Math G2 Practices 10

Multivariable functions II.: directional derivative, tangent plane, local extrema

1 Directional derivative

The directional derivative of function f in the direction of vector v can be calculated as

$$\partial_v f = \text{grad}(f) \cdot \frac{v}{|v|},$$

where $\cdot$ is a scalar product and $|v|$ is the length of a vector v.

1. Calculate the directional derivative of $f(x, y, z) = e^{-(x^2+y^2)} - z$ in the direction of vector $v = (3, 2, -5)^T$ at point $P(1, 0, 1)$.

   Solution: The gradient is

$$\text{grad}(f) = \begin{pmatrix} \dfrac{\partial f(x,y,z)}{\partial x} \\ \dfrac{\partial f(x,y,z)}{\partial y} \\ \dfrac{\partial f(x,y,z)}{\partial z} \end{pmatrix} = \begin{pmatrix} e^{-(x^2+y^2)}(-2x) \\ e^{-(x^2+y^2)}(-2y) \\ -1 \end{pmatrix}$$

   Also,

$$\frac{v}{|v|} = \frac{(3,\ 2,\ 5)}{\sqrt{3^2+2^2+5^2}} = \frac{1}{\sqrt{38}}(3,\ 2,\ 5).$$

   The derivative is only needed at point $(1, 0, 1)$, and here

$$\text{grad}(f) = \begin{pmatrix} e^{-1}(-2) \\ 0 \\ -1 \end{pmatrix}.$$

   Then, the directional derivative is

$$\partial_v f = \text{grad}(f) \cdot \frac{v}{|v|} = \begin{pmatrix} e^{-1}(-2) \\ 0 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{38}}(3,\ 2,\ 5) =$$

$$= \frac{1}{\sqrt{38}} \left(-2e^{-1} \cdot 3 + 0 \cdot 2 + (-1) \cdot (-5)\right) \approx 0.453$$

2 Tangent plane

The equation of a tangent plane at point (x_0, y_0) of the surface given by the function $f(x, y)$ is given as

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0).$$

2. Give the equation of the tangent plane of the surface $xy^2 + z^3 = 12$ at point $P(1, 2, 2)$.

   Solution: The function we have to observe is given by the equation

$$f(x, y) = z(x, y) = \sqrt[3]{12 - xy^2}.$$

   We can either differentiate this function, or consider the initial equation of the surface, namely

$$xy^2 + (z(x, y))^3 = 12,$$

and by considering it as an implicit function we can calculate the partial derivatives: by differentiating it with respect to variable x:

$$y^2 + 3z^2 \frac{\partial z(x,y)}{\partial x} = 0,$$

$$\frac{\partial z(x,y)}{\partial x} = -\frac{y^2}{3z^2} = -\frac{4}{3 \cdot 4} = -\frac{1}{3}.$$

Similarly, the partial derivative with respect to y:

$$2yz + 3z^2 \frac{\partial z(x,y)}{\partial y} = 0,$$

$$\frac{\partial z(x,y)}{\partial y} = -\frac{2xy}{3z^2} = -\frac{1}{3}.$$

Then, the equation of the tangent is

$$z - 2 = -\frac{1}{3}(x-1) - \frac{1}{3}(y-2).$$

3. Calculate the equation of the tangent plane at point $\frac{1}{\sqrt{13}}(8,4,1)$ of the ellipsoid given by the equation

$$\frac{x^2}{8} + \frac{y^2}{4} + z^2 = 1.$$

Solution: The partial derivatives are

$$\frac{2x}{8} + 2z\frac{\partial z(x,y)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z(x,y)}{\partial x} = -\frac{1}{8}\frac{x_0}{z_0} = -1,$$

$$\frac{2y}{4} + 2z\frac{\partial z(x,y)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z(x,y)}{\partial y} = -\frac{1}{4}\frac{y_0}{z_0} = -1.$$

Then, the equation of the plane is given by

$$z - \frac{1}{\sqrt{13}} = -(x - \frac{8}{\sqrt{13}}) - (y - \frac{4}{\sqrt{13}}),$$

$$z = -x - y + \sqrt{13}.$$

3 Local extrema

Local extrema can be find at those points where the partial derivatives are zero. Moreover, let us consider the Hessian (or Jacobian) of $f(x_1, x_2, \ldots, x_n)$ given by

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Then, if at point p all the partial derivatives are zero, and

- the principal minors (the determinants of those square sub-matrices which have their upper left corner at the upper left corner of the original matrix) of the Hessian are all positive, then function f has a local minimum at point p.
- the principal minors of the Hessian have the sign pattern $- + - + - + \ldots$, then function f has a local minimum at point p.
- the principal minors of the Hessian are all non-zero, and the sign pattern is neither of the previous two, then there is no extrema at this point.
- one of the principal minors is zero, then we do not know what happens there (can be minimum, maximum or neiter).

The special case of two-variable functions can be formulated as follows: if at point p all the partial derivatives are zero, and

- the determinant of the Hessian is positive, and
  - $\frac{\partial^2 f}{\partial x_1^2} > 0$, then function f has a local minimum at point p.
  - $\frac{\partial^2 f}{\partial x_1^2} < 0$, then function f has a local minimum at point p.
- the determinant of the Hessian is negative, then it has no extrema there.
- the determinant is zero, then we do not know what happens there (can be minimum, maximum or neiter).

Another condition is the following: if at point p all the partial derivatives are zero, and

- the eigenvalues of the Hessian are all positive, then function f has a local minimum at point p.
- the eigenvalues of the Hessian are all negative, then function f has a local minimum at point p.
- the eigenvalues of the Hessian have different signs and none of them is zero, then function f has no extremum at point p.
- one of the eigenvalues of the Hessian is zero, then we do not know what happens there (can be minimum, maximum or neiter).

4. Calculate the local minima and maxima of the function $f(x, y) = x^3 + y^3 - 3xy$.

   Solution: The partial derivatives are

$$\frac{\partial f(x, y)}{\partial x} = 3x^2 - 3y,$$

$$\frac{\partial f(x, y)}{\partial y} = 3y^2 - 3x,$$

so the system we have to solve is

$$3x^2 - 3y = 0,$$
$$3y^2 - 3x = 0.$$

The solutions of this equation are $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 1)$.
The Hessian is

$$\begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

At $(x_1, y_1) = (0, 0)$, it is $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$, with determinant -9, so it has no extrema here. (Also, the eigenvalues are ± 3, so it has no local extrema here.) At $(x_1, y_1) = (1, 1)$, it is $\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$, with determinant 27 and $\dfrac{\partial^2 f}{\partial x_1^2} = 6 > 0$ so it has a local minimum here. (Also, the eigenvalues are 3 and 9, so it has local minimum here.)

5. Calculate the local minima and maxima of the function $f(x, y) = \ln(x) + \ln(y) - x - y$.
   Solution: The partial derivatives are

$$\frac{\partial f(x, y)}{\partial x} = \frac{1}{x} - 1,$$
$$\frac{\partial f(x, y)}{\partial y} = \frac{1}{y} - 1,$$

so the system we have to solve is

$$\frac{1}{x} - 1 = 0,$$
$$\frac{1}{y} - 1 = 0.$$

The solution of this equation is $(x, y) = (1, 1)$.
The Hessian is

$$\begin{pmatrix} \frac{-1}{x^2} & 0 \\ 0 & \frac{-1}{y^2} \end{pmatrix}.$$

At $(x, y) = (1, 1)$, it is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, with determinant 1 and $\dfrac{\partial^2 f}{\partial x_1^2} = -1 < 0$ so it has a local maximum here. (Also, the eigenvalues are -1 and -1, so it has local maximum here.)

6. Calculate the local extrema of the following function!

$$f(x, y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4.$$

Solution: Here

$$f'_x = 6x^2 + 6x - 12,$$

which has two roots: $x_1 = 1$ and $x_2 = -2$. Moreover,

$$f'_y = 3y^2 - 3,$$

which also has two roots: $y_1 = 1$ and $y_2 = -1$. Then, the possible extrema of this function are: $(1, 1)$, $(1, -1)$, $(-2, 1)$ and $(-2, -1)$. Let us compute the Hessian of this function:

$$H(x, y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 12x + 6 & 0 \\ 0 & 6y \end{pmatrix}.$$

For $(1, 1)$, the Hessian has the form:

$$H(1, 1) = \begin{pmatrix} 18 & 0 \\ 0 & 6 \end{pmatrix},$$

which has a determinant of 108, which is positive, and also $f_x x'' = 18 > 0$, so the function has a minimum there.

For $(1, -1)$, the Hessian has the form:

$$H(1,1) = \begin{pmatrix} 18 & 0 \\ 0 & -6 \end{pmatrix},$$

which has a determinant of -108, which is negative, so the function has a saddle point there.

For $(-2, 1)$, the Hessian has the form:

$$H(1,1) = \begin{pmatrix} -18 & 0 \\ 0 & 6 \end{pmatrix},$$

which has a determinant of -108, which is negative, so the function has a saddle point there.

For $(-2, -1)$, the Hessian has the form:

$$H(1,1) = \begin{pmatrix} -18 & 0 \\ 0 & -6 \end{pmatrix},$$

which has a determinant of 108, which is positive, and also $f_x x'' = -18 < 0$, so the function has a maximum there.

7. Calculate the local minima and maxima of the function $f(x, y, z) = x^3 + xy + y^2 + 2z^2 + 4z + 1 - x$.

Solution: The partial derivatives are

$$\frac{\partial f(x,y,z)}{\partial x} = 3x^2 + y - 1,$$
$$\frac{\partial f(x,y,z)}{\partial y} = x + 2y,$$
$$\frac{\partial f(x,y,z)}{\partial z} = 4z + 4,$$

so the system we have to solve is

$$3x^2 + y - 1 = 0,$$
$$x + 2y = 0,$$
$$4z + 4 = 0.$$

The solutions of this equation are $(x_1, y_1, z_1) = \left(\frac{2}{3}, -\frac{1}{3}, -1\right)$ and $(x_2, y_2, z_2) = \left(-\frac{1}{2}, \frac{1}{4}, -1\right)$.

The Hessian is

$$\begin{pmatrix} 6x & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

At $(x_1, y_1, z_1) = \left(\frac{2}{3}, -\frac{1}{3}, -1\right)$, it is

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

with determinant $28 > 0$, second principal minor $7 > 0$ and $\frac{\partial^2 f}{\partial x_1^2} = 4 > 0$ so it has a local minimum here. (Also, the eigenvalues are 4, $\sqrt{2} + 3$ and $-\sqrt{2} + 3$, so it has local minimum here.) At $(x_2, y_2, z_2) = \left(-\frac{1}{2}, \frac{1}{4}, -1\right)$, it is

$$\begin{pmatrix} -3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

with determinant $-3 < 0$, second principal minor $-7 < 0$ and $\frac{\partial^2 f}{\partial x_1^2} = -3 < 0$ so it has no local extremum here. (Also, the eigenvalues are 4, $\frac{-\sqrt{29} - 1}{2}$ and $\frac{\sqrt{29} - 1}{2}$, so it has no local extremum here.)

8. Consider the following matrix:

$$\begin{pmatrix} -6 & 1 & -3 & -1 \\ 1 & -4 & -4 & 4 \\ -3 & -4 & -8 & 4 \\ -1 & 4 & 4 & -4 \end{pmatrix}$$

Suppose this a Hessian of a 4-variable function calculated at some point p. Is this point an extrema?

Solution: If someone starts to calculate the determinants, the first minor is $-6 < 0$, so it might be a maximum. Then, the second is

$$\det \begin{pmatrix} -6 & 1 \\ 1 & -4 \end{pmatrix} = 23 > 0,$$

so the sign pattern is still fine. The third one is

$$\det \begin{pmatrix} -6 & 1 & -3 \\ 1 & -4 & -4 \\ -3 & -4 & -8 \end{pmatrix} = (-6) \begin{vmatrix} -4 & -4 \\ -4 & -8 \end{vmatrix} + (-1) \begin{vmatrix} 1 & -4 \\ -3 & -8 \end{vmatrix} + (-3) \begin{vmatrix} 1 & -4 \\ -3 & -4 \end{vmatrix} = -28 < 0,$$

so it still might be a minimum.

For the last one, let us realize that the second and last rows are dependent: if we add the second row to the last one, we get the matrix

$$\begin{pmatrix} -6 & 1 & -3 & -1 \\ 1 & -4 & -4 & 4 \\ -3 & -4 & -8 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which has the same determinant as the original one, but since it has a zero row, this value is zero. This means that at this point there is no guarantee for an extremum.

4 Application of local extrema

7. Give the volume of the largest right rectangular prism that we can get inside the elliptic paraboloid given by the equation

$$z = 2x^2 + y^2$$

in a way that the top side of the prism is in the plane $z = 5$.

Solution: Let us assume that the bottom side of the prism is at height w, and let us assume that one of its vertices is at point (u, v, w). Then, the lengths of the sides of the prism are $5 - w$, $2u$ and $2v$, meaning that its volume is $4uv(5 - w)$.

However, we know that the point (u, v, w) is on the paraboloid, meaning that the equation

$$w = 2u^2 + v^2$$

also holds, so the volume is

$$\text{Vol} = 4uv(5 - 2u^2 - v^2).$$

So we are searching for the maximum of the function

$$f(u, v) = 4uv(5 - 2u^2 - v^2).$$

The partial derivatives are

$$\frac{\partial f}{\partial u} = 4v(5 - 2u^2 - v^2) + 4uv(-4u),$$

$$\frac{\partial f}{\partial v} = 4u(5 - 2u^2 - v^2) + 4uv(-2v),$$

so the system we have to solve is

$$\begin{aligned} 4v(5 - 2u^2 - v^2) + 4uv(-4u) &= 0, \\ 4u(5 - 2u^2 - v^2) + 4uv(-2v) &= 0. \end{aligned}$$

If we divide the first equation by $4v$ and the second by $4u$, we get a more simple form:

$$5 - 2u^2 - v^2 = 4u^2,$$
$$5 - 2u^2 - v^2 = 2v^2.$$

By subtracting the first equation from the second one, we get

$$2v^2 - 4u^2 = 0,$$

from which $u = \frac{v}{\sqrt{2}}$. From the first equation, by substitution we get

$$5 - 2\frac{v^2}{2} - v^2 = 4\frac{v^2}{2},$$

meaning that $5 = 4v^2$ and $v = \frac{\sqrt{5}}{2}$. Similarly, $u = \frac{\sqrt{5}}{2\sqrt{2}}$.

The Hessian has the form

$$\begin{pmatrix} -16uv - 32uv & 20 - 8u^2 - 12v^2 - 16u^2 \\ 20 - 24u^2 - 4v^2 - 8v^2 & -8uv - 16uv \end{pmatrix},$$

which can also be written as

$$\begin{pmatrix} -48uv & 20 - 24u^2 - 12v^2 \\ 20 - 24u^2 - 12v^2 & -24uv \end{pmatrix}.$$

At the point $(u, v) = \left(\frac{\sqrt{5}}{2\sqrt{2}}, \frac{\sqrt{5}}{2}\right)$, it has the form

$$\begin{pmatrix} -48\frac{5}{4\sqrt{2}} & 20 - 24\frac{5}{8} - 12\frac{5}{4} \\ 20 - 24\frac{5}{8} - 12\frac{5}{4} & -24\frac{5}{4\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-60}{\sqrt{2}} & -10 \\ -10 & \frac{-30}{\sqrt{2}} \end{pmatrix}$$

Here the determinant is

$$900 - 100 > 0$$

and the upper left element is negative, so we have a maximum at this point.

The maximal volume is

$$\text{Vol} = f\left(\frac{\sqrt{5}}{2\sqrt{2}}, \frac{\sqrt{5}}{2}\right) = 4\frac{\sqrt{5}}{2\sqrt{2}}\frac{\sqrt{5}}{2}\left(5 - 2\left(\frac{\sqrt{5}}{2\sqrt{2}}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2\right) = \frac{25}{2\sqrt{2}}$$

8. Determine the distance of the curves $y = x^2$ and $y = 1 - (x+2)^2$.

Solution: At a given point x_1, the first curve is at $(x_1, y_1) = (x_1, x_1^2)$ and the second at a given point x_2 is $(x_2, y_2) = (x_2, 1 - (x_2 + 2)^2)$. The distance between two curves is the smallest possible distance between some points which are on the given curves.

Since the distance has a minimum where the square of the distance has a minimum, we will observe the minimum of the square instead.

The square of the distance is

$$f(x_1, x_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x_2)^2 + (x_1^2 - 1 + (x_2 + 2)^2)^2$$

We are searching for the minimum of this function.

The partial derivatives are:

$$\frac{\partial f}{\partial x_1} = 2(x_1 - x_2) + 2(x_1^2 - 1 + (x_2 + 2)^2)(2x_1)$$

$$\frac{\partial f}{\partial x_2} = -2(x_1 - x_2) + 2(x_1^2 - 1 + (x_2 + 2)^2)(2(x_2 + 2))$$

The system we have to solve is

$$2(x_1 - x_2) + 2(x_1^2 - 1 + (x_2 + 2)^2)(2x_1) = 0,$$
$$-2(x_1 - x_2) + 2(x_1^2 - 1 + (x_2 + 2)^2)(2(x_2 + 2)) = 0.$$

Let us modify the equations by moving the long bracket to the left-hand side in both cases:

$$4x_1(x_1^2 - 1 + (x_2 + 2)^2) = -2(x_1 - x_2),$$
$$(4x_2 + 8)(x_1^2 - 1 + (x_2 + 2)^2) = 2(x_1 - x_2).$$

Then,

$$\frac{-2x_1 + 2x_2}{4x_1} = \frac{2x_1 - 2x_2}{4x_2 + 8},$$
$$\frac{4x_1}{4x_2 + 8} = \frac{-2x_1 + 2x_2}{2x_1 - 2x_2} = -1,$$

meaning that $x_1 = -x_2 - 2$ and $x_2 = -x_1 - 2$. Let us substitute this formula to the second equation:

$$(4(-x_1 - 2) + 8)(x_1^2 - 1 + ((-x_1 - 2) + 2)^2) = 2(x_1 - (-x_1 - 2)),$$
$$(-4x_1)(x_1^2 - 1 + x_1^2) = 2(2x_1 + 2),$$
$$(-4x_1)(2x_1^2 - 1) = 4x_1 + 4,$$
$$-8x_1^3 + 4x_1 = 4x_1 + 4,$$

meaning that $x_1^3 = -\frac{1}{2}$, so $x_1 = -\frac{1}{\sqrt[3]{2}}$ and $x_2 = \frac{1}{\sqrt[3]{2}} - 2$.

The second derivatives are:

$$\frac{\partial^2 f}{\partial x_1^2} = 2 + 4(x_1^2 - 1 + (x_2 + 2)^2) + 4x_1(2x_1) = 12x_1^2 + 4x_2^2 + 8x_2 + 14,$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -2 + 8x_1(x_2 + 2) = 8x_1x_2 + 14$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2 + 4(x_1^2 - 1 + (x_2 + 2)^2) + 4(x_2 + 2)(2(x_2 + 2)) = 4x_1^2 - 2 + 12(x_2 + 2)^2$$

So the Hessian is

$$\begin{pmatrix} 12x_1^2 + 4x_2^2 + 8x_2 + 14 & 8x_1x_2 + 14 \\ 8x_1x_2 + 14 & 4x_1^2 - 2 + 12(x_2 + 2)^2 \end{pmatrix}$$

The determinant at point $(x_1, x_2) = \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} - 2\right)$ is $48\sqrt[3]{2}(\sqrt[3]{2} - 1) > 0$ and the upper left element is $14 + 8\sqrt[3]{2} - 2 \cdot 2^{2/3} \approx 17.73 > 0$, so it is indeed a minimum.

The distance is

$$\sqrt{f\left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} - 2\right)} =$$

$$= \sqrt{\left(-\frac{1}{\sqrt[3]{2}} - \left(\frac{1}{\sqrt[3]{2}} - 2\right)\right)^2 + \left(\left(-\frac{1}{\sqrt[3]{2}}\right)^2 - 1 + \left(\frac{1}{\sqrt[3]{2}}\right)^2\right)^2} \approx 0.656.$$