Math G2 Midterm 1

2022

1. Calculate the following determinants! (3+2 points)

   a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$,

   b) $\det\left\{\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}\right]^3\right\}$.

   Solution: (a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 40 \end{vmatrix}.$$

   Then, the determinant is $1 \cdot (-1) \cdot (-4) \cdot 40 = 160$.

   (b) Since $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2$ and $\det\begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} = 2$, then the value is $((-2) \cdot 2)^3 = -64$.

2. Does the following matrix have an inverse? If yes, calculate it by using one of the methods we learned. (5 points)

$$A = \begin{pmatrix} 2 & 0 & 7 \\ -1 & 4 & 5 \\ 3 & 1 & 2 \end{pmatrix}$$

   Solution: This matrix has an inverse, since its determinant (by using the first row) is

$$\det(A) = 2 \cdot \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} + 7 \cdot \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = 2 \cdot 3 + 7 \cdot (-13) = -85 \neq 0.$$

   Then, we proceed further by using the method of the adjoint: the matrix of the minors is

$$\begin{pmatrix} \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 5 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 7 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 7 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 7 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -17 & -13 \\ -7 & -17 & 2 \\ -28 & 17 & 8 \end{pmatrix}$$

   Then, we switch the signs:

$$\begin{pmatrix} 3 & 17 & -13 \\ 7 & -17 & -2 \\ -28 & -17 & 8 \end{pmatrix}$$

   and then take the transpose:

$$\begin{pmatrix} 3 & 7 & -28 \\ 17 & -17 & -17 \\ -13 & -2 & 8 \end{pmatrix}$$

Then, the inverse is

$$A^{-1} = \frac{1}{-85} \begin{pmatrix} 3 & 7 & -28 \\ 17 & -17 & -17 \\ -13 & -2 & 8 \end{pmatrix}$$

3. Solve the following SLAE! How many solutions does it have depending on the value of λ? (5 points)

$$\begin{cases} x_1 + x_2 + x_3 = 1, \\ x_1 + x_2 + \lambda x_3 = \lambda^2, \\ x_1 + \lambda x_2 + x_3 = \lambda. \end{cases}$$

Solution: The extended matrix is in the form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & \lambda & \lambda^2 \\ 1 & \lambda & 1 & \lambda \end{array}\right)$$

The steps of the Gaussian elimination are

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & \lambda - 1 & \lambda^2 - 1 \\ 0 & \lambda - 1 & 0 & \lambda - 1 \end{array}\right)$$

If $\lambda \neq 1$, then the system has the form

$$\begin{cases} x_1 + x_2 + x_3 = 1, \\ (\lambda - 1)x_3 = \lambda^2 - 1, \\ (\lambda - 1)x_2 = \lambda - 1. \end{cases}$$

From this we get $x_2 = 1$, $x_3 = \lambda + 1$ and $x_1 = 1 - x_2 - x_3 = -\lambda - 1$.

If $\lambda = 1$, then the only equation we have is

$$x_1 + x_2 + x_3 = 1$$

and we have two free variables, meaning that if $x_1 = p_1$ and $x_2 = p_2$, then the solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ 1 - p_1 - p_2 \end{pmatrix}$$

4. Calculate the eigenvalues and eigenvectors of the following matrix! (5 points)

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$$

Solution: The characteristic equation is

$$(6 - \lambda)(6 - \lambda) - 4 = 0$$

$$(6 - \lambda)^2 = 4$$

$$6 - \lambda = \pm 2$$

The eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 4$.

The eigenvector of $\lambda_1 = 8$ is the solution of

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equation we have to solve is

$$-2v_1 + 2v_2 = 0,$$

which has a solution $\begin{pmatrix} p \\ p \end{pmatrix}$, which can be chosen to be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenvector of $\lambda_2 = 4$ is the solution of

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equation we have to solve is

$$2v_1 + 2v_2 = 0,$$

which has a solution $\begin{pmatrix} p \\ -p \end{pmatrix}$, which can be chosen to be $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.