Optimal arrangements of unit vectors in Hilbert spaces I:

New constructions of few-distance sets and biangular lines

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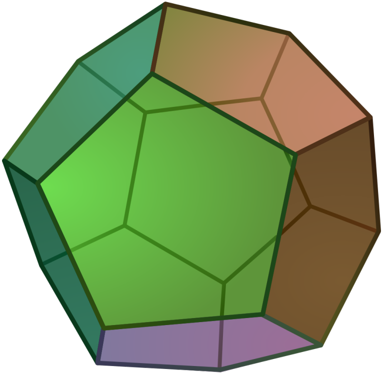
A motivating example

The 20 vertices of the platonic dodechadron form a 5-distance set in $\mathbb{S}^2$. The 10 main diagonals (going through antipodal vertices) form a system of 10 biangular lines in $\mathbb{R}^3$.



In this talk we explore analogous configurations in higher dimensions.

What this talk is about

Recent results on vectors in the 12-dimensional Euclidean space.

Theorem[P.R.J. Östergård and F. Sz., 2018]

The largest set of equiangular lines with common angle $1/5$ in $\mathbb{R}^{12}$ is 20. There are exactly 32 pairwise nonisometric configurations.

The following are new results in $\mathbb{R}^3$.

Theorem[F. Sz., 2019+]

The 20 vertices of the platonic dodecahedron is the unique maximum 5-distance set in $\mathbb{S}^2$.

Theorem[M. Ganzhinov and F. Sz., 2019+]

The 10 main diagonals of the dodecahedron is the unique maximum set of biangular lines in $\mathbb{R}^3$.

In this talk I develop a computational approach for proving statements of this kind, and report on our experimental results.

The basics

As always, $d \geq 1, s \geq 1$ are integers, μ is the Euclidean distance.

Euclidean few-distance sets

A finite set of points $\mathcal{X} \subset \mathbb{R}^d$ is an s-distance set, if the set of distances

$$\{\mu(x_i, x_j)\colon x_i \neq x_j \in \mathcal{X}\}$$

is of cardinality *at most s*.

Multiangular lines

A finite set of (pairwise non-antipodal) points $\mathcal{X} \subset \mathbb{S}^{d-1}$ form a set of m-angular lines, if the set of common angles

$$\{\left|\left\langle x_i, x_j\right\rangle\right| : x_i \neq x_j \in \mathcal{X}\}$$

is of cardinality *at most m*.

Multiangular lines are switching classes of certain few-distance sets.

Spherical s-distance sets

For *conceptual simplicity* I discuss s-distance sets on $\mathbb{S}^{d-1}$, but everything carries over to $\mathbb{R}^d$ and to the multi-angular setting after proper modifications. Also, instead of distances, I will talk about inner products, as

$$\mu(u,v) = \sqrt{\sum_i u_i^2 + \sum_i v_i^2 - 2\langle u,v\rangle} = \sqrt{2-2\langle u,v\rangle},$$

$$\langle u,v\rangle = 1 - \mu(u,v)^2/2.$$

So from now on, we are interested in unit vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$.

Examples:

- The $d+1$ vertices of the regular simplex in $\mathbb{R}^d$ is a 1-distance set
- The midpoints of the edges of the regular simplex in $\mathbb{R}^d$, and a set of unit vectors spanning equiangular lines are 2-distance sets
- The codewords of a $(d,0,s)$ binary constant weight code (embedded into $\mathbb{R}^d$ in the obvious way) is an s-distance set

Gram matrices

The coordinates of the vectors v_i depend on the choice of basis. To avoid this inconvenience, we pass on to the Gram matrix:

$$G(v_1, \dots, v_n) := [\langle v_i, v_j \rangle]_{i,j=1}^{n}$$

The elements of G are now basis independent, since

$$\langle v_i, v_j \rangle = \langle Ov_i, Ov_j \rangle$$

for any orthogonal matrix O (i.e., an isometry of the Euclidean space).

So from now on, we consider Gram matrices of spherical few-distance sets, which determine these sets up to isometry.

The vectors v_i can be uniquely recovered (up to isometry) from G via the Cholesky decomposition.

Properties of the Gram matrices

The Gram matrix G of unit vectors forming an s-distance set has various combinatorial and algebraic properties.

Combinatorial:

- has constant diagonal 1
- is symmetric (i.e., $G = G^T$)
- it has *at most s* distinct off-diagonal entries

Algebraic:

- it has rank *at most d*
- it is positive semidefinite

Note that G is a matrix with real entries.

The main conceptual difficulty in understanding these Gram matrices is the lack of control on their elements. In particular, for (typical) n fixed, the set of $n \times n$ Gram matrices is *not* a finite set. For d fixed and n large this set is empty by Ramsey theory (Bannai–Bannai–Stanton).

Candidate Gram matrices

The Goal of the talk is to set up a framework to describe all large set of few-distance sets in a systematic way.

It would be sufficient to describe the Gram matrices...

...but it is too difficult, so we introduce a weaker concept, capturing the *combinatorial* properties of Gram matrices.

Definition[Candidate Gram matrices]

A matrix $C(a, b, \dots, s)$, over the symbol set $\{1, a, b, c, \dots, s\}$ with

- constant diagonal 1
- $C = C^T$
- the off-diagonal entries belong to the set $\{a, b, \dots, s\}$

is called a *candidate Gram matrix*.

Note that every Gram matrix gives rise to a candidate Gram matrix. For n, s fixed the set of $n \times n$ candidate Gram matrices is finite.

Symmetries of Gram matrices

We attempt to describe the candidate Gram matrices...

...but there are too many of those, and instead we introduce an equivalence relation first.

The order of the n vectors forming a few-distance set is irrelevant.

Definition

Two Gram matrices G_1 and G_2 (of the same size) are equivalent, if $G_1 = PG_2P^T$ for some permutation matrix P.

Definition

Two candidate Gram matrices C_1 and C_2 (of the same size, over the same symbol set) are equivalent, if $C_1(a, b, \ldots, s) = PC_2(\sigma(a), \sigma(b), \ldots, \sigma(s))P^T$ for some permutation matrix P and for some permutation σ interchanging the symbols among themselves.

Candidate Gram matrices, up to equivalence

We attempt to describe the candidate Gram matrices up to equivalence, by realizing that this is essentially boils down to isomorph-free exhaustive generation of graphs...

Lemma

An equivalence class of $n \times n$ candidate Gram matrices over the symbol set $\{1, \dots, s\}$ is in one-to-one correspondence with the graph isomorphism classes of the at-most-s-edge-colored complete graphs on n vertices (where permutation of the colors is allowed).

This is obvious, as the candidate Gram matrix may be thought as the graph-adjacency matrix where same symbols specify edges of the same color. Every equivalence class of Gram matrices correspond to a *unique* equivalence class of candidate Gram matrices (with the number of symbols in C matching the number of distinct entries of G).

...but of course, this is too difficult to do for $n > 13$, and at this point it is still unclear how this would lead to a classification of Gram matrices.

Using the algebraic properties

So far we have focused on the combinatorial properties of Gram matrices.

The way step forward is to leverage on $\operatorname{rank} G \leq d$ (and we ignore positive semidefiniteness for a while).

We endow our candidate Gram matrices with algebraic properties by embedding them into the matrix ring $\mathcal{M}_n(\mathbb{Q}[x_1, \dots, x_s])$ (in the obvious way) where $x_1, \dots, x_s$ are pairwise commuting indeterminates.

Note that every [representative of a] Gram matrix [equivalence class] is the evaluation of some [representative of a] candidate Gram matrix [equivalence class] at a real s-tuple $(\widehat{x}_1, \dots, \widehat{x}_s) \in \mathbb{R}^s$.

Lemma

Let G be the Gram matrix of some spherical s-distance set in $\mathbb{R}^d$, with candidate Gram matrix C. Then we have $G = C(\widehat{x}_1, \dots, \widehat{x}_s)$ and in particular $\operatorname{rank} C(\widehat{x}_1, \dots, \widehat{x}_s) \leq d$, for some $(\widehat{x}_1, \dots, \widehat{x}_s) \in \mathbb{R}^s$.

The determinantal variety

The goal is now to understand candidate Gram matrices of certain rank. The rank of a matrix is conveniently characterized by its vanishing minors.

Lemma

Let $M \in \mathcal{M}(\mathbb{C})$. Then $\mathrm{rank}\, M \leq d$ if and only if all $(d+1) \times (d+1)$ minors of M are vanishing.

Proposition

Let $C(x_1, \dots, x_s) \in \mathcal{M}_n(\mathbb{Q}[x_1, \dots, x_s])$ be a candidate Gram matrix. Then there exists a $(\widehat{x_1}, \dots, \widehat{x_s}) \in \mathbb{R}^s$ so that $\mathrm{rank}\, C(\widehat{x_1}, \dots, \widehat{x_s}) \leq d$ if and only if it is a real solution of the system of polynomial equations

$$\{\det M(x_1, \dots, x_s) = 0 \colon M \text{ is a } (d+1) \times (d+1) \text{ submatrix of } C\}.$$

But this is a vacuous condition, since $\mathrm{rank}\, C(1, 1, \dots, 1) = 1$ always.

The quotient ideal

The goal is to understand candidate Gram matrices of certain rank, and deal with the inconvenience that $\operatorname{rank} C(1,1,\dots,1)=1$.

It turns out, that we should specify the algebraic conditions $\widehat{x}_i\neq\widehat{x}_j$, $\widehat{x}_i\neq 1$ for $i\neq j$. These conditions capture the triviality that the off-diagonal entries of a Gram matrix are never equal to 1, and the number of distinct off-diagonal entries in G matches exactly the number of underlying symbols in C.

Proposition

Let $C(x_1,\dots,x_s)\in\mathcal{M}_n(\mathbb{Q}[x_1,\dots,x_s])$ be a candidate Gram matrix. Then there exists a $(\widehat{x}_1,\dots,\widehat{x}_s)\in\mathbb{R}^s$ with $\widehat{x}_i\neq\widehat{x}_j$, $\widehat{x}_i\neq 1$ for $i\neq j$ so that $\operatorname{rank} C(\widehat{x}_1,\dots,\widehat{x}_s)\leq d$ if and only if there exists a $\widehat{u}\in\mathbb{R}$ so that $(\widehat{u},\widehat{x}_1,\dots,\widehat{x}_s)\in\mathbb{R}^{s+1}$ solves the system of polynomial equations

$$\begin{cases}\det M(x_1,\dots,x_s)=0\colon\ M \text{ is a } (d+1)\times(d+1) \text{ submatrix of } C\\ u\prod_{i\neq j}(x_i-x_j)\prod_k(x_k-1)=1, \text{where } u \text{ is an auxiliary variable.}\end{cases}$$

The main technical tool

The goal is to exhaustively generate all candidate Gram matrices with small rank. We can eliminate unsuitable candidate Gram matrices.

Main Theorem

Let $d \geq 2$ be fixed, and let $C(x_1, \ldots, x_s) \in \mathcal{M}_n(\mathbb{Q})[x_1, \ldots, x_s]$ be a candidate Gram matrix. If the system of polynomial equations

$$\begin{cases} \det M(x_1, \ldots, x_s) = 0\colon M \text{ is a } (d+1) \times (d+1) \text{ submatrix of } C \\ u \prod_{i \neq j}(x_i - x_j) \prod_k (x_k - 1) = 1, \text{where } u \text{ is an auxiliary variable} \end{cases}$$

has no solution in $\mathbb{C}^{s+1}$, then C does not correspond to any Gram matrix representing a spherical s-distance set in $\mathbb{R}^d$. In particular, [the matrix equivalence class of] C cannot be a submatrix of any candidate Gram matrix corresponding to an actual Gram matrix.

At this point we are *only* interested in the *existence* of a *complex* solution, and *not* in *what* these solution(s) are, or whether they are real.

A Roadmap for classifying spherical few-distance sets

- Fix s and d.
- Generate all isomorphism classes of the at-most-s-edge-colorings of the complete graph on $n = d + 1$ vertices (candidate Gram matrices).
- Use Main Theorem (by computing a Gröbner basis) to test which candidate Gram matrix [equivalence class representative] C has small rank, and store the survivors in a list L_{d+1}.
- For every $C \in L_{d+1}$, enlarge C up to equivalence in every possible way with one more row and column, and then use Main Theorem to test the enlarged matrix. Store the survivors in a list L_{d+2}.
- Repeat, until new matrices are being discovered.

Once the process ends (it does, by B–B–S), we have a finite list of large candidate Gram matrices so that each can be evaluated to a matrix with rank at most d. The positive semidefinite matrices are *the* Gram matrices of maximum spherical s-distance sets in $\mathbb{R}^d$.

Note: the distances come out as a by-product of the classification.

A case study

Theorem[F.Sz., 2019+]

The dodecahedron is *the* maximum spherical 5-distance set in $\mathbb{R}^3$.

Proof: by Roadmap.

- Fix $s = 5$ and $d = 3$.
- Generate all 370438 isomorphism classes of the at-most-5-edge-colorings of the complete graph on $n = 6$ vertices (candidate Gram matrices).
- Use Main Theorem to test which candidate Gram matrix [equivalence class representative] C has small rank, and store the survivors in a 19566-element list L_6 ($\approx 5\%$ of the cases survive).
- Enlarge, Test, Store, and Repeat, until L_{21}, which is empty.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
#	19566	18073	3358	1281	1047	827	638	383	211	89	38	11	4	1	1	0

The single largest candidate Gram matrix corresponds to the dodecahedron. The second largest example is formed by two 7-cycles.

A miracle happens here

Why does Main Theorem work?

- It is somewhat surprising, that the *only* candidate Gram matrix surviving the tests is *the* candidate Gram matrix corresponding to the dodecahedron.
- In principle, there could have been other candidate Gram matrices which while have the correct rank, are *not* positive semidefinite.

Earlier Lisoněk classified the two-distance sets in $\mathbb{R}^7$, and he remarked the following:

Theorem[Menger, 1950s]

The congruence order of $\mathbb{R}^d$ is $d+3$.

This means, that a given $n \times n$ candidate Gram matrix corresponds to a Gram matrix in $\mathbb{R}^d$ if and only if each of its principal $(d+3) \times (d+3)$ submatrices evaluated at *the same point* correspond to some Gram matrix. This suggests that *there should be no junk beyond* $n = d + 3$.

Summary

A combination of isomorph-free exhaustive generation of graphs and Gröbner basis computation yielded new classification results for few-distance sets.

The framework presented here can be adjusted so that to classify nonspherical few-distance sets as well. In particular, we have:

Theorem[P.R.J. Östergård, F. Szöllősi, 2019+]

The unique maximum (nonspherical) 3-distance set in $\mathbb{R}^4$ is a 16-element subset of the integer grid, namely:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix},$$

with set of distances $\{1, \sqrt{2}, \sqrt{3}\}$.

The team at Aalto University

Biangular lines

"Switching classes" of certain spherical 4-distance sets.

Definition

A set of n lines, represented by the real unit vectors $v_1, \ldots, v_n$, is called biangular, if $\{\langle v_i, v_j\rangle : i \neq j \in \{1, \ldots, n\}\} = \{\pm\alpha, \pm\beta\}$ with $0 \leq \alpha, \beta < 1$.

It is clear, that on the plane one may have at most 5 biangular lines:

- take any line, passing through the origin
- at most two additional lines are at angle $\pm\alpha$
- at most two additional lines are at angle $\pm\beta$
- there is no room for more lines
- this is realizable by the pentagon

Or, by algebra, the coordinates of v_i:

$$\begin{bmatrix} 1 & \alpha & \alpha & \beta & \beta \\ 0 & \sqrt{1-\alpha^2} & -\sqrt{1-\alpha^2} & \sqrt{1-\beta^2} & -\sqrt{1-\beta^2} \end{bmatrix}$$

The aim of this talk is to explore what happens in higher dimension.

The kissing number problem

There are several reasons for studying biangular lines:

- The equiangular case is now ‘solved’
- The Equiangular Tight Frame community is moving towards Biangular Tight Frames, and possibly beyond
- The vectors of several low-dimensional kissing arrangements form biangular lines

The kissing number problem

What is the maximum number of unit vectors $\tau(d)$ in $\mathbb{R}^d$ so that

$$\langle v_i, v_j \rangle \leq 1/2 \qquad \text{for all } i \neq j \in \{1, \dots, \tau(d)\}?$$

One may place solid spheres of radius $1/2$ at the location v_i, each touching a central unit sphere of radius $1/2$.

The dream: improving on $\tau(d)$ by the methods presented today.

Constructions: From spherical 2-distance sets

Canonical examples:

- Any set of equiangular lines with angle set $\{\pm\alpha\}$
  In $\mathbb{R}^d$ this gives roughly $2/100 \cdot d^2$ lines
- The midpoints of the regular simplex with angle set
  $\{(d-3)/(2d-2), -4/(2d-2)\}$
  In $\mathbb{R}^d$ this gives $\binom{d+1}{2} = d(d+1)/2 \approx 1/2 \cdot d^2$ lines for $d \geq 4$
- Several other, sporadic examples

Upper bounds, from linear programming.

Theorem[Delsarte–Goethals–Seidel]

The maximum cardinality of a set of biangular lines is at most $\binom{d+3}{4} \approx 1/24 \cdot d^4$.

There is a considerable gap between the lower and upper bounds.

Constructions: from lattices

Example:
The shortest vectors of the D_d lattices in $\mathbb{R}^d$ are of the form:

$$\{\sigma([\pm 1, \pm 1, 0, \ldots, 0]/\sqrt{2})\colon \sigma \in \mathfrak{S}_n\}$$

This gives rise to a set of $2\binom{d}{2} = d(d-1) \approx 1 \cdot d^2$ biangular lines in $\mathbb{R}^d$ with angle set $\{0, \pm 1/2\}$.

From the exceptional lattices E_6, E_7, E_8, one has a system of

- 36 biangular lines in $\mathbb{R}^6$
- 63 biangular lines in $\mathbb{R}^7$
- 120 biangular lines in $\mathbb{R}^8$

The last two examples show that it is possible to have more than d^2 biangular lines in $\mathbb{R}^d$.

Constructions: from binary codes

Consider a (d, δ, w) binary constant-weight code $\mathcal{C}$ of length d, weight w, and minimum distance δ. One can embed such a code on the sphere by the following map (which acts entrywise on the codewords):

$$S_x\colon \mathbb{F}_2 \mapsto \mathbb{R}, \qquad S_x(1) = x, \qquad S_x(0) = \sqrt{(1 - wx^2)/(d - w)},$$

where x is to be specified later.
It is easy to see, that if $c_1, c_2 \in \mathcal{C}$ with Hamming distance $\Delta(c_1, c_2)$, then

$$\langle S_x(c_1), S_x(c_2) \rangle = 1 - \Delta(c_1, c_2)(x - S_x(0))^2/2$$

therefore if the S_x-image of $\mathcal{C}$ is biangular, then $\mathcal{C}$ necessarily have at most four distinct distances between codewords.

There are several studies of constant-distance codes in the literature.

Problem

What are the constant weight codes with having at most four distinct distances?

Constructions: from binary codes (ctd.)

Actually, the distances between the codewords should satisfy some additional condition so that to have a spherical 4-distance set which is a biangular line system. In particular, the spherical embedding should be S_a, with

$$a = \frac{-2(d-w) + \sqrt{4w^2 - 4dw + d\Delta}}{d\sqrt{\Delta}},$$

as only this guarantees that $\langle S_a(c_1), S_a(c_2)\rangle = 0$ for codewords at Hamming distance $\Delta(c_1, c_2) > 0$.

Theorem[M. Ganzhinov, F. Sz., 2019+]

Let $d \geq 7$, and let $\mathcal{C}$ be a $(d, 2w-6, w)$ code with $w \in \{\lceil\sqrt{3d/2}\rceil, \dots, \lfloor d/2 \rfloor\}$. Then with $\Delta := 4w - 6$, the S_a-image of $\mathcal{C}$ is a set of biangular lines with angle set $\{\pm 1/(2w-3), \pm 3/(2w-3)\}$.

- $(9, 2, 4) \rightsquigarrow 126$ lines in $\mathbb{R}^9$ with angle set $\{\pm 1/5, \pm 3/5\}$
- $(10, 2, 4) \rightsquigarrow 210$ lines in $\mathbb{R}^{10}$ with angle set $\{\pm 1/5, \pm 3/5\}$
- $(15, 4, 5) \rightsquigarrow 242$ lines in $\mathbb{R}^{15}$ with angle set $\{\pm 1/7, \pm 3/7\}$
- $(16, 4, 5) \rightsquigarrow 322$ lines in $\mathbb{R}^{16}$ with angle set $\{\pm 1/7, \pm 3/7\}$

Constructions: from codes (ctd.)

When $\mathcal{C}$ has fewer than four distances, then some flexibility arise, and in particular for $d \leq 17$ from $(d, 0, 3)$ codes, (that is, from all weight 3 vectors) we can get $\binom{d}{3}$ biangular lines.

Example:

- 286 lines in $\mathbb{R}^{13}$
- 364 lines in $\mathbb{R}^{14}$
- 455 lines in $\mathbb{R}^{15}$
- 560 lines in $\mathbb{R}^{16}$
- 816 lines in $\mathbb{R}^{17}$

This last example is biangular tight frame with more than 17^2 lines.

Classification of maximum biangular lines

By Roadmap, mentioned earlier, with some modifications.

Biangular lines are special spherical 4-distance sets. Their Gram matrix have four distinct off-diagonal entries. So we set up a framework to search for candidate Gram matrices over the symbol set $\{1, 0, \pm x_1, \pm x_2\}$ of the form: $C(x_1, x_2) := C(x_1, x_2, -x_1, -x_2)$, and we exclude the cases $x_1 \neq \pm x_2$, $x_1 \neq \pm 1$, $x_2 \neq \pm 1$, $x_1 \neq 0$, $x_2 \neq 0$.

In particular, from the determinantal variety we eliminate cases by intersecting it with the polynomial

$$ux_1x_2(x_1^2 - x_2^2)(x_1^2 - 1)(x_2^2 - 1) - 1.$$

Equivalence of candidate Gram matrices is also modified slightly: two candidate Gram matrices, C_1 and C_2 are equivalent, iff

$$C_1(x_1, x_2) = PDC_2(\sigma(x_1), \sigma(x_2))DP^T$$

for some permutation matrix P, monomial diagonal matrix D, and a relabeling permutation σ.

Results on biangular lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem

The 5 lines passing through the antipodal vertices of the convex regular 10-gon is the unique maximum biangular set in $\mathbb{R}^2$.

Proof:

n	2	3	4	5	6
#	2	3	2	1	0

Theorem

The 10 lines passing through the vertices of the platonic dodecahedron is the unique maximum biangular set in $\mathbb{R}^3$.

Proof:

n	2	3	4	5	6	7	8	9	10	11
#	2	5	22	23	12	5	2	1	1	0

Results on biangular lines in $\mathbb{R}^4$

Theorem

The maximum number of biangular lines is 12 in dimension 4. There are exactly four distinct configurations up to isometry.

Proof:

n	2	3	4	5	6	7	8	9	10	11	12	13
$\#$	2	5	25	191	701	184	69	27	14	3	3	0

The largest sets are

- Shortest vectors of the D_4 lattice with angle set $\{0, \pm 1/2\}$
- Doubling the shortest vectors of D_3 with angle set $\{\pm 1/5, \pm 3/5\}$
- Two 3-distance sets with angle set $\{\pm x_1, x_2\}$, where

$$x_1 = \frac{-4 \pm \sqrt{5}}{11}, \qquad x_2 = \frac{-3 \mp 2\sqrt{5}}{11}$$

Results on biangular lines in $\mathbb{R}^5$

I had a direct construction of 20 biangular lines with angle set $\{\pm 1/5, \pm 3/5\}$ and got surprised to find the following.

Theorem

The maximum number of biangular lines is 24 in dimension 5. There is a unique configuration realizing this up to isometry.

Proof:

n	2	3	4	...	11	12	13	...	22	23	24	25
#	2	5	25	...	54750	56548	52246	...	4	1	1	0

The candidate Gram matrix (if chosen well) carries the structure of a 4-class association scheme, and was found earlier by Hanaki.
More precisely: the Bose–Mesner algebra of this association scheme contains [a representative of] this Gram matrix.
It turns out, that this configuration can be obtained by doubling the shortest vectors of the D_4 lattice.

Doubling

Several important lessons learned from computation.

Theorem[M. Ganzhinov, F. Sz., 2019+]

Let $\mathcal{B}$ be a system of n biangular lines in $\mathbb{R}^{d-1}$ with angle set $\{0, \pm 1/2\}$. Then there exists a system of $2n$ biangular lines in $\mathbb{R}^d$ with angle set $\{\pm 1/5, \pm 3/5\}$.

Proof: Consider the set $\{[1, \pm 2v_i]/\sqrt{5} \colon v_i \in \mathcal{B}\}$. We have:

$$\left\langle [1, \pm 2v_i], [1, \pm 2v_j] \right\rangle = 1 \pm 4 \left\langle v_i, v_j \right\rangle = \begin{cases} 5, -3 & \text{for } i = j \\ 1 & \text{for } \left\langle v_i, v_j \right\rangle = 0 \\ 3, -1 & \text{for } \left\langle v_i, v_j \right\rangle = \pm 1/2 \end{cases}$$

Corollary

The number of biangular lines in $\mathbb{R}^d$ is at least $2(d-1)(d-2) \approx 2 \cdot d^2$.

Proof: Double the $(d-1)(d-2)$ shortest vectors of the D_{d-1} lattice.

Results on biangular lines in $\mathbb{R}^6$

The E_6 root system gives 36 biangular lines, doubling D_5 gives 40.

Theorem

The maximum number of biangular lines is 40 in dimension 6. There is a unique configuration realizing this up to isometry.

Proof: This case is roughly 1000-times more difficult than the case $\mathbb{R}^5$.

n	2	3	4	...	19	20	21	...	38	39	40	41
$\#$	2	5	25	...	$\approx 35M$	$\approx 36M$	$\approx 35M$	...	5	1	1	0

In dimension $d = 7$ we have 63 lines from E_7, and $2 \cdot 36 = 72$ from E_6. This case *might* be doable, but we don't expect anything more interesting to be found.

Summary

We have the following lower and upper bounds on the number N of biangular lines in $\mathbb{R}^d$.

$$4\binom{d-1}{2} \leq N \leq \binom{d+3}{4}, \qquad d \geq 2.$$

Exact values and constructions from exceptional lattices:

d	2	3	4	5	6	7	8	9	10	11	12
N	5	10	12	24	40	72-	126-	240-	256-	256-	256-

Constructions from 3-distance sets:

d	13	14	15	16	17	18	19	20	21	22	23
N	286-	364-	455-	560-	816-	816-	816-	816-	896-	1408-	2300-

Despite our efforts, the intriguing question of the existence of more than d^2 biangular lines in $\mathbb{R}^d$ remains open.

References

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The next part is tomorrow in building G, lecture room 507 at 10:00AM.