Probability 2.
5. Exercise sheet
2010.04.19.

Characteristic functions and applications. Central Limit Theorem

1. $X_1, \dots, X_n$ are i.i.d. $\mathrm{U}(0,1)$ distributed random variables and we denote their maximum by $Y_n = \max\{X_1, \dots, X_n\}$. Determine the characteristic function of Y_n!

2. $X_1, X_2, \dots$ random variables are independent, and their distribution is given by

$$\begin{aligned}\mathbf{P}(X_n = \pm(n+1)) &= \frac{1}{2(n+1)^2}\\ \mathbf{P}(X_n = 0) &= 1 - \frac{1}{(n+1)^2}\end{aligned}$$

Determine the characteristic function of $X_1 + X_2 + \cdots + X_m$!

3. Show that the difference of two i.i.d. random variables (of arbitrary distribution) cannot be $U(-1,1)$-distributed! (Hint: determine their characteristic functions and prove that their cannot be the same!)

4. A random variable Y is called *infinitely divisible* if $\forall n$ there exists a characteristic function $\psi_n(t)$, such that $\mathbf{E}\left(e^{itY}\right) = [\psi_n(t)]^n$. Show (again) that $\mathrm{POI}(\lambda)$ and $\mathrm{N}(0,1)$ are infinitely divisible distributions!

5. $\nu, X_1, X_2, \dots$ are independent random variables, $X_1, X_2, \dots$ are i.i.d. with characteristic function $\psi(t)$. Determine the characteristic function of $Y_\nu = X_1 + \cdots + X_\nu$ if
   (a) $\nu \sim POI(\lambda)$ distributed
   (b) $\mathbf{P}(\gamma = k) = pq^k$ $(k = 0, 1, 2, \dots)$.
   (c) Prove that the distribution in exercise (a) is infinitely divisible.

6. $\psi(t)$ is a characteristic function. Is it true that $Re\psi(t)$, and $Im\psi(t)$ are characteristic functions as well?

7. Calculate the characteristic function of the distributions below, if their density function is given by $(a > 0)$:
   (a)
   $$f(x) = \frac{a}{2}e^{-a|x|}$$
   (b)
   $$f(x) = \begin{cases}\frac{1}{a^2}(a - |x|) & \text{if } |x| \le a\\ 0 & \text{otherwise}\end{cases}$$

8. The density function of the Cauchy-distribution is

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Determine its characteristic function! (hint: use the residue theorem.)

9. *Laplace distribution:* X_1, X_2 are independent $EXP(\lambda)$ distributed, Y is independent of them, $\mathbf{P}(Y = \pm 1) = \frac{1}{2}$.
   (a) Show that the distribution of $X_1 - X_2$ and $Y \cdot X_1$ is the same, called Laplace distribution!
   (b) Using the properties of inverse Fourier transform, determine the characteristic function of the Cauchy distribution with no more integration needed.

10. (a) $X_1, \ldots, X_n$ are i.i.d. Cauchy-distributed random variables. Determine the density function of $Z_n = \frac{1}{n}(X_1 + \ldots + X_n)$!

    (b) Give an example for the statement: the fact that the characteristic function of the sum of random variables is the product of the characteristic function of the variables does not imply independence, i.e. $\mathbf{E}(e^{it(X+Y)}) = \mathbf{E}(e^{itX})\mathbf{E}(e^{itY})$ does not imply independence of X and Y. (use the previous...)

11. $X \sim \mathit{Cauchy}(b, a), a > 0, b \in \mathbb{R}$ has density function

$$\frac{1}{\pi} \frac{a}{a^2 + (x-b)^2}.$$

the standard case is $a = 1$, $b = 0$.

   (a) Show that $\mathit{Cauchy}(b, a)$ is a linear transform of the standard Cauchy distribution!

   (b) Using the fact that the characteristic function of $\mathit{Cauchy}(0, 1)$ is $e^{-|t|}$, calculate the characteristic function of $\mathit{Cauchy}(b, a)$ (without any integration!).

   (c) Prove that the convolution of $\mathit{Cauchy}(b_1, a_1)$ and $\mathit{Cauchy}(b_2, a_2)$ random variables has distribution $\mathit{Cauchy}(b_1 + b_2, a_1 + a_2)$!

   (d) Could you prove the same using moment generating functions? Why?

   (e) Is the function $f(t) = e^{-(|t|+1)^2} \cdot e$ a characteristic function? If yes, express its distribution using well-known distributions.

12. $X_1, X_2, X_3 \ldots$ are independent random variables, such that the distribution function of $X_1, X_3, X_5, \ldots$ is F, and the distribution function of $X_2, X_4, X_6, \ldots$ is G. Let's assume $\mathbf{E}(X_1) = m_1$, $\mathbf{E}(X_2) = m_2$, $\mathbf{D}^2(X_1) = \mathbf{D}^2(X_2) = \sigma^2$. Show that

$$\frac{\sum\limits_{j=1}^{n} (X_j - \mathbf{E}(X_j))}{\sqrt{n}\sigma} \Rightarrow N(0,1)$$

13. Peter travels to work by bus. The bus is 5 minutes late in 10% of the days, 10 minutes in 40% of the days, 15 minutes in 50% of the days. Assuming Peter travels to work 230 days yearly, is it probable that he will spend more than 50 hours altogether, waiting for the bus (which is late)?

14. $X_1, X_2, X_3, \ldots$ are independent of distribution $\mathbf{P}(X_i = 1) = p_i$, $\mathbf{P}(X_i = 0) = q_i = 1 - p_i$ such that $\sum_{i=1}^{\infty} p_i q_i = \infty$. Let $S_n = X_1 + x_2 + \cdots + X_n$. Prove the Law of Large Numbers and Central Limit Theorem for S_n!

15. $X_1, X_2, X_3, \ldots$ independent, $\mathrm{U}(0,1)$ distributed random variables. Prove that if $n \to \infty$, then

$$Y_n = \frac{\sum\limits_{k=1}^{n} kX_k - \frac{n^2}{4}}{\frac{1}{6} n^{\frac{3}{2}}} \Rightarrow N(0,1)$$

16. Calculate

$$\lim_{n \to \infty} \underset{\substack{0 \le x_j \le 1 \\ x_1 + \ldots + x_n < \frac{n}{2} + \sqrt{\frac{n}{12}}}}{\int \cdots \int} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

1. Prove the following identities using probabilistic arguments!

   (a)

$$\frac{\sin t}{t} = \frac{\sin \frac{t}{2}}{\frac{t}{2}} \cos \frac{t}{2}$$

   (b)

$$\frac{\sin t}{t} = \prod_{k=1}^{\infty} \cos \frac{t}{2^k}$$