Probability 2.
2. Homework and Exercises
2009.02.25.

1. Show that the commodity chain on Homework 1 is irreducible and aperiodic.

2. The **Wright-Fisher model** is a Markov-chain on the state space $S = \{0, 1, \dots, N\}$, $N \geq 1$, where $0, N$ are absorbing states. Given, that $X_0 = i$, what is the probability that the chain is absorbed in state N?

   In the Wright-Fisher model there are N genes in each generation, each of them of type A or a. In the $n+1$th generation, every gene among the N genes inherits one of the genes from the nth generation with equal probability, i.e, the transition matrix is given by

   $$p_{i,j} = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}$$

   for $i, j \in S$.

3. In tennis, a game is won by the player who reaches 4 points first, except when the score is $4 : 3$, when the game is continued until someone reaches 2 points advantage. In a game, always player A serves (he is the server) and the probability, that the point is won by the server is always 0.6. (So, they do not switch who is serving!) We start following the game when the score is already $3 : 3$.

   (a) What is the probability now that the server wins?

   (b) Then, the next point is won by the server, so now the score of the game is $4 : 3$. What is the winning probability of him now?

   (c) What would be in the other case, i.e. from $3 : 4$?

   (d) (continue) In average, how many points are needed to finish the game, starting from score $3 : 3$?

4. In Utopy city the local government organizes a program called "free bicycles for the inhabitants". The bikes are placed in front of the library (L), the pub (P) ore the grocery store (G). The organizer of the program has made some statistics and he declared that the bikes lent from near the library are given back at the pub with prob. 0.2, and with prob 0.3 at the grocery store. Similarly, the bikes lent from near the pub are given back at the library with prob. 0.4 and 0.1 at the grocery store, respectively. Finally, the bikes lent from near the grocery store are given back at the library and the pub with prob.-s $0.25, 0.25$, respectively.

   On Sunday there are equal number of bikes everywhere.

   (a) What is the proportion of bikes at each of the three places on Tuesday?

   (b) What is this proportion on next Sunday?

   (c) After a long time, what is the proportion of bikes at the places?

5. *Umbrella chain.* The professor has three umbrellas, and keeps them at the university or at home. He commutes every day between the two buildings. If it is raining when he leaves the building (his house or the math institute), picks up an umbrella if there is any (if there's none there, unfortunately, he will be wet). Each time, the probability of rain is 0.2, independently of everything else. Let us denote the number of umbrellas at the current building (i.e. every odd number is at home and every even is at the university).

(a) Determine the transition matrix of the Markov chain;

(b) Determine the proportion of times of getting wet during a long time period.

6. The employees of the company Macrosoft are either programmers or project-managers. The 20% of the programmers is promoted to become a manager, 10% of them is fired, the position of the others remain unchanged. 5 % of the project managers is fired, the position of the others remain unchanged. For how many years does a programmer work for the company in average?

7. For a finite state Markov chain, state j is said to be accessible from a state i – denoted by $i \to j$, if $\exists n > 0$, such that $p_n(i, j) > 0$. The states i, j are communicating – denoted by $i \leftrightarrow j$, if both $i \to j, j \to i$. Show that

   (a) on the set of non-negligible states, $i \leftrightarrow j$ is an equivalence-relation; (the equivalence-classes are called communicating classes of the chain)

   (b) the period of every state in the same communicating class is the same.

8. $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d sequence of dice-throws. Let us denote $S_n = X_1 + \cdots + X_n$, and $T_1 = \min\{n \geq 1 | S_n \equiv 0 \pmod 8\}$, és $T_2 = \min\{n \geq 1 | S_n \equiv 1 \pmod 8\}$. Calculate $\mathbf{E}T_1$ and $\mathbf{E}T_2$!

9. Let X_n be an irreducible Markov chain on finite state space S, with transition matrix $\mathbf{P}$, starting from state i. Denote

$$T := \min\{n > 0 | X_n = i\}$$

   For the state j, denote

$$r(j) = \mathbf{E}\left(\sum_{n=0}^{T-1} \mathbb{I}_{\{X_n = j\}}\right)$$

   (obviously $r(i) = 1$).

   (a) Let $\mathbf{r}$ be a vector consisting of $r(j)$-s. Show that, $\mathbf{r}^*\mathbf{P} = \mathbf{r}^*$;

   (b) Show that

$$\mathbf{E}(T) = \sum_{j \in S} r(j)$$

   (c) Conclude that $\mathbf{E}(T) = \pi(i)^{-1}$, where π is the vector of stationary distribution.

10. A fair coin is thrown sequently (Heads and Tails with equal probability . What is the expectation of the waiting time for HHH? What is it for HTH?

    Hint: It is useful to formulate an 8-state space Markov chain, starting after the 3rd throw at some (random) state. Mathematica or Maple can be used for solving the corresponding linear equation system.

11. Let us consider a simple random walk (SRW) on the graph with vertex set $\{A,\ B,\ C,\ D,\ E\}$ and (undirected) edge set $\{AB, AC, BC, CD, BD, BE, DE\}$.

    (a) Suppose the walker starts at A. What is the expected number of steps before reaching C first?

    (b) Suppose the walker starts at C. What is the expected number of steps taken before the first return to C? *(E.g. conditioning on the first step taken helps)*

(c) Suppose the walker starts at A. What is the expected number of visits of E before reaching C first?

(d) Suppose the walker starts at B. What is the probability of reaching A before reaching C?

12. *Horse on a chessboard = SRW on a graph*

    A horse starts at a corner of a chessboard, and in each step, it jumps a proper horse-step, choosing the direction uniformly among the possibilities. What is the expected number of steps before first return to the initial corner? What is the expected number of visits of the opposite corner before the first return?

13. *Generalization or hint to the previous exercise: SRW on a finite graph*

    Let us consider SRW on a (undirected) graph $G(V, E)$, i.e. at each step, the walker chooses uniformly among the neighbors and jumps to one of them. Find the stationary distribution of the chain. Show that the chain is reversible. *Hint: If the degree of vertex i is denoted by $d(i)$, show that the distribution which is proportional to the degrees, is reversible.*

I. **Theorem, Perron-Frobenius** A matrix P with positive entries has a unique largest real eigenvalue $\lambda(P)$, for which

(a) $\lambda(P) > 0$ and for $Ph = \lambda(P)h$, the vector $h > 0$ (the inequalities for vectors and matrices are meant coordinatewise);

(b) $\lambda(P)$ has multiplicity 1;

(c) for every other eigenvalue κ, $|\kappa| < \lambda$;

(d) P has no other positive eigenvector except h.

hint: denote by $l(P) := \{\lambda \geq 0 | \exists x \neq 0, x \geq 0, \text{ such that } Px \geq \lambda x\}$. Show that if P is positive, then

(a) $l(P) \cap (0, \infty) \neq \emptyset$;

(b) $l(P)$ bounded;

(c) $l(P)$ closed.

Secondly, denote by $\lambda_{\text{MAX}} := \max\{\lambda | \lambda \in l(P)\}$. Show that this will be the eigenvalue $\lambda(P)$.