Math G2 Practice 1

Matrices: operations, determinant.

1. Consider the following matrices:

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix}.$$

Calculate the following expressions!

(a) $(A+B)^2$.

Solution: We first calculate $A+B$:

$$A + B = \begin{pmatrix} 5+1 & -3+4 \\ 2+2 & 1-3 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & -2 \end{pmatrix}.$$

Then, we multiply $A+B$ by itself:

$$(A+B)\cdot(A+B) = \begin{array}{cc} & \begin{pmatrix} 6 & 1 \\ 4 & -2 \end{pmatrix} \\ \begin{pmatrix} 6 & 1 \\ 4 & -2 \end{pmatrix} & \begin{pmatrix} 6\cdot 6 + 1\cdot 4 & 6\cdot 1 + 1\cdot(-2) \\ 6\cdot 4 + 4\cdot(-2) & 4\cdot 1 + (-2)\cdot(-2) \end{pmatrix} \end{array} = \begin{pmatrix} 40 & 4 \\ 16 & 8 \end{pmatrix}$$

So the final answer is $\begin{pmatrix} 40 & 4 \\ 16 & 8 \end{pmatrix}$.

(b) $A^2 + 2AB + B^2$.

Solution: An important remark here is that this is not the same as the previous calculation: the reason for this is that the previous one has the form

$$(A+B)^2 = (A+B)\cdot(A+B) = A^2 + AB + BA + B^2,$$

and AB and BA are not the same.
First let us calculate $2AB$:

$$2AB = 2\cdot \begin{array}{cc} & \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 4 \\ 2 & -3 \end{pmatrix} & \begin{pmatrix} -1 & 29 \\ 4 & 5 \end{pmatrix} \end{array} = \begin{pmatrix} -2 & 58 \\ 8 & 10 \end{pmatrix}.$$

Similarly, we get that $A^2 = \begin{pmatrix} 19 & -18 \\ 12 & -5 \end{pmatrix}$ and $B^2 = \begin{pmatrix} 9 & -8 \\ -4 & 17 \end{pmatrix}$, so the final solution is

$$A^2 + 2AB + B^2 = \begin{pmatrix} 19 & -18 \\ 12 & -5 \end{pmatrix} + \begin{pmatrix} -2 & 58 \\ 8 & 10 \end{pmatrix} + \begin{pmatrix} 9 & -8 \\ -4 & 17 \end{pmatrix} = \begin{pmatrix} 26 & 32 \\ 16 & 22 \end{pmatrix}$$

(c) $A^2 + BA - AB - B^2$.

Since this is the same as Exercise (d), we will solve it there.

(d) $(A+B)(A-B)$

Solution: We already know from exercise (a) that $A + B = \begin{pmatrix} 6 & 1 \\ 4 & -2 \end{pmatrix}$, and the difference can be calculated similarly: $A - B = \begin{pmatrix} 4 & -7 \\ 0 & 4 \end{pmatrix}$. Then,

$$(A+B)(A-B) = \begin{pmatrix} 24 & -38 \\ 16 & -36 \end{pmatrix}.$$

(e) $A^2 - BA + AB - B^2$

Solution: The thing that you have to realize here is that this is $(A-B)(A+B)$. We already know these matrices, so after the multiplication we get $\begin{pmatrix} -4 & 18 \\ 16 & -8 \end{pmatrix}$.

(f) $A^2 - B^2$

Solution: It is important to realize that this is not the same as Exercise (d), since that one is Exercise (c). However, we already know A^2 and B^2 from Exercise (b), meaning that

$$A^2 - B^2 = \begin{pmatrix} 19 & -18 \\ 12 & -5 \end{pmatrix} - \begin{pmatrix} 9 & -8 \\ -4 & 17 \end{pmatrix} = \begin{pmatrix} 10 & -10 \\ 16 & -22 \end{pmatrix}.$$

2. [1] Prove that an arbitrary square matrix A can be written as a sum of a symmetric and an anti-symmetric matrix, and that this decomposition is unique.

Solution: Let us first remind ourselves the definition of symmetric and anti-symmetric matrices:

- Symmetric matrix: for every element $a_{i,j} = a_{j,i}$. Or, in other words, $A = A^T$ (where A^T is the transpose of A).
- Anti-symmetric matrix: for every element $a_{i,j} = -a_{j,i}$ and for its main diagonal $a_{j,j} = 0$. Or, in other words, $A = -A^T$.

One such decomposition is the following: let $S = \frac{1}{2}(A + A^T)$ and $Z = \frac{1}{2}(A - A^T)$. Then, it is easy to see that $S + Z = A$, since

$$S + Z = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T = A.$$

Moreover, S is symmetric, since

$$s_{i,j} = \frac{1}{2}(a_{i,j} + a_{j,i}) = \frac{1}{2}(a_{j,i} + a_{i,j}) = s_{j,i}.$$

Similarly, Z is anti-symmetric, since

$$z_{i,j} = \frac{1}{2}(a_{i,j} - a_{j,i}) = -\frac{1}{2}(a_{j,i} - a_{i,j}) = -z_{j,i}.$$

Now we prove that this decomposition is unique. Let us assume that there is a decomposition $A = A_1 + A_2$ where A_1 is symmetric and A_2 is anti-symmetric. Then it means that $A^T = A_1^T + A_2^T = A_1 - A_2$, where we used the definition of symmetric and anti-symmetric matrices. Then,

$$S = \frac{1}{2}(A + A^T) = \frac{1}{2}(A_1 + A_2 + A_1 - A_2) = A_1,$$

and similarly

$$Z = \frac{1}{2}(A - A^T) = \frac{1}{2}(A_1 + A_2 - A_1 + A_2) = A_2,$$

meaning that there is only one such decomposition.

3. Apply the method of Exercise 2. to the following matrix:

$$A = \begin{pmatrix} -4 & 2 & 3 \\ 1 & -1 & 2 \\ 8 & -6 & 5 \end{pmatrix}$$

Solution: Here

$$A^T = \begin{pmatrix} -4 & 1 & 8 \\ 2 & -1 & -6 \\ 3 & 2 & 5 \end{pmatrix},$$

meaning that

$$S = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{pmatrix} -8 & 3 & 11 \\ 3 & -2 & -4 \\ 11 & -4 & 10 \end{pmatrix} = \begin{pmatrix} -4 & 3/2 & 11/2 \\ 3/2 & -1 & -2 \\ 11/2 & -2 & 5 \end{pmatrix}$$

[1] This is a harder exercise, in the midterm/exam you will not be required to prove theorems.

and similarly

$$Z = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{pmatrix} 0 & 1 & -5 \\ -1 & 0 & 4 \\ 5 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & -5/2 \\ -1/2 & 0 & 4 \\ 5/2 & -4 & 0 \end{pmatrix}.$$

It can be seen that S is symmetric, Z is anti-symmetric, and also $A = S + Z$.

4. Compute the following determinant!

$$\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix} = (1+\sqrt{2})(1-\sqrt{2}) - (2-\sqrt{3})(2+\sqrt{3}) = (1-2) - (4-3) = -2,$$

where we used the identity $(a-b)(a+b) = a^2 + b^2$.

5. Compute the following determinant!

$$\begin{vmatrix} a+bi & c+di \\ c-di & a-bi \end{vmatrix}$$

Solution:

$$\begin{vmatrix} a+bi & c+di \\ c-di & a-bi \end{vmatrix} = (a+bi)(a-bi) - (c+di)(c-di) = a^2+b^2-(c^2+d^2) = a^2+b^2-c^2-d^2,$$

where we used the fact that $i^2 = -1$.

6. [2] Prove that the determinant of an upper-triangular matrix is the product of the elements in its main diagonal.

Solution: Let us consider the determinant of an n-by-n upper triangular matrix:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{vmatrix} =$$

Let us use the first column to compute the determinant: then, the terms we get are:

$$= a_{1,1} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} + 0 \cdot (-1) \cdot \ldots + 0 \cdot (+1) \cdot \ldots + \cdots =$$

Here we observe that since every element (apart from $a_{1,1}$) is zero in the first column, all the terms apart from the first one will be zero, since all of them are multiplied by zero. This means that

$$= a_{1,1} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} =$$

Now we apply the same process: we compute the determinant using the first column.

$$= a_{1,1} \cdot \left(a_{2,2} \cdot \begin{vmatrix} a_{3,3} & a_{3,4} & \dots & a_{3,n} \\ 0 & a_{4,4} & \dots & a_{4,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} + 0 \cdot (-1) \cdot \ldots + 0 \cdot (+1) \cdot \ldots + \ldots \right) =$$

[2] This is a harder exercise, in the midterm/exam you will not be required to prove theorems.

Our argument is the same as in the previous case: every element (apart from $a_{2,2}$) is zero in the first column, meaning that all the terms apart from the first one will be zero, since all of them are multiplied by zero. This means that

$$= a_{1,1} \cdot a_{2,2} \cdot \begin{vmatrix} a_{3,3} & a_{3,4} & \dots & a_{3,n} \\ 0 & a_{4,4} & \dots & a_{4,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} =$$

This process can be continued until we get the following form:

$$= a_{1,1} \cdot a_{2,2} \cdot \ldots \cdot a_{n,n},$$

so we got the product of the elements in the main diagonal, so the statement is proved.

Remark: This statement can also be proved using the first row of the matrix: in this case we should argue that apart from the first minor, every other minor contains an all-zero column, thus they are all zero.

Remark: This statement is also true for lower triangular, and diagonal matrices. The proofs are very similar in those cases.

7. Calculate the following determinant!

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$$

Solution: Instead of using the usual formula for 4-by-4 matrices, we will use the following proposition from the lecture:

Proposition: If we add a row of the matrix (possibly multiplied by a real number) to some other row, then the determinant remains the same.

So our goal in this exercise is to somehow transform the determinant using the steps mentioned in the proposition resulting in an upper-triangular form: the reason for this goal is that by Exercise 6., we can calculate the determinant of such matrices pretty easily: we only have to take the product of their elements in the main diagonal.

This means that our goal is to make the elements marked by red disappear (i.e. to make them zero):

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \color{red}{1} & 2 & 3 & 4 \\ \color{red}{1} & \color{red}{3} & 6 & 10 \\ \color{red}{1} & \color{red}{4} & \color{red}{10} & 20 \end{vmatrix}.$$

To make the red ones in the first column disappear, we can use the first row: if we subtract the first row from all the other rows, they are going to be zero. For example, if we subtract the first row from the second one:

$$\begin{array}{r} \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \\ -\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix} \end{array}$$

So our determinant has the following form:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$$

Similarly, when we subtract the first one from the third one:

$$\begin{array}{r} \begin{pmatrix} 1 & 3 & 6 & 10 \end{pmatrix} \\ -\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 2 & 5 & 9 \end{pmatrix} \end{array}$$

Then, the current form of the determinant is:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 1 & 4 & 10 & 20 \end{vmatrix}$$

Also, we subtract the first row from the last one:

$$\begin{array}{r} \begin{pmatrix} 1 & 4 & 10 & 20 \end{pmatrix} \\ -\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 3 & 9 & 19 \end{pmatrix} \end{array}$$

So the first column now looks as it should be:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

For the second column, we are going to use the second row of the matrix: for $a_{3,2} = 2$ to be zero, we are going to multiply the second row by 2 and going to subtract it from the third one:

$$\begin{array}{r} \begin{pmatrix} 0 & 2 & 5 & 9 \end{pmatrix} \\ -\begin{pmatrix} 0 & 2 & 4 & 6 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & 1 & 3 \end{pmatrix} \end{array}$$

So the determinant looks like this:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

Also, we are going to multiply the second row by 3 and subtract it from the last one:

$$\begin{array}{r} \begin{pmatrix} 0 & 3 & 9 & 19 \end{pmatrix} \\ -\begin{pmatrix} 0 & 3 & 6 & 9 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & 3 & 10 \end{pmatrix} \end{array}$$

So the determinant is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix}$$

Now we deal with the last problematic element, i.e. $a_{4,3} = 3$. We are going to multiply the third row and subtract it from the last one:

$$\begin{array}{r} \begin{pmatrix} 0 & 0 & 3 & 10 \end{pmatrix} \\ -\begin{pmatrix} 0 & 0 & 3 & 9 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then, the final form of the determinant is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

As we can see, this is now an upper triangular matrix, so its determinant is $1 \cdot 1 \cdot 1 \cdot 1 = 1$, and this is the same as the determinant of the initial matrix, so its determinant was one.

Remark: The process which we followed here is called *Gaussian elimination*, and we will use it in Practice 3 to solve systems of linear equations.

8. Calculate the following determinant:

$$\begin{vmatrix} a+x & a & a \\ a & a+x & a \\ a & a & a \end{vmatrix}$$

Solution: Instead of using the usual definition, we are going to apply the method discussed in the previous exercise.

First let us subtract the last row from the first one:

$$\begin{vmatrix} x & 0 & 0 \\ a & a+x & a \\ a & a & a \end{vmatrix}$$

Then, let us subtract the last row from the second one:

$$\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ a & a & a \end{vmatrix}$$

We got a lower triangular matrix, meaning that its determinant is $x \cdot x \cdot a = x^2 a$ (by the remark after Exercise 6.).

9. Calculate the following determinant!

$$\det\left\{\left[\begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}\right]^3\right\}$$

Solution: Instead of doing the calculations, we are going to use the following theorem stated on the lecture:

Theorem: For n-by-n square matrices A and B, we have

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

Let us use the notations $A = \begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$. Then, the thing we have to calculate is

$$\det((A \cdot B)^3) =$$

Now we use the fact that $\det(M^3) = \det(M \cdot M \cdot M) = \det(M) \cdot \det(M) \cdot \det(M) = (\det(M))^3$.

$$= (\det(A \cdot B))^3 = (\det(A) \cdot \det(B))^3 = (\det(A))^3 \cdot (\det(B))^3$$

So we only have to calculate the determinants of matrices A and B: these are $\det(A) = 9$ and $\det(B) = -4$, so the final answer is $9^3 \cdot (-4)^3 = -46656$.