Math G2 Practices 11

Constrained optimization
(Conditional extrema)

Main idea: we are searching for the local minimum/maximum of a function $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ when $\boldsymbol{x} \in A$, $A \subset \mathbb{R}^n$.

The extremum might be

- either inside the set, and then we can search for it using the methods described in the previous practice session, or
- it might be at the boundary of the set: to find these, we are using the method of Lagrange multipliers.

Method of Lagrange multipliers: Let us assume that the set A can be described by the inequality

$$g(\boldsymbol{x}) \leq b.$$

Then, let us define the **Lagrange function** as

$$L(\boldsymbol{x}) = f(\boldsymbol{x}) - \lambda(g(\boldsymbol{x}) - b).$$

Here $\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**. Then, the possible local extrema on the boundary are those points where the partial derivatives of function L are zero. Not all such points are minima/maxima, but that one where the value of the function is the largest will be the maximum, and where the value of the function is the smallest will be the minimum.

1. Find the maximum and minimum of the function $f(x, y) = x^2 - y^2$ on the set $x^2 + y^2 \leq 4$.

   Solution: Consider the possible extrema of the function inside the set. For this, let us calculate the partial derivatives:

   $$\frac{\partial f(x,y)}{\partial x} = 2x,$$

   $$\frac{\partial f(x,y)}{\partial x} = -2y,$$

   meaning that the equations we have to solve are

   $$2x = 0,$$
   $$-2y = 0.$$

   So the possible extremum is $(x_1, y_1) = (0, 0)$. Here the Hessian is

   $$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

   This means that we have an extremum here (the determinant is negative).

   Now consider the boundary. The Lagrange function is

   $$L(x, y) = f(x, y) - \lambda(g(x, y) - b) = x^2 - y^2 - \lambda(x^2 + y^2 - 4).$$

   The partial derivatives are

   $$\frac{\partial L(x,y)}{\partial x} = 2x - \lambda 2x,$$
   $$\frac{\partial L(x,y)}{\partial y} = -2y - \lambda 2y,$$

   so the system of equations we have to solve is

   $$2x - \lambda 2x = 0,$$
   $$-2y - \lambda 2y = 0,$$
   $$x^2 + y^2 = 4.$$

   From the first equation we get $2x(1 - \lambda) = 0$, and the second can also be rewritten as $-2y(1 + \lambda) = 0$, meaning that we have the following cases:

- *Case 1*: $x = 0$. Then, from the third one we get $y = \pm 2$ and here $\lambda = -1$, thus resulting in the possible candidates $(x_2, y_2) = (0, 2)$ and $(x_3, y_3) = (0, -2)$.
- *Case 2*: $y = 0$, then from the third equation we get $x = \pm 2$ and from the first one $\lambda = 1$, resulting in the possible candidates $(x_4, y_4) = (2, 0)$ and $(x_5, y_5) = (-2, 0)$.

Consequently, the possible extrema are $(x_2, y_2) = (0, 2)$, $(x_3, y_3) = (0, -2)$, $(x_4, y_4) = (2, 0)$ and $(x_5, y_5) = (-2, 0)$. The values of the function at these points are $f(0, 2) = -4$, $f(0, -2) = -4$, $f(2, 0) = 4$, $f(-2, 0) = 4$, meaning that the points (x_2, y_2) and (x_3, y_3) are the conditional minima and the points (x_4, y_4) and (x_5, y_5) are the maxima.

Remark: one can also solve the second part of this exercise as a one-dimensional problem: the boundary $x^2 + y^2 = 4$ can also be expressed as those points for which $y^2 = 4 - x^2$ and $-2 \leq x \leq 2$, and then the previous ,function f can be viewed as a one-variable function

$$h(x) = x^2 - y^2 = x^2 - (4 - x^2)^2$$

on the interval $x \in [-2, 2]$.

2. Search for the maximum or minimum of the function

$$f(x, y) = x^2 - 2x + y^2$$

inside set A where $A = B \cap C$, in which B is the disc with radius 2 centered at the origin, and C is the union of the first, third and fourth quadrants of the coordinate system.

Solution: Let us first consider the inside of the set. First, calculate the partial derivatives:

$$\frac{\partial f(x, y)}{\partial x} = 2x - 2,$$

$$\frac{\partial f(x, y)}{\partial x} = 2y,$$

meaning that the equations we have to solve are

$$\begin{aligned} 2x - 2 &= 0, \\ 2y &= 0. \end{aligned}$$

So the possible extremum is $(x_1, y_1) = (1, 0)$. Here the Hessian is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This means that since the determinant is positive along with the upper left element, we have a local minimum at $(x_1, y_1 = 1, 0)$.

Now let us consider the boundary. It has three sections: on one hand we have the outer circle given by the equation $x^2 + y^2 = 4$, and also the sections of the x and y axis. Let us consider the circle first.

Here $g(x, y) = x^2 + y^2$ and $b = 4$, meaning that the Lagrange function is

$$L(x, y) = x^2 - 2x + y^2 - \lambda(x^2 + y^2 - 4).$$

The partial derivatives are

$$\frac{\partial L(x, y)}{\partial x} = 2x - 2 - \lambda 2x,$$

$$\frac{\partial L(x, y)}{\partial y} = 2y - \lambda 2y,$$

so the system of equations we have to solve is

$$\begin{aligned} 2x - 2 - \lambda 2x &= 2((1 - \lambda)x - 1) = 0, \\ 2y - \lambda 2y &= 2y(1 - \lambda) = 0, \\ x^2 + y^2 &= 4. \end{aligned}$$

From the second equation it is evident that we have two cases: either $\lambda = 1$, or $y = 0$. If $\lambda = 1$, then from the first equation we get $-2 = 0$ which is clearly a contradiction. Then

$y = 0$, so from the third one we get $x = \pm 2$, meaning that we have two further candidates at $(x_2, y_2) = (2, 0)$ and $(x_3, y_3) = (-2, 0)$.

Now let us consider the other sections of the boundary, namely the sections on the two axes. If we are on the x-axis, it means that $y = 0$, so we must search for the local extrema of the function $f(x) = x^2 - 2x$ when $x \in [0, 2]$. It is easy to see (using the methods learned in G1) that it has a minimum at $x = 1$, resulting in the point $(1, 0)$, but we have already found that one (this was (x_1, y_1)). Also, we might have possible extrema at the endpoints of this section, which are $(2, 0)$ and $(0, 0)$: the former was already found, so let us add the origin to the possible candidates as $(x_4, y_4) = (0, 0)$.

If we are on the y-axis, it means that $x = 0$, so we must search for the extrema of the function $f(y) = y^2$ if $y \in [0, 2]$. It has a minimum at $y = 0$ and a maximum at $y = 2$: the first one is the origin which,h we have already found, but the second one was not yet fund: call it $(x_5, y_5) = (0, 2)$.

Then, the possible candidates for a col extrema are $(x_1, y_1 = 1, 0)$, $(x_2, y_2) = (2, 0)$, $(x_3, y_3) = (-2, 0)$, $(x_4, y_4) = (0, 0)$ and $(x_5, y_5) = (0, 2)$. The values of the function at ,these points are

$$f(x_1, y_1) = f(1, 0) = -1,$$

$$f(x_, y_2) = f(2, 0) = 0,$$
$$f(x_3, y_3) = f(-2, 0) = 8,$$
$$f(x_4, y_4) = f(0, 0) = 0,$$
$$f(x_5, y_5) = f(0, 2) = 4.$$

This means that the maximum is at $(x_3, y_3) = (-2, 0)$ and the minimum is at $(x_1, y_1) = (1, 0)$.

Applications

3. Consider a quadrilateral with sides a, b, c and d and two angles (which are at opposites vertices) α and β. With what choice of the angles will the area of the quadrilateral be maximal?

   Solution: Let us call the vertex between aa and d A, the vertex between a and b B, and so on, the other two are C and D. Also, let us call the angle at vertex A α, and the angle at vertex C β. Let us connect vertices B and D: then, we get two smaller triangles, whose areas can be calculated from their sides and one of their angles: the sum of these gives the area of the quadrilateral:

$$T = \frac{ab\sin(\alpha)}{2} + \frac{cd\sin(\beta)}{2}.$$

   Let us maximize $2T$ (this is the same as the maximization of T), and let us use the notation

$$f(\alpha, \beta) = ab\sin(\alpha) + cd\sin(\beta).$$

   However, the values of α and β cannot be arbitrary: let us use the law of cosines for the triangles: let us call the common side of the triangles x. Then,

$$a^2 + b^2 - 2ab\cos(\alpha) = x^2,$$

$$c^2 + d^2 - 2cd\cos(\beta) = x^2.$$

   Consequently,

$$a^2 + b^2 - 2ab\cos(\alpha) = c^2 + d^2 - 2cd\cos(\beta),$$

   so the condition in this case is

$$g(\alpha, \beta) = a^2 + b^2 - 2ab\cos(\alpha) - c^2 - d^2 + 2cd\cos(\beta) = 0.$$

   The Lagrange function in this case is

$$L(\alpha, \beta) = ab\sin(\alpha) + cd\sin(\beta) - \lambda(a^2 + b^2 - 2ab\cos(\alpha) - c^2 - d^2 + 2cd\cos(\beta))$$

   The derivatives are

$$\frac{\partial L}{\partial \alpha} = ab\cos(\alpha) + 2ab\lambda\sin(\alpha),$$

$$\frac{\partial L}{\partial \beta} = cd\cos(\beta) - 2cd\lambda\sin(\beta),$$

so the equations we have to solve are

$$\begin{aligned}
ab\cos(\alpha) + 2ab\lambda\sin(\alpha) &= 0,\\
cd\cos(\beta) - 2cd\lambda\sin(\beta) &= 0.
\end{aligned}$$

From the first one we get $\cos(\alpha) = -2\lambda\sin(\alpha)$, and then $\cot(\alpha) = -2\lambda$. Similarly, from the second one $\cos(\beta) = 2\lambda\sin(\beta)$, and then $\cot(\beta) = 2\lambda$. Then, it means that

$$\cot(\alpha) = -\cot(\beta),$$

meaning that

$$\cot(\alpha) = \cot(-\beta).$$

Since the cotangent function is periodic, this can only hold if $\alpha = -\beta + k\pi$. However, since we are talking about a quadrilateral, only the $k = 1$ case can be real, meaning that $\alpha = -\beta + \pi$ so $\alpha + \beta = \pi$, so the area is maximal if we have a cyclic quadrilateral.

4. Assume that the point $P(a, b, c) \in \mathbb{R}^3$ $(a, b, c > 0)$ is on the plane L. For which plane will the volume of the space between the plane and the three axes be maximal?

   Solution: Assume that the equation of the plane is

$$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1.$$

Since the point $P(a, b, c)$ is on the plane, we also know that

$$\frac{a}{A} + \frac{b}{B} + \frac{c}{C} = 1.$$

Then, the volume in question is given by

$$V = \frac{1}{6}ABC.$$

Let us consider $6V$ instead, so maximize the function

$$f(A, B, C) = ABC.$$

So we have to maximize this function if the constraint

$$\frac{a}{A} + \frac{b}{B} + \frac{c}{C} = 1$$

holds.

The Lagrange function is

$$L(A, B, C) = ABC - \lambda\left(\frac{a}{A} + \frac{b}{B} + \frac{c}{C}\right).$$

The derivatives are

$$\begin{aligned}
\frac{\partial L}{\partial A} &= BC + \lambda\frac{a}{A^2},\\
\frac{\partial L}{\partial B} &= AC + \lambda\frac{b}{B^2},\\
\frac{\partial L}{\partial C} &= AB + \lambda\frac{c}{C^2}.
\end{aligned}$$

The equations we have to solve are

$$\begin{aligned}
BC + \lambda\frac{a}{A^2} &= 0,\\
AC + \lambda\frac{b}{B^2} &= 0,\\
AB + \lambda\frac{c}{C^2} &= 0.
\end{aligned}$$

Then, by some rearrangements we get

$$\lambda\frac{a}{A} = -\frac{ABC}{\lambda},$$
$$\lambda\frac{b}{B} = -\frac{ABC}{\lambda},$$
$$\lambda\frac{c}{C} = -\frac{ABC}{\lambda}.$$

By the constraint, we get

$$-\frac{ABC}{\lambda} - \frac{ABC}{\lambda} - \frac{ABC}{\lambda} = 1.$$

From which $-\frac{ABC}{\lambda} = \frac{1}{3}$. Then, from the previous equations $A = 3a$, $B = 3b$, $C = 3c$.

5. Let us assume that we have some point masses m_i at points $P_i(x_i, y_i, z_i)$ $(i = 1, 2, \ldots n)$. At which point is the second moment of area minimal?

   Solution: Let us assume that the point we are looking for is $P_0(x, y, z)$. Then, the distance between the points P_i and P_0 is

   $$d_i = (x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2.$$

   The second moment of area is given by $\Theta = \sum_{i=1}^n \Theta_i = \sum_{i=1}^n m_i d_i^2$, so the function we have to minimize is

   $$\Theta = \sum_{i=1}^n \Theta_i = \sum_{i=1}^n m_i \left((x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2\right).$$

   The partial derivatives are

   $$\frac{\partial\Theta}{\partial x} = \sum_{i=1}^n m_i 2(x_i - x) = 2\sum_{i=1}^n m_i x_i - 2x\sum_{i=1}^n m_i,$$
   $$\frac{\partial\Theta}{\partial y} = \sum_{i=1}^n m_i 2(y_i - y) = 2\sum_{i=1}^n m_i y_i - 2y\sum_{i=1}^n m_i,$$
   $$\frac{\partial\Theta}{\partial z} = \sum_{i=1}^n m_i 2(z_i - z) = 2\sum_{i=1}^n m_i z_i - 2z\sum_{i=1}^n m_i,$$

   If all of these are zeros, then we get the equations

   $$x_0 = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \qquad y_0 = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}, \qquad z_0 = \frac{\sum_{i=1}^n m_i z_i}{\sum_{i=1}^n m_i}.$$

   The Hessian matrix in this case is

   $$\begin{pmatrix} 2\sum_{i=1}^n m_i & 0 & 0 \\ 0 & 2\sum_{i=1}^n m_i & 0 \\ 0 & 0 & 2\sum_{i=1}^n m_i \end{pmatrix},$$

   which has eigenvalues $2\sum_{i=1}^n m_i$ which are positive, so the point is indeed a minimum. (The point is actually a centroid.)