^2 \ (= x)$,

$$\begin{aligned}\mathbb{P}(\xi_{t+}^1 = (x+z^1, v_+^1), \xi_{t+}^2 =& (x+z^2, v_+^2); v_+^1 \in A^1, v_+^2 \in A^2 \\ &|\xi_{t-}^1 = (x, v_-^1), \xi_{t-}^2 = (x, v_{t-}^2)) \\ &= C_{z^1,z^2}(v_-^1, v_-^2, A^1, A^2)\end{aligned}$$

is the collision kernel. We assume that C satisfies conservation of energy: $(v_-^1)^2 + (v_-^2)^2 = (v_+^1)^2 + (v_+^2)^2$ (momentum is not conserved since the collision kernel contains averaging over normal of impact). We can and do assume that $(v^1)^2 + (v^2)^2 = 1$. Therefore the state space of the two particle process is isomorphic to $\left(\mathbb{Z}^2 \times S\right)^2 \times \tilde{\mathcal{I}}$ where $\tilde{\mathcal{I}} = [0,1]$. It is worth noting that the concrete form of the collision kernel for the mechanical two disk model is calculated in Appendix A of [GG 08].

Theorem 2.1. *(Joint with Zs. Pajor-Gyulai, [P-GySz 09])*
For every initial distribution of (ξ_0^1, ξ_0^2), *the density function of the weak limit law of* $(\frac{1}{\sqrt{t}}\eta_{t,\lambda_0}^1, \varepsilon_t^1, \frac{1}{\sqrt{t}}\eta_{t,\lambda_0}^2, \varepsilon_t^2)$ *exists and is equal to*

$$h(x_1, v_1, x_2, v_2) = \frac{\rho(v_1)\rho(v_2)}{(2\pi)^2|\sigma|}\int_0^1 \frac{1}{\lambda\sqrt{1-\lambda^2}} e^{-\frac{1}{2}\left(\frac{x_1^T\sigma^{-1}x_1}{\lambda} + \frac{x_2^T\sigma^{-1}x_2}{\sqrt{1-\lambda^2}}\right)} d\rho_s(\lambda)$$

where $\rho_s(\lambda)$ *is the outgoing stationary distribution of the speed of the first particle in the collision Markov chain,* ρ *is the stationary density of the internal states on* S *of the component RWwIS's.*

We note that in earlier models of joint motion of two identical particles the diffusive limits of the two motions were either independent or got glued together (cf. [Sz 80], [KV 86]).

3. Interacting Markov chain of energies from deterministic dynamics

[GG 08] considered a quasi-1D version of the localized hard ball system of [BLPS 92]. It is a chain of 2D disks each performing a billiard dynamics in its cell and *interacting with the neighboring disks rarely - under their special choice of parameters*. Concretely the parameters are: box size: l (with periodic b. c.'s along y-axis); chain length $= N$ (along the x-axis, with free or periodic boundary conditions); radius of scatterers (shaded circles)$= \rho_f$; radius of moving disks (empty circles) $= \rho_m$; $\rho_f + \rho_m = \rho$ is being kept fixed; condition of localization: $\rho > l/2$ (and, of course, $\rho < l/\sqrt{2}$); condition of conductivity: $\rho_m > \rho_{crit} = \sqrt{(\rho^2 - (l/2)^2}$; finally the small parameter in the model is small $\varepsilon = \rho_m - \rho_{crit}$. Under these conditions they, in the limit $\varepsilon \to 0$, derive a master equation for the time evolution of the energies of the disks from the kinetic equation of the mechanical motion. (Moreover, they also treat the master equation for obtaining the coefficient of heat conductivity: $\kappa = \sqrt{T}$ (T being the temperature).

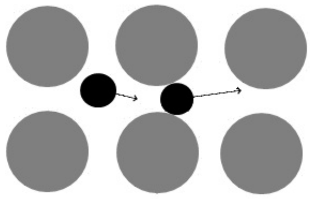

Fig. 2. A chain of length $N = 2$. Dark circles - moving disks. Grey ones - scatterers.

The underlying fact is that, in the limit $\varepsilon \to 0$, the disk chain becomes an uncoupled system of Sinai billiards. In other words, by denoting the wall collision rate by $\nu_{\text{wall},\varepsilon}$ (i. e. for the collisions of the disks with the fixed boundaries of the domain) and the binary collision rate by $\nu_{\text{bin},\varepsilon}$ (i. e. that of the inter-disk-collisions), one has a separation of time scales since as $\varepsilon \to 0$, $\nu_{\text{wall},\varepsilon}(\sim \nu_{wall,crit} > 0) \gg \nu_{\text{bin},\varepsilon} \to 0$. Though the original N-disk system is a semi-dispersing billiard in a $2N-1$-dimensional configuration space, in the given small coupling limit most of the time the disks evolve according to uncoupled, 2D dispersing billiard dynamics. Thus - on the time scale $\nu^{-1}_{\text{wall},\varepsilon}$ - there is an averaging and the binary collisions only occur on the time scale $\nu_{\text{bin},\varepsilon}$.

Theorem 3.1. *(joint work with IP Tóth, [SzT 09])*
$N = 2$, free boundary along x-axis. Dynamics: $(M_\varepsilon = \{q_1, v_1; q_2, v_2 | dist(q_1, q_2) \geq 2\rho_m, v_1^2 + v_2^2 = 1\}), S^{\mathbb{R}}, \mu_\varepsilon)$. Denote by $0 < \tau_{1,\varepsilon} < \tau_{2,\varepsilon} < \dots$ the successive binary collision times of the two disks. Then, as $\varepsilon \to 0$

- *$(E_1(\nu_{\text{bin},\varepsilon}t), E_2(\nu_{\text{bin},\varepsilon}t))$ converges to a jump Markov process on the state space $E_1 + E_2 = 1$ where $E_j(t) = \frac{1}{2}v_j^2(t); j = 1, 2$*
- *the transition kernel $k(E_1^+|E_1^-)$ can be calculated; it is, in fact, a verification of Boltzmann's 'microscopic chaos' property for this model (cf. G-G, '08, and also Theorem 2.1).*

In other words the intercollision times for binary collisions are asymptotically exponential where the rate of the exponential clock depends on the energies of the disks. It is also worth mentioning that in the case $N = 2$ $\nu_{\text{bin},\varepsilon} \sim \text{const } \varepsilon^3$.

Idea of proof.

- since binary collisions are rare, most of the time the two disks evolve independently
- between two binary collisions - with an overwhelming probability - there is averaging in each of the in-cell, 2D billiard dynamics
- for these typically long time intervals it is natural to apply the Chernov-Dolgopyat averaging, cf. [ChD 09]
- for that purpose
  - one checks that for an incoming proper family of stable pairs, so is

the outgoing family, cf. [ChD 09]

– one applies martingale approximation for jump processes (á la Ethier-Kurtz, [EK 86]).

Theorem 3.2. *(joint work with IP Tóth, [SzT 09])*
$N \geq 2$. Dynamics: $(M_\varepsilon = \{q_1, v_1; \dots, q_N, v_N | dist(q_j, q_j + 1) \geq 2\rho_m, v_1^2 + \dots v_2^N = 1\}), S^{\mathbb{R}}, \mu_\varepsilon)$.

- $(E_1(\nu_{\mathrm{bin},\varepsilon} t), \dots, E_N(\nu_{\mathrm{bin},\varepsilon} t))$ *converges to a jump Markov process on the state space* $E_1 + \dots E_N = 1$ *where* $E_j(t) = \frac{1}{2} v_j^2(t); j = 1, \dots, N$
- *the transition kernel* $k(E_1^+, \dots, E_N^+ | E_1^-, \dots, E_N^-)$ *can be expressed as the sum of the binary collision kernels of Theorem 3.1.*

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