Math G2 Midterm 2

1 December 2022

Solutions

1. Determine whether the following series converges or diverges! (4 points)

$$2022 + \frac{2022^2}{2!} + \frac{2022^3}{3!} + \frac{2022^4}{4!} + \dots$$

Solution: The series is

$$\sum_{n=1}^{\infty} \frac{2022^n}{n!}. \qquad \text{(1 point)}$$

Let us use the ratio criterion (1 point):

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2022^{n+1}}{(n+1)!}}{\frac{2022^n}{n!}} = \lim_{n\to\infty} \frac{2022^{n+1}}{2022^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2022}{n+1} = 0, \qquad \text{(1 point)}$$

which is smaller than one so it converges (1 point).

(It can be also solved by the root criterion, and then we should use that $\sqrt[n]{n!} \to \infty$ as $n \to \infty$.)

2. Determine the Taylor series of the function $f(x) = \frac{1}{\sqrt[3]{(x+1)^2}}$. What is the coefficient of the third term? (4 points)

Solution:

$$f(x) = \frac{1}{\sqrt[3]{(x+1)^2}} = (x+1)^{-2/3} = \sum_{k=0}^{\infty} \binom{-2/3}{k} x^k. \qquad \text{(3 points)}$$

The coefficient of the third term is either

$$\binom{-2/3}{2} = \frac{(-2/3)(-2/3-1)}{2} = \frac{\frac{2}{3}\frac{5}{3}}{2} = \frac{5}{15}$$

or

$$\binom{-2/3}{3} = \frac{(-2/3)(-2/3-1)(-2/3-2)}{3} = \frac{\frac{2}{3}\frac{5}{3}\frac{8}{3}}{3} = \frac{80}{81}. \qquad \text{(1 point)}$$

3. Let us consider function $f : [0, \pi] \to \mathbb{R}$ which is defined as $f(x) = 1$. Define the extension of this function f in a way that its extension is defined for every $x \in \mathbb{R}$ and the Fourier series of the extension is a sine Fourier series. Calculate the coefficients of this series! (4 points)

Solution: Since we want a sine Fourier series, we would like to have an odd function. Then, the extension on $[-\pi, \pi]$ is

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi], \\ -1 & \text{if } x \in [-\pi, 0), \end{cases}$$

and also $f(x) = f(x + 2\pi)$. (1 point)

The coefficients are $a_0 = 0$, $a_k = 0$ (1 point) and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \cdot \sin(kx) dx + \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \sin(kx) dx =$$

$$= -\frac{1}{\pi} \left[\frac{-\cos(kx)}{k} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{-\cos(kx)}{k} \right]_{0}^{\pi} =$$

$$= \frac{1}{\pi k}\left(-(-\cos(0)+\cos(-k\pi)) + (-\cos(k\pi)+\cos(0))\right) =$$

$$= \frac{2}{\pi k}(\cos(0)-\cos(k\pi)) = \begin{cases} 0 & \text{if k is even} \\ \dfrac{4}{\pi k} & \text{if k is odd} \end{cases} \qquad \text{(2 points)}$$

4. Calculate the following limit! (4 points)

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{x^2-4x+5+y^2}-1}{x^2-4x+4+y^2}$$

Solution:

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{x^2-4x+5+y^2}-1}{x^2-4x+4+y^2} = \lim_{(x,y)\to(2,0)} \frac{\sqrt{(x-2)^2+1+y^2}-1}{(x-2)^2+y^2} =$$

Let us use the coordinates $x = r\cos(\varphi)+2$ and $y = r\sin(\varphi)$ (1 point).

$$= \lim_{r\to 0} \frac{\sqrt{r^2\cos^2(\varphi)+1+r^2\sin^2(\varphi)}-1}{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)} =$$

$$= \lim_{r\to 0} \frac{\sqrt{r^2+1}-1}{r^2} = \lim_{r\to 0} \frac{\frac{1}{2}(r^2+1)^{-1/2}2r}{2r} = \frac{1}{2}. \qquad \text{(3 points)}$$

(It can be solved in other ways too, like rewriting the numerator in a product form.)

5. Calculate the local extrema of the following function! (4 points)

$$f(x,y) = x^3+y^3-9xy+27.$$

Solution:

$$\frac{\partial f(x,y)}{\partial x} = 3x^2-9y = 0,$$
$$\frac{\partial f(x,y)}{\partial y} = 3y^2-9x = 0.$$

(0.5 point) From the first equation $y = \frac{1}{3}x^2$, and then from the second one:

$$3\frac{1}{9}x^4 - 9x = 0,$$

$$x\left(\frac{1}{3}x^3-9\right) = 0.$$

So the two solutions are $x_1 = 0$ and $x_2 = 3$ (0.25 point), meaning that $y_1 = 0$ and $y_2 = 3$ (0.25 point).

The Hessian is

$$\begin{pmatrix} 6x & -9 \\ -9 & 6y \end{pmatrix} \qquad \text{(1 point)}.$$

At $(x_1, y_1) = (0,0)$ it is $\begin{pmatrix} 0 & -9 \\ -9 & 0 \end{pmatrix}$ with determinant $-81 < 0$, so it has no extrema (the eigenvalues are ± 9) (1 point).

At $(x_2, y_2) = (3,3)$ it is $\begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix}$ with determinant 243 and $18 > 0$, so it has a minimum there (the eigenvalues are 9 and 27) (1 point)