Lorentz Process with shrinking holes in a wall

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November 26, 2011

Abstract

We ascertain the diffusively scaled limit of a periodic Lorentz process in a strip with an almost reflecting wall at the origin. Here, almost reflecting means that the wall contains a small hole waning in time. The limiting process is a Quasi-reflected Brownian Motion, which is Markovian but not strong Markovian.

The periodic Lorentz process is a fascinating non-linear, chaotic model that has been deeply investigated in the last decades. The model is very simple: a massless point particle moves freely in the plane (or in our case, in a strip) until it hits one of the periodically situated smooth convex scatterers, when it is reflected. The limit of the diffusively scaled trajectory of the particle is known to be the Brownian motion. Further, if the particle is restricted to a half strip, then the scaling limit is going to be the so-called reflected Brownian motion. Here we introduce a time-dependent scatterer configuration (by adding a vertical wall with a shrinking hole) that almost confines the particle to the half strip in such a way that the scaling limit is

*The support of the Hungarian National Foundation for Scientific Research grant No. K 71693 is gratefully acknowledged. The authors also thank the kind hospitality of the Fields Institute (Toronto) where - during June 2011 - part of this work was done.

a quasi-reflected Brownian motion, a natural generalization of both the Brownian motion and the reflected Brownian motion.

1 Introduction

In the last decade after a broad and thorough study of Sinai billiards - or equivalently of periodic Lorentz processes - the non-homogeneous case got also widely examined. Here, non-homogeneity may appear either in time (cf. [3] as to a mechanical model of Brownian motion or [7], [4], [9] as to models of Fermi acceleration) or in space (cf. [6] as to local perturbations of periodic Lorentz processes). In the present work we investigate a question with non-homogeneity in time. Consider a periodic Lorentz Process with a finite horizon given in a horizontal strip, where the scatterer configuration is assumed to be symmetric to a vertical axis - through the origin, say. Now, put a vertical wall at the symmetry axis and a tiny hole onto the wall. The hole is getting smaller and smaller with time, thus giving the particle less and less chance to cross the wall. Our aim is to find a non trivial scaling limit of the trajectory of the particle. Here, non trivial means that it is neither Brownian motion (BM), nor Reflected Brownian motion (RBM) since, if the hole was of full size or absent, then these two processes would appear in the limit (see [6]).

Indeed, if one takes the hole arbitrarily small, but fixed of size $\varepsilon > 0$, then the limiting process is a BM whereas if the hole is empty, then it is a RBM. The essence of this observation is that the limiting process does not change continuously as $\varepsilon \to 0$ and our goal is precisely to understand the situation when the limit is taken in a more delicate, time-dependent way.

To be more precise, let the configuration space in the absence of the wall be $\mathcal{T} := (\mathbb{R} \times [0,1]) \setminus \cup_{i=1}^{\infty} O_i$ where O_i is a $\mathbb{Z}$-periodic configuration of open, strictly convex, pairwise disjoint, smooth scatterers, the union of which is assumed to be symmetric to the y-axis. The wall without the hole is $W_\infty = \{(x,y) \in \mathcal{T} |\, x = 0\} = \cup_{k=1}^{K} (\mathcal{J}_{k,l}, \mathcal{J}_{k,r})$ where the subintervals of the y axis, denoted by $(\mathcal{J}_{k,l}, \mathcal{J}_{k,r})$, are the connected components of W_∞. The holes will be subintervals $I_n \subset W_\infty$, thus we will be considering a sequence $\{W_n = W_\infty \setminus I_n\}_n$ of walls. Since we will work with the Poincaré section, the so-called billiard ball map of the Lorentz process, in the presence of the wall W_n its configuration space will, of course, be $\cup_i \partial O_i \cup W_n$. Further, define $\mathcal{J} \subset \cup \partial O_i \times [-\pi/2, \pi/2]$ as the points from which before the forthcoming collision, the particle crosses $\cup_{i=1}^{K} (\mathcal{J}_{i,l}, \mathcal{J}_{i,r})$.

Write $\mathcal{I}_n = \{I_k\}_{1 \le k \le n}$ and denote by $\mathcal{F}_n : \mathcal{T}_n \to \mathcal{T}_{n+1}$ the billiard ball map and $\kappa_n : \mathcal{T}_n \to \mathcal{T}_{n+1}$ the projection to the horizontal direction of the free flight vector from the billiard table $\mathcal{T}_n$ to $\mathcal{T}_{n+1}$, and write

$$S_n(x, \mathcal{I}_n) = S_n(x) = \sum_{k=1}^{n} \kappa_k \mathcal{F}_{k-1} \dots \mathcal{F}_1(x),$$

where $x = (q, v)$ is an element of the usual Poincaré section.

What remains, is the definition of the holes I_n. For this, fix some sequence $\underline{\alpha}$ with $\alpha_n \to 0$ and, independently of each other, choose uniformly distributed points ξ_n, $n \ge 1$ on $\cup_{i=1}^{K} (\mathcal{J}_{i,l}, \mathcal{J}_{i,r} - \alpha_n)$, (if $\mathcal{J}_{i,r} - \alpha_n < J_{i,l}$, then $(\mathcal{J}_{i,l}, \mathcal{J}_{i,r} - \alpha_n) = \emptyset$ by definition). We will use the following three special choices:

1. Define the holes as $I_n = (\xi_n, \xi_n + \alpha_n)$ which makes sense for n large enough. With this particular choice write

$$S_n^{\searrow}(x, \underline{\alpha}) = S_n^{\searrow}(x) = S_n(x, \mathcal{I}_n)$$

and

$$\mathcal{F}_n^{\searrow} = \mathcal{F}_n.$$

2. For each $1 \le k \le n$, let the random variables $\xi_n^{(k)}$ be independent and distributed like ξ_n. Define $\mathcal{I}^{(n)} = \{(\xi_k^{(n)}, \xi_k^{(n)} + \alpha_n)\}_{1 \le k \le n}$. With this

particular choice write

$$S_n^{\equiv}(x,\underline{\alpha}) = S_n^{\equiv}(x) = S_n(x,\mathcal{I}^{(n)}).$$

3. Let $I_n = [0,1]$. With this special choice, write

$$S_n^{(per)}(x) = S_n(x,\mathcal{I}_n)$$

and

$$\mathcal{F}^{(per)} = \mathcal{F}_1.$$

Here the first choice - the only really time dependent - is the most interesting one. The second choice is a sequence of billiards (a double array, in other words), while the third one is just a usual periodic Lorentz Process.

Now, we turn to the definition of the limiting processes. Since we are going to have two very similar processes, we call both Quasi-reflected Brownian Motions and distinguish between them only in the abbreviation.

Consider a BM $\mathcal{B} = (B_t)_{0\le t\le 1}$ with parameter σ on $[0,1]$. Its local time at the origin is denoted by $(L_t)_{0\le t\le 1}$. Now, given $\mathcal{B}$, consider a Poisson Point Process Π with intensity measure cdL with some positive constant c. Then with probability one, the support of the measure cdL is Z, where $Z = \{s : 0 \le s \le 1 : B_s = 0\}$ is the zero set of $\mathcal{B}$. Denote the points of Π by $P_1, P_2, ...$ in decreasing order. In fact, Π has finitely many points. If it has m points, then put $P_{m+1} = P_{m+2} = ... = 0$. Further, put $P_0 = 1$ and introduce a Bernoulli distributed random variable η with parameter $1/2$ that is independent of B_t and Π.

Now, the process Q on $[0,1]$ with $Q_0 = 0$ and

$$Q_t = \begin{cases} (-1)^{\eta}|\mathcal{B}_t| & \text{if } \exists n \in \mathbb{Z}_+ \cup \{0\} : t \in (P_{2n+1}, P_{2n}] \\ (-1)^{1-\eta}|\mathcal{B}_t| & \text{otherwise} \end{cases}$$

is called the quasi-reflected Brownian Motion with parameters c and σ, and denoted by qRBM(c,σ).

The definition of QRBM is similar to that of qRBM. The difference is that cdL should now be replaced by $c\frac{1}{\sqrt{t}}dL_t$. As a result, the Poisson process will have infinitely many points, which accumulate only at the origin. Now, denote by $P_1, P_2, \ldots$ these points in decreasing order (N.B. there is no smallest one among them), put $P_0 = 1$ and define η and QRBM(c,σ) as before.

Remark 1. *One can easily check the following statements. The qRBM(c,σ) is almost surely continuous on $[0,1]$, homogeneous Markovian but not strong Markovian (think of the stopping time $T = \min\{t > 1/2 : Q_t = 0\} \wedge 1$) and Q_t has Gaussian distribution with mean zero and variance $t\sigma^2$.*
The QRBM, similarly to the qRBM, is continuous, Markovian (however not time homogeneous), not strong Markovian, and has the same one dimensional distributions as qRBM. Contrary to the qRBM, the QRBM is self similar in the following sense: if Q_t is a QRBM, then

$$(Q_t)_{t\in[0,1/p]} =^d \left(\frac{1}{\sqrt{p}}Q_{pt}\right)_{t\in[0,1/p]},$$

where $1 < p$.
Further, one can easily extend the definition of both processes to $\mathbb{R}_+$.

As usual, $C[0,1]$ will denote the space of continuous functions and $D[0,1]$ the Skorokhod space over $[0,1]$. We will also use evident modifications, for instance, $D_{\mathbb{R}^2}[t_0,1]$ will denote the Skorokhod space of vector valued functions over an interval $[t_0,1]$.
Let the function $\mathbf{W}_n^{\searrow}$ be the following: $\mathbf{W}_n^{\searrow}(k/n) = S_k^{\searrow}/\sqrt{n}$ for $0 \le k \le n$ and define $\mathbf{W}_n^{\searrow}(t)$ for $t \in [0,1]$ as its piecewise linear, continuous extension. Let $\mu_n^{\searrow}$ denote the measure on $C[0,1]$ induced by $\mathbf{W}_n^{\searrow}$, where the initial distribution, i.e. the distribution of x, is the Liouville measure on the phase space of the billiard map on $\mathcal{T}$ restricted to the two neighboring tori to the origin. Analogously, define $\mu_n^{\equiv}$ with $\mathbf{W}_n^{\equiv}$, where $\mathbf{W}_n^{\equiv}(k/n) = S_k^{\equiv}/\sqrt{n}$.
Now we can formulate our main result.

Theorem 1. *There are positive constants σ and c_1 depending only on the periodic scatterer configuration, such that*

1. *if $\exists c > 0 : \alpha_n \sqrt{n} \to c$, then $\mu_n^{\searrow}$ converges weakly to the measure induced by $QRBM(c_1 c, \sigma)$.*

2. *if $\exists c > 0 : \alpha_n \sqrt{n} \to c$, then $\mu_n^{\equiv}$ converges weakly to the measure induced by $qRBM(c_1 c, \sigma)$.*

3. *if $\alpha_n \sqrt{n} \to 0$, then both $\mu_n^{\searrow}$ and $\mu_n^{\equiv}$ converge weakly to the convex combination of the measures induced by RBM and -RBM with weights $1/2$.*

4. *if $\alpha_n \sqrt{n} \to \infty$, then both $\mu_n^{\searrow}$ and $\mu_n^{\equiv}$ converge weakly to the Wiener measure.*

Instead of introducing the holes on the wall one could think about the wall as a trapdoor, i.e. sometimes it is open and then the particle crosses it without collisions, other times it is closed. If one opens the door randomly with probability α_n, then obtains the same result.

The analogue of Theorem 1 for random walks is, of course, easy to formulate in the following way. Define the stochastic process $\mathbb{S}_n$ by: $Prob(\mathbb{S}_0 = 1) = Prob(\mathbb{S}_0 = -1) = 1/2$ and for $k > 0$:

$$Prob(\mathbb{S}_{k+1} = \mathbb{S}_k + 1 | \mathbb{S}_k \neq 0) = Prob(\mathbb{S}_{k+1} = \mathbb{S}_k - 1 | \mathbb{S}_k \neq 0) = 1/2,$$

and

$$Prob(\mathbb{S}_{k+1} = \mathbb{S}_{k-1} | \mathbb{S}_k = 0) = 1 - \epsilon, \tag{1}$$

$$Prob(\mathbb{S}_{k+1} = -\mathbb{S}_{k-1} | \mathbb{S}_k = 0) = \epsilon. \tag{2}$$

Here - and also in the sequel - $Prob$ stands for some abstract probability measure.

In the definition of $\mathbb{S}_k$ put first $\epsilon = \alpha_k$ and denote by $\nu_n^{\searrow}$ the measure on $C[0,1]$ induced by $\mathbf{W}_n$, where $\mathbf{W}_n(k/n) = \mathbb{S}_k / \sqrt{n}$ for $0 \leq k \leq n$ and is linearly interpolated in between. Analogously, define $\nu_n^{\equiv}$ for each n with the choice

$\epsilon = \alpha_n$. Then, if we replace each μ with ν in Theorem 1, then the statement remains true and can be proven the same way as we prove Theorem 1.

Let $\mathbf{P}$ denote the probability measure generated by the particular choice of x which is the Liouville measure on the phase space of the billiard map on $\mathcal{T}$ restricted to the two neighboring tori to the origin, and let ϕ denote the density of the standard normal law.

The following statement is a reformulation of Theorem 4.2 in [13] and Theorem 3.4 in [5].

Proposition 1. *Let k be any integer, and $b_m^{(1)}, b_m^{(2)}, ..., b_m^{(k)}$, sequences such that $b_m^{(i)}/\sqrt{m} \to b^{(i)} \in \mathbb{R}$ and let $\mathcal{Z}$ be any subset of the set $\{i : \forall m : b_m^{(i)} = 0\}$. If $n_1, ..., n_k \to \infty$, then*

$$\begin{aligned}
\mathbf{P} \quad & \big(\lfloor S_{n_1}^{(per)}(x) \rfloor = b_{n_1}^{(1)}, \lfloor S_{n_1+n_2}^{(per)}(x) \rfloor = b_{n_1}^{(1)} + b_{n_2}^{(2)}, \ldots, \\
& \lfloor S_{n_1+n_2+\ldots n_k}^{(per)}(x) \rfloor = b_{n_1}^{(1)} + b_{n_2}^{(2)} \ldots, + b_{n_k}^{(k)}, \forall j \in \mathcal{Z} : \left(\mathcal{F}^{(per)}\right)^{n_j}(x) \in \mathcal{J}\big) \\
\sim \quad & c_0^{|\mathcal{Z}|} \prod_{i=1}^{k} \frac{\phi(\frac{b^{(i)}}{\sigma})}{\sigma\sqrt{n_i}},
\end{aligned}$$

where σ and c_0 are universal constants depending only on the periodic scatterer configuration.

The proof of Theorem 1 is presented in Section 2.

2 Proof of Theorem 1

Note that, though in its spirit our statement is very close to the results of [6], their proof cannot be applied here since the limiting process is not strong Markovian (see Remark 1) thus leaving no chance to apply the martingale method.

2.1 Proof of case 1

Our actual proof consists of four steps.

Step 1

Choose $0 < t_0 < 1$ and a_n such that $a_n/\sqrt{n} \to a \in \mathbb{R} \setminus \{0\}$. Restrict the measure $\mathbf{P}$ to such points x where $\lfloor S^{(per)}_{\lfloor nt_0 \rfloor}(x) \rfloor = a_n$ and rescale it to obtain a probability measure. The resulting measure is denoted by $\mathbf{P_n}$.

Denote by L_{nt}, $t \in [t_0, 1]$ the number of visits to $\mathcal{J}$ in the time interval $[\lfloor nt_0 \rfloor, \lfloor nt \rfloor]$.

In step 1, we are going to prove the following statement.

Lemma 1.

$$\left(\frac{S^{(per)}_{nt}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}} \right)_{t \in [t_0, 1]} \Rightarrow (\mathcal{W}_t, c_0 \mathcal{L}_t)_{t \in [t_0, 1]},$$

as $n \to \infty$ where the left hand side is understood as a random variable with respect to the probability measure $\mathbf{P_n}$; $\mathcal{W}$ is a Brownian motion with parameter σ starting from a at time t_0, and $\mathcal{L}$ is its local time at the origin. Further, $\Rightarrow$ stands for weak convergence in the Skorokhod space $D_{\mathbb{R}^2}[t_0, 1]$.

Note that $S^{(per)}_{nt}$ is continuous by definition, but L_{nt} is just a càdlàg process, that is why the Skorokhod space is needed.

Proof of Lemma 1. As usual, one has to check the convergence of finite dimensional distributions and the tightness.

For the first, we are going to prove the following statement.

Sublemma 1. *For any sequence b_n, such that $b_n/\sqrt{n} \to b \in \mathbb{R} \setminus \{0\}$ and any $t_0 < t_1 \leq 1$, as $n \to \infty$,*

$$\frac{\sigma \sqrt{n(t_1 - t_0)}}{\phi(\frac{b-a}{\sigma\sqrt{t_1 - t_0}})} \mathbf{P}_n \left(\lfloor S^{(per)}_{nt_1} \rfloor = b_n, L_{nt_1} < c_0 \sqrt{n} z \right) \tag{3}$$

converges to the probability that the local time at the origin of a Brownian bridge (with parameter σ) of length $t_1 - t_0$ starting from a and arriving at b is smaller than z.

Proof of Sublemma 1. Denote by $\mathbf{Q_n}$ the conditional measure of $\mathbf{P_n}$ to such points where $\lfloor S_{nt_1}^{(per)} \rfloor = b_n$. Proposition 1 yields that it is enough to prove that $L_{nt_1}/\sqrt{n}$ - as a random variable with respect to $\mathbf{Q_n}$ - weakly converges to the appropriate limit.

Similarly to the proof of Theorem 9 in [5] we are going to use the "method of moments".

Proposition 1 implies that for any $\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < ... < n_k < n_{k+1} = \lfloor nt_1 \rfloor$,

$$\begin{aligned}
& \mathbf{Q_n}(\{\left(\mathcal{F}^{(per)}\right)^{n_i} x, 1 \le i \le k\} \subset \mathcal{J}) \\
\sim \quad & \frac{c_0^k}{\sigma^k (2\pi)^{\frac{k-1}{2}}} \phi\left(\frac{a_n}{\sigma\sqrt{n_1 - n_0}}\right) \phi\left(\frac{b_n}{\sigma\sqrt{n_{k+1} - n_k}}\right) \frac{\sqrt{n(t_1 - t_0)}}{\phi\left(\frac{b_n - a_n}{\sigma\sqrt{n(t_1 - t_0)}}\right)} \prod_{i=1}^{k+1} \frac{1}{\sqrt{n_i - n_{i-1}}}.
\end{aligned}$$

Hence, for $k \ge 2$,

$$\begin{aligned}
& \mathbb{I}_n^k := \int (L_{nt_1})^k d\mathbf{Q_n} \\
\sim \quad & k! \sum_{\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < ... < n_k < n_{k+1} = \lfloor nt_1 \rfloor} \mathbf{Q_n}(\{\left(\mathcal{F}^{(per)}\right)^{n_i} x, 1 \le i \le k\} \subset \mathcal{J}) \\
\sim \quad & n^{\frac{k}{2}} k! c_0^k \sigma^k (2\pi)^{-\frac{k-1}{2}} \sqrt{t_1 - t_0} \left[\phi\left(\frac{b-a}{\sigma\sqrt{t_1 - t_0}}\right)\right]^{-1} \\
& \int \cdots \int_{0 < s_1 < s_2 < ... < s_k < t_1 - t_0} d\underline{s} \\
& \phi\left(\frac{a}{\sigma\sqrt{s_1}}\right) \frac{1}{\sqrt{s_1}} \frac{1}{\sqrt{s_2 - s_1}} \cdots \frac{1}{\sqrt{s_k - s_{k-1}}} \frac{1}{\sqrt{t_1 - t_0 - s_k}} \phi\left(\frac{b}{\sigma\sqrt{t_1 - t_0 - s_k}}\right) \\
\sim \quad & c_0^k \sigma^k n^{\frac{k}{2}} k! (2\pi)^{-\frac{k-1}{2}} \sqrt{t_1 - t_0} \left[\phi\left(\frac{b-a}{\sigma\sqrt{t_1 - t_0}}\right)\right]^{-1} \left[\Gamma\left(\frac{1}{2}\right)\right]^{k-1} \frac{1}{\Gamma\left(\frac{k}{2}\right)} \\
& \int\int_{0 < s_1 < s_2 < t_1 - t_0} ds_1 ds_2 \phi\left(\frac{a}{\sigma\sqrt{s_1}}\right) \phi\left(\frac{b}{\sigma\sqrt{s_2 - s_1}}\right) \frac{1}{\sqrt{s_1}} \frac{1}{\sqrt{s_2 - s_1}} (1 - s_2)^{\frac{k}{2} - \frac{3}{2}}
\end{aligned}$$

where $\underline{s} = (s_1, \ldots, s_k)$.

Thus, it would be enough to prove that $\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2} = \mathbb{J}^k$ where $\mathbb{J}^k$ is the k-th moment of the local time at the origin of a Brownian bridge. The probability of this local time being larger than y is

$$\exp\left[-\frac{1}{2\sigma^2(t_1-t_0)}\left(\left(|a|+|b|+\sigma^2 y\right)^2-(b-a)^2\right)\right]$$

(see [2] or [11]), hence its moments can be expressed with an integral. Note that, by using the above asymptotics, one can easily deduce

$$\limsup_{k\to\infty}\left(\frac{\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2}}{k!}\right)^{1/k} < \infty \tag{4}$$

yielding that the weak limit does exist. However, proving $\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2} = \mathbb{J}^k$ turns out to be a nontrivial computation.

Thus we need to argue in a more complicated way. That is, we are going to prove the statement of the Sublemma for random walks and then - since the moments for the random walk have the same asymptotic behavior, and these moments do converge - we arrive at the original statement.

To be more precise, pick a one dimensional simple symmetric random walk that starts from a_n and denote its position after $n(t_1-t_0)$ steps by X_n. Similarly, its total number of visits to the origin until time $n(t_1-t_0)$ is denoted by Y_n. Further, denote by (X,Y) the position X and the local time Y until t_1-t_0 at the origin of a standard BM starting from a. We denote by $f_X(x)$ the probability density function of X. Similarly, $F_{Y|X}(y|x)$ stands for the conditional cumulative distribution function of Y under the condition $X=x$. Let x_n be a sequence of integers such that $x_n/\sqrt{n}\to x$

The following two statements are well known for random walks (see for example [2] and [10]):

$$Prob\left(X_n < x_n, \frac{Y_n}{\sqrt{n}} < y\right) \to Prob(X<x, Y<y), \tag{5}$$

and

$$\frac{\sqrt{n}}{2} Prob\left(X_n \in \{x_n, x_n+1\}\right) \to f_X(x). \tag{6}$$

We want to prove that

$$\frac{\sqrt{n}}{2} Prob\left(\frac{Y_n}{\sqrt{n}} < y, X_n \in \{x_n, x_n+1\}\right) := p_n(x,y) \to f_X(x) F_{Y|X}(y|x). \quad (7)$$

Note that (7) is proved in [12] for the case $x = 0$. In the general case, using (6), one easily sees that (7) is equivalent to the statement that under the condition $X_n \in \{x_n, x_n+1\}$, $Y_n/\sqrt{n}$ converges weakly to $Y|X = x$. But the moments of $Y_n/\sqrt{n}$ under the condition $X_n \in \{x_n, x_n+1\}$, have the same asymptotics as $\mathbb{I}_n^k c_0^{-k} n^{-k/2}$, with b replaced by x, and $\sigma = 1$. Thus (4) implies that $p(x,y) := \lim_{n\to\infty} p_n(x,y)$ exists. Now suppose that there exist some x, y such that $p(x,y) \neq f_X(x) F_{Y|X}(y|x)$. For this fixed y both $p(x,y)$ and $f_X(x) F_{Y|X}(y|x)$ are continuous in x since the integral representation for $\mathbb{I}_n^k c_0^{-k} n^{-k/2}$ is continuous in a and b. Thus one can find an interval $I = [x-\delta, x+\delta]$ such that $\int_I p(x,y)dx \neq \int_I f_X(x) F_{Y|X}(y|x)dx$, which is a contradiction to (5). Thus we have verified (7). But (7) together with (4) implies $\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2} = \mathbb{J}^k$. □

Since the limiting process has independent increments, using Proposition 1, one can prove the convergence of any finite dimensional distribution in the same way as we did for the one dimensional ones.

Now we turn to the proof of tightness. It is well known that the first coordinate converges weakly to the desired limit (in $C[t_0, 1]$ thus in $D[t_0, 1]$ as well), hence is tight, too. Thus we only need to establish the tightness of the local times.

Since the process L_{nt} is nondecreasing in t, tightness can be deduced from the convergence of finite dimensional distributions. Namely, Theorem 15.2 in [1] yields that we only have to verify the following two statements:

1. For each $\eta > 0$ there is a b such that

$$\mathbf{P_n}\left(\frac{L_n}{\sqrt{n}} > b\right) < \eta, \ \ n \geq 1. \quad (8)$$

2. For each positive η and ε there is a δ, $0 < \delta < 1$ and an integer n_0 such

that

$$\mathbf{P_n}\left(w_{\frac{L_{nt}}{\sqrt{n}}}(\delta) \geq \varepsilon \right) \leq \eta,\ n \geq n_0. \tag{9}$$

Here,

$$w_\psi(\delta) = \inf_{\{t_i\}} \max_{0<i\leq r} (\lim_{\tau \nearrow t_i} \psi(\tau) - \psi(t_{i-1})),$$

where the infimum is taken over finite sets $\{t_i\}$, for which $t_0 < t_1 < ... < t_r = 1$, $t_i - t_{i-1} > \delta$ for all i.

Since we have just verified that $L_n/\sqrt{n}$ converges weakly, (8) follows.

Again, the convergence of finite dimensional distributions implies that for fix $\eta > 0$ and $\varepsilon > 0$ one can find $\delta > 0$ and n_0 such that for all $n > n_0$, $0 \leq k_1 \leq \lfloor 1/\delta \rfloor$

$$\mathbf{P_n}\left(\frac{\#\{k : nk_1\delta < k < n(k_1+1)\delta, S_k^{(per)} \in \mathcal{J}\}}{\sqrt{n\delta}} > \frac{\varepsilon}{\sqrt{\delta}} \right) < \eta\delta.$$

Now the equidistant partition $\{t_i\}$ is enough to verify (9).

□

Step 2

In this step we prove the following statement which is a stronger version of Lemma 1:

Lemma 2. *With the notations of Lemma 1,*

$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}}, \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\frac{L_{n\tau}}{\sqrt{n}} \right)_{t\in[t_0,1]} \Rightarrow \left(\mathcal{W}_t, c_0\mathcal{L}_t, c_0 \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathcal{L}_\tau \right)_{t\in[t_0,1]},$$

where $\Rightarrow$ *stands for weak convergence on the* $D_{\mathbb{R}^3}[t_0, 1]$ *Skorokhod space.*

Note that we needed to introduce a and t_0 in order that the above stochastic integrals be finite. This is a technical difficulty which does not appear in the case of $\mu^{\equiv}$.

Proof of Lemma 2. To prove the lemma we again need a Sublemma.

Sublemma 2. *Let E and F be any Polish spaces, X, X_n any random variables taking values in the space E such that $X_n \Rightarrow X$. Then for any continuous function $f : E \to F$ one has $(X_n, f(X_n)) \Rightarrow (X, f(X))$ in the product topology.*

Proof. Pick any $U \subset E \times F$ open set and define $V = \{x \in E : (x, f(x)) \in U\}$. If $x \in V$, then one can find an open product set $U_x = E_x \times F_x \subset U$ containing $(x, f(x))$. Since $f^{-1}(F_x)$ is open, there is an open set D_x with $x \in D_x \subset f^{-1}(F_x)$. Now $x \in E_x \cap D_x \subset V$ implies that V is open, too. Thus

$$Prob((X_n, f(X_n)) \in U) = Prob(X_n \in V)$$

yields the statement. □

Use Lemma 1 and Sublemma 2 with the choice

$$\begin{aligned} E &= \{\psi = (\psi_1, \psi_2) \in D_{\mathbb{R}^2}[t_0, 1] : \psi_2 \text{ is non decreasing}\}, \\ F &= D[t_0, 1] \\ f((\psi_1, \psi_2)) &= \left(\int_{\tau = t_0}^{t} \frac{1}{\sqrt{\tau}} d\psi_2 \right)_{t \in [t_0, 1]} \end{aligned}$$

to obtain Lemma 2. □

Step 3

In this step we are going to prove a simplified version of the first statement of Theorem 1, namely, the convergence of the measures $\mu_n^{\searrow}$ restricted to $C[t_0, 1]$. Note that one can think about our model as having two sources of randomness. The first one is the choice of x and the second is the choice of ξ's. In the first two steps we were only dealing with the first source, but now we are going to treat the second one, as well.

It would be more convenient to consider $S_n^{\searrow}$ as if the time instants of the reflections on the wall W_k $(0 < k \leq n)$ were not computed. Since Proposition 1 implies that $|\{i \leq n : \mathcal{F}_i^{\searrow} \dots \mathcal{F}_1^{\searrow} \in \mathcal{J}\}|$ is asymptotically of order $\sqrt{n}$, the diffusively scaled limits of $S_n^{\searrow}$ and of this "modified $S_n^{\searrow}$" (i.e. when we do not count the reflections on the wall) have the same limit. Thus it is sufficient to prove our statement for the modified $S_n^{\searrow}$.

To do so we will need the following elementary lemma from probability theory.

Lemma 3. *For every* $\varepsilon > 0$, $L^{(n)} \in \mathbb{N}$, $L^{(n)} \to \infty$, $0 < p_i^{(n)} \le 1$ *for* $1 \le i \le L^{(n)}$ *the following assertion holds true: if* $\exists C$ *such that* $\forall n \in \mathbb{N}$

$$\frac{\max_{1\le i\le L^{(n)}} p_i^{(n)}}{\min_{1\le i\le L^{(n)}} p_i^{(n)}} < C$$

and

$$\sum_{i=1}^{L^{(n)}} p_i^{(n)} \to \lambda$$

as $n \to \infty$, *then*

$$\prod_{i=1}^{L^{(n)}} \left(1 - p_i^{(n)}\right) \to \exp(-\lambda)$$

and for $k \ge 1$,

$$\sum_{B\subset\{1,2,\dots L^{(n)}\},|B|=k} \prod_{i\in B} p_i^{(n)} \prod_{i\in B^c} \left(1 - p_i^{(n)}\right) \to \frac{\lambda^k}{k!}\exp(-\lambda)$$

as $n \to \infty$, *and the convergence in uniform for* $\lambda \in [\varepsilon, 1/\varepsilon]$.

Proof. The statement is an easy consequence of Le Cam's famous inequality, [8]: Assume Σ_n is the sum of n independent Bernoulli random variables ε_j; $1 \le j \le n$ such that $Prob(\varepsilon_j = 1) = p_j$. Then

$$\sum_{k=1}^{n} \left| Prob(\Sigma_n = k) - e^{-\lambda}\lambda^k/k! \right| \le \sum_{j=1}^{n} p_j^2.$$

□

Lemma 2 infers that - by Skorokhod's representation theorem, cf. [1] - there exists a probability space $(\Omega, \mathbb{Q})$ together with random variables $\tilde{X}_n, \tilde{Y}_n, \tilde{X}, \tilde{Y}$ having the same distribution as

$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}\right)_{t\in[t_0,1]} \left(\int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\frac{L_{n\tau}}{\sqrt{n}}\right)_{t\in[t_0,1]}, (\mathcal{W}_t)_{t\in[t_0,1]}, \left(c_0 \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathcal{L}\right)_{t\in[t_0,1]}$$

with respect to $\mathbf{P_n}$, respectively, such that $(\tilde{X}_n, \tilde{Y}_n) \to (\tilde{X}, \tilde{Y})$ $\mathbb{Q}$-almost surely. Now, for $\mathbb{Q}$-almost all $\omega \in \Omega$ one can define measures $\nu(\omega)$, $\nu_n(\omega)$ on $C[t_0, 1]$ in the following way. Consider the modulus of $\tilde{X}(\omega)$, i.e. $|\tilde{X}(\omega)| \in C[t_0, 1]$ (if $a > 0$; otherwise consider $-|\tilde{X}(\omega)|$), pick a Poisson point process with intensity $\tilde{Y}(\omega)$ and then reflect the subgraph of $|\tilde{X}(\omega)|[P_i, 1]$ to the origin for each point P_i of the Poisson point process (if $a > 0$; otherwise reflect $-|\tilde{X}(\omega)|$). The distribution of the resulting random function is denoted by $\nu(\omega)$. The construction of $\nu_n(\omega)$ is analogous, the difference is that instead of the Poisson point process, one introduces independent Bernoulli random variables with parameters of the size of jump of $\tilde{Y}_n(\omega)$. The resulting measure is denoted by $\nu_n(\omega)$.
Lemma 3 implies that for $\mathbb{Q}$-almost all ω we have $\nu_n(\omega) \Rightarrow \nu(\omega)$ on $C[t_0, 1]$.
Define

$$\varrho(A) = \int_\Omega \nu(\omega)(A) d\mathbb{Q}(\omega),$$

and analogously $\varrho_n(A)$. Using Fubini's theorem and dominated convergence, one can prove $\varrho_n(A) \Rightarrow \varrho(A)$ on $C[t_0, 1]$. This implies the convergence of the measures $\mu_n^{\searrow}$ restricted to $C[t_0, 1]$ to the desired limit (in fact, with the constant $c_1 = c_0 / \sum_{k=1}^K \mathcal{J}_{k,r} - \mathcal{J}_{k,l})$.

Step 4

Now, one can easily prove the first part of the Theorem. The convergence of finite dimensional distributions is implied by Proposition 1 and the result of Step 3. The tightness is also easy since the moduli of the random functions are tight.

The above proof with trivial modifications is easily applicable to cases 2 and 3. The only non trivial modification is needed in case 4.

2.2 Proof of case 4

We are going to prove the convergence of two dimensional marginals. The convergence of any finite dimensional marginals can be proven the same way, while the tightness is trivial since the moduli of the random functions are tight. In the proof we use the notations of the previous subsection. One can assume $b > 0$ and we also assume that $a > 0$ (the case of negative a can be treated identically). Now put

$$q = Prob(\forall s : t_0 < s < t_1 : \mathcal{W}_s > 0 || \mathcal{W}_{t_1}| = b)$$

and introduce the notation $\mathbf{Q}_{\mathbf{n},|.|}$ for the conditional measure of $\mathbf{P}_\mathbf{n}$ under the condition that $|S^{(per)}_{\lfloor nt_1 \rfloor}| = b_n$. Sublemma 1 implies that

$$\mathbf{Q}_{\mathbf{n},|.|}\left(\forall s : t_0 < s < t_1 : S^{(per)}_{\lfloor ns \rfloor} > 0\right) \to q.$$

Hence, for the convergence of the two dimensional marginals, it would be enough to prove

$$\mathbf{P}\left(S^{\searrow}_{\lfloor nt_1 \rfloor} = b_n | \left(S^{\searrow}_{\lfloor nt_0 \rfloor} = a_n \& \exists s : t_0 < s < t_1, S^{\searrow}_{\lfloor ns \rfloor} = 0 \& |S^{\searrow}_{\lfloor nt_1 \rfloor}| = |b_n|\right)\right) \to \frac{1}{2}. \tag{10}$$

On the other hand, Sublemma 1 also yields that for every $\varepsilon > 0$ there exists $\delta > 0$ and N such that for all $n > N$,

$$\mathbf{Q}_{\mathbf{n},|.|}\left(L_{nt_1} > \delta\sqrt{n} | \exists s : t_0 < s < t_1, S^{(per)}_{\lfloor ns \rfloor} = 0\right) > 1 - \varepsilon. \tag{11}$$

Introduce the modification of $S^{\searrow}_n$ as in Step 3 of the previous subsection. Then $|S^{(per)}_{\lfloor nt_1 \rfloor}| = |S^{\searrow}_{\lfloor nt_1 \rfloor}|$. Hence

$$\mathbf{Q}_{\mathbf{n},|.|}\left(S^{\searrow}_{\lfloor nt_1 \rfloor} = b_n | \exists s : t_0 < s < t_1, S^{(per)}_{\lfloor ns \rfloor} = 0\right) \tag{12}$$

is just the probability that the number of crosses of the origin in the interval $[\lfloor nt_0 \rfloor, \lfloor nt_1 \rfloor]$ is even. The latter can be estimated the following way. Consider the Markov transition matrices

$$A_p = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

on the space $\{+,-\}$ and a time dependent Markov chain M_k such that $M_0 = +$ and the transition between k and $k+1$ is described by A_{p_k} with some numbers p_k. Now, for a fix x on the phase space of the billiard map on $\mathcal{T}$ restricted to the two neighboring tori to the origin, define $J_k(x)$ as the k-th leftmost discontinuity of the function $s \to L_{\lfloor ns \rfloor}(x)$ on $s \in [t_0, t_1]$. With the choice $p_k(x) = \alpha_{nJ_k(x)}/\sum_{k'=1}^{K} \mathcal{J}_{k',r} - \mathcal{J}_{k',l}$, $1 \le k \le L_{\lfloor nt_1 \rfloor}(x)$ for each x, one easily sees that the probability in (12) is asymptotically equal to

$$\frac{1}{\mathbf{Q}_{\mathbf{n},|.|}(x : L_{\lfloor nt_1 \rfloor}(x) > 0)} \int_{\{x : L_{\lfloor nt_1 \rfloor}(x) > 0\}} Prob(M_{L_{\lfloor nt_1 \rfloor}(x)} = +) d\mathbf{Q}_{\mathbf{n},|.|}(x). \quad (13)$$

If fact, this is only true for the case of $\mu_n^{\searrow}$, while in the case of $\mu_n^{\equiv}$, one needs to set $p_k(x) = \alpha_n/\sum_{k'=1}^{K} \mathcal{J}_{k',r} - \mathcal{J}_{k',l}$.

On the other hand, elementary computations show that if one selects sequences $A(n) \to \infty, m(n) \to \infty$ and non-negative numbers $p_{k,n}, 1 \le n, 1 \le k \le m(n)$, then with the transition matrices corresponding to $p_{1,n}, ..., p_{m(n),n}$,

$$\begin{aligned} & Prob(M_{m(n)} = +) \\ = & \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=1}^{m(n)} (1 - 2p_{k,n}) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ = & 1/2 + o(1), \end{aligned}$$

as $n \to \infty$. Further, the convergence is uniform in m and p_k if $m(n) > \delta\sqrt{n}$ and $\min_{1 \le k \le m(n)} \{p_{k,n}\sqrt{n}\} > A(n)$. This estimation together with (11) and (13) yields (10).

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